CO 355 Mathematical Optimization (Fall 2010)
Assignment 2

Due: Thursday October 14th, in class.

Policy. No collaboration is allowed. You may use the course notes / textbook and the lecture slides, but **please be very specific** when using citing results found there. (Don't just say "from some claim in class we know....".) Every other resource that you might stumble upon must be properly referenced. You are welcome to seek help from the current instructor and TAs for CO 355.

The Fundamental Theorem of Linear Programming: We now complete the proof of this theorem.

Question 1: (10 points)
In class we proved the following variant of Farkas' Lemma:

$$Ax \leq b \text{ has no solution} \quad \iff \quad \exists y \geq 0 \text{ such that } y^{\mathsf{T}}A = 0 \text{ and } y^{\mathsf{T}}b < 0 \tag{1}$$

Using Eq. (1), prove the following variant of Farkas' Lemma:

$$Mu = d,\ u \geq 0 \text{ has no solution} \quad \iff \quad \exists w \text{ such that } w^{\mathsf{T}}M \geq 0 \text{ and } w^{\mathsf{T}}d < 0. \tag{2}$$

Question 2: (10 points)
Consider the LP $\max\left\{ c^{\mathsf{T}}x : Ax \leq b \right\}$. Suppose that it is feasible but its dual is *in*feasible. Use the variant of Farkas' lemma in Eq. (2) to prove that the (primal) LP is unbounded.

Properties of Convex Combinations.

Question 3: (15 points)
Let $S \subseteq \mathbb{R}^n$ be arbitrary. The ***convex hull*** of S, denoted by $\text{conv}(S)$, is defined to be the intersection of all convex sets that contain S. Prove that $\text{conv}(S)$ is precisely the set of all convex combinations of points in S. (This is Proposition 2.25 in the course notes.)

Claims needed for the Ellipsoid method.

Question 4: (15 points)
Consider the rank-1 update matrix $M = I + \alpha zz^{\mathsf{T}}$, where $\alpha \in \mathbb{R}$, $z \in \mathbb{R}^n$ and $z \neq 0$. Suppose that $\alpha \geq -1/z^{\mathsf{T}}z$. Prove that M is positive semi-definite.

Note. If you use a definition of a positive semi-definite matrix M other than the one given in class (i.e., $M = V^{\mathsf{T}}V$ for some matrix V) then you must formally cite a reference proving (or give your proof of) the equivalence of the definitions.

Question 5: (10 points)
In our discussion of "covering half-ellipsoids by ellipsoids", we defined an invertible, affine map f such that $f(B) = E$ and $f(H_u) = H_a$. We then defined $E' = f(B')$, where B' is an ellipsoid covering $B \cap H_u$. Prove that $E \cap H_a \subseteq E'$. (This is Claim 5 in the notes for Lecture 7.)

Note. Prove this carefully!

Swapping max and min.

Question 6: (10 points)
Let $f(x, y)$ be any real-valued function of the vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be arbitrary. Prove that

$$\sup_{x \in P} \inf_{y \in Q} f(x, y) \;\leq\; \inf_{y \in Q} \sup_{x \in P} f(x, y).$$

(If you don't know what sup and inf mean, just assume sup means max and inf means min, and don't worry about the max or min not being achieved.)

Question 7: (20 points)
Let

$$f(x, y) \;=\; c^{\mathsf{T}} x - y^{\mathsf{T}} A x + y^{\mathsf{T}} b,$$

where $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are vectors and A is an $m \times n$ matrix. Suppose that the LP $\max \{\, c^{\mathsf{T}} x \,:\, Ax \leq b \,\}$ and its dual $\min \{\, b^{\mathsf{T}} y \,:\, A^{\mathsf{T}} y = c,\, y \geq 0 \,\}$ are both feasible. Prove that

$$\sup_{x \in \mathbb{R}^n} \inf_{\substack{y \in \mathbb{R}^m \\ y \geq 0}} f(x, y) \;=\; \inf_{\substack{y \in \mathbb{R}^m \\ y \geq 0}} \sup_{x \in \mathbb{R}^n} f(x, y).$$

Hint. Consider optimal solutions for the LP and its dual.

Remark. This equality is true if either the LP or its dual is feasible, but false if both are infeasible.