CO 355 Mathematical Optimization (Fall 2010)
Assignment 4

Due: Thursday, November 11th, in class.

Policy. No collaboration is allowed. You may use the course notes / textbook and the lecture slides, but **please be very specific** when using citing results found there. (Don't just say "from some claim in class we know....".) Every other resource that you might stumble upon must be properly referenced. You are welcome to seek help from the current instructor and TAs for CO 355.

Question 1: (10 points) [Exercise 3.2.3]
Let $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R}$ be a convex function. For any points $x^1, ..., x^m \in S$ and scalars $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, prove that

$$f\left(\sum_{i=1}^m \lambda_i x^i\right) \;\leq\; \sum_{i=1}^m \lambda_i f(x^i).$$

Question 2: (15 points) [Exercise 3.2.7]

- **(a):** Let $\mathbb{R}_{++} = \{\, x \in \mathbb{R} \,:\, x > 0 \,\}$. Let $g : \mathbb{R}_{++} \to \mathbb{R}$ be defined by $g(x) = -\log(x)$. (This is the natural logarithm.) Prove that g is convex.
- **(b):** Prove that $\log y \leq y - 1$ for all $y > 0$.
- **(c):** For any scalars $x^1, ..., x^m > 0$ and scalars $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, prove that

$$\sum_{i=1}^m \lambda_i x^i \;\geq\; \prod_{i=1}^m (x^i)^{\lambda_i}.$$

Question 3: (10 points) [Exercise 3.3.2]
Let $S \subseteq \mathbb{R}^n$ be an arbitrary set and let $f : S \to \mathbb{R}$ be a function. Recall that f is *coercive* if, for every $\alpha \in \mathbb{R}$, the sublevel set $\{\, x \,:\, f(x) \leq \alpha \,\}$ is bounded. Prove that being coercive is equivalent to the statement: for every sequence $x_1, x_2, ... \in S$ such that $\lim_{i\to\infty} \|x_i\| = \infty$ then $\lim_{i\to\infty} f(x_i) = \infty$.

Question 4: (10 points) [Exercise 3.4.2]
Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$ be arbitrary. We are interested in the constrained optimization problem:

$$\begin{array}{llll} & \min & f(x) & \\ \text{(NLP)} & \text{s.t.} & g_i(x) \leq 0 & \forall i = 1, ..., p \\ & & x \in S & \end{array}$$

where $g_1, ..., g_p : S \to \mathbb{R}$. For notational convenience, let $g : S \to \mathbb{R}^p$ be defined by $g(x) = (g_1(x), ..., g_p(x))$. Assume that

- $\bar{x}$ is feasible for (NLP),
- there exists $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that $\bar{x}$ minimizes the Lagrangian function $f(x) + \lambda^{\mathsf{T}} g(x)$ over S, and

- complementary slackness holds, meaning that for every $i = 1, ..., p$, either $g_i(\bar{x}) = 0$ or $\lambda_i = 0$, or both.

Let $z \in \mathbb{R}^p$ be arbitrary. Prove that

$$\inf \{ \ f(x) \ : \ g(x) \le z, \ x \in S \ \} \ \ \ge \ f(\bar{x}) - \lambda^{\mathsf{T}} z.$$

NOTE: The course notes have an error: they incorrectly write "+" instead of "−".

Question 5: (15 points) [Exercise 3.4.3]

The optimal value of (NLP) is $\inf \{ \ f(x) \ : \ x \text{ feasible} \ \}$. The *dual function* for (NLP) is

$$\Phi(\mu) \ = \ \inf \Big\{ \ f(x) + \mu^{\mathsf{T}} g(x) \ : \ x \in S \ \Big\},$$

the *dual problem* is

$$\begin{array}{lll} & \max & \Phi(\mu) \\ \text{(NLP-dual)} & \text{s.t.} & 0 \le \mu \in \mathbb{R}^p \end{array}$$

and the dual optimal value is $\sup \{ \ \Phi(\mu) \ : \ \mu \ge 0 \ \}$.

(a): Prove that the optimal value of (NLP) is at least the dual optimal value.

(b): Write down and simplify the dual to the linear program:

$$\begin{array}{llll} \min & c^{\mathsf{T}} x & & \\ \text{s.t.} & (a^j)^{\mathsf{T}} x - b_j & \le 0 & \quad j = 1, ..., p \\ & x & \in \mathbb{R}^n & \end{array}$$

(c): Make the same three assumptions as in Question 4. Prove that $(\bar{x}; \lambda)$ is a *saddlepoint* of the Lagrangian: for all $x \in S$ and vectors $\mu \ge 0$ in $\mathbb{R}^p$, we have

$$f(\bar{x}) + \mu^{\mathsf{T}} g(\bar{x}) \le f(\bar{x}) + \lambda^{\mathsf{T}} g(\bar{x}) \le f(x) + \lambda^{\mathsf{T}} g(x).$$

Deduce that the vector λ is optimal for (NLP-dual), with optimal value equalling the optimal value of (NLP).

Question 6: (20 points) [Exercise 3.4.5]

Show that there is no Lagrange multiplier for the problem

$$\begin{array}{lll} \min & e^{x_2} & \\ \text{s.t.} & \|x\| - x_1 & \le 0 \\ & x & \in \mathbb{R}^2. \end{array}$$

In other words, show that for any number $\lambda \ge 0$ in $\mathbb{R}$, no optimal solution of the problem minimizes the Lagrangian $e^{x_2} + \lambda(\|x\| - x_1)$.