CO 355 Mathematical Optimization (Fall 2010)
Assignment 3

Due: Tuesday, November 2nd, in class.

Policy. No collaboration is allowed. You may use the course notes / textbook and the lecture slides, but **please be very specific** when using citing results found there. (Don't just say "from some claim in class we know....".) Every other resource that you might stumble upon must be properly referenced. You are welcome to seek help from the current instructor and TAs for CO 355.

Let C be a non-empty convex set. Let A be an affine space such that $C \subseteq A$ and $\dim A = \dim C$. In fact, A is unique, and it is called the ***affine hull*** of C. We say that a point $x \in C$ is a ***relative interior*** point of C if there exists a closed ball $B(x, \epsilon)$ centered at x of radius $\epsilon > 0$ such that $B(x, \epsilon) \cap A \subseteq C$. (In other words, $y \in A$ and $\|x - y\| \leq \epsilon$ jointly imply $y \in C$.) The set of all relative interior points of C is denoted $\mathrm{ri}(C)$. The relative interior is often very useful because $\mathrm{ri}(C)$ is non-empty and convex (assuming C is non-empty and convex). You may use this fact anywhere in this assignment.

Question 1: (10 points)
Suppose that C_1 and C_2 are non-empty convex sets such that $C_1 \subseteq C_2$. Is $\mathrm{ri}(C_1) \subseteq \mathrm{ri}(C_2)$? Give a proof or a counterexample.

Let $P = \left\{ x : a_i^\mathsf{T} x \leq b_i \ \forall i = 1...m \right\}$ be a polyhedron. Consider the following three properties that a set $F \subseteq P$ might satisfy.

$$\exists c \in \mathbb{R}^n, \gamma \in \mathbb{R} \text{ such that } P \subseteq \left\{ x : c^\mathsf{T} x \leq \gamma \right\} \text{ and } F = P \cap \left\{ x : c^\mathsf{T} x = \gamma \right\} \tag{1}$$

$$F \text{ is convex, and whenever } y, z \in P,\ \alpha \in (0,1) \text{ satisfy } \alpha y + (1-\alpha) z \in F \text{ then both } y, z \in F \tag{2}$$

$$\exists I \subseteq \{1, ..., m\} \text{ such that } F = P \cap \left\{ x : a_i^\mathsf{T} x = b_i \ \forall i \in I \right\} \tag{3}$$

In class, we called a set satisfying (1) a ***face*** of the polyhedron P. In the next four questions, we prove that properties (1)-(3) are equivalent. Assume that $\emptyset \neq F \subseteq P$.

Question 2: (10 points)
Prove (1) implies (2).

Question 3: (10 points)
Prove (3) implies (1).

Question 4: (15 points)
Let F be a set satisfying (2). (Recall our assumption $F \neq \emptyset$.) Define

$$\mathcal{I}_F = \left\{ i : a_i^\mathsf{T} x = b_i \ \forall x \in F \right\} \qquad \text{and} \qquad \mathcal{A}_F = \left\{ a_i : i \in \mathcal{I}_F \right\}.$$

Prove that $\operatorname{rank} \mathcal{A}_F = n - \dim F$.

Hint: Find a point $y \in F$ such that $a_i^\mathsf{T} y < b_i$ for all $i \notin \mathcal{I}_F$.

Question 5: (15 points)
Prove that (2) implies (3).

Let $P = \{ \ x \ : \ a_i^\intercal x \leq b_i \ \forall i = 1 \ldots m \ \}$ be a polyhedron.

Question 6: (15 points)
Recall that a face of P is a set satisfying (1). Let F be a face of P. Prove that

$$\{ \ F' \ : \ \ F' \text{ is a face of } F \ \} \ = \ \{ \ F' \ : \ F' \text{ is a face of } P, \ \text{ and } \ F' \subseteq F \ \}.$$

Question 8: (10 points)
Find a convex but **non-polyhedral** set P and a non-empty set $F \subseteq P$ such that F satisfies (2) but F does not satisfy (1).

Question 7: Optional Bonus Question (20 points)
Assume that $\dim P = d$. Recall that a facet of P is a face of dimension $d - 1$. Let F_1 be a facet of P and let F' be a facet of F_1. Prove that there exists a facet F_2 of P such that $F' = F_1 \cap F_2$.