CO 355 Mathematical Optimization (Fall 2010)
Assignment 1

Due: Tuesday September 28th, in class.

Policy. No collaboration is allowed. You may only use the course notes / textbook and the lecture slides. Every other resource that you might stumble upon must be properly referenced. You are welcome to seek help from the current instructor and TAs for CO 355.

Question 1: (15 points)
(You do not need to be very rigorous for this question.)

In class we claimed the "Fundamental Theorem of LPs": every linear program either has an optimal solution, is unbounded, or is infeasible.

Consider the following linear program in $\mathbb{R}^2$, whose objective function has not yet been specified.

$$\begin{array}{lll} \min & ax + by & \\ \text{s.t.} & x - y & \leq 2 \\ & -2x + y & \leq 2 \\ & x + y & \geq 2 \end{array}$$

- **(a):** Find an objective function (i.e., values of a and b) for which the linear program has a unique optimal solution.
- **(b):** Find an objective function for which the linear program has infinitely many optimal solutions.
- **(c):** Is there an objective function for which the linear program is unbounded?
- **(d):** Is there an objective function for which the linear program is infeasible?

Question 2: (10 points)

Prove that the conclusions of the "Fundamental Theorem" are not satisfied by the following mathematical program:

$$\begin{array}{lll} \max & x_2 & \\ \text{s.t.} & x_1 & \geq 1 \\ & x_1 x_2 & \leq -1 \end{array}$$

In other words, it is not infeasible, is not unbounded and does not have any optimal solution. (If you know what "sup" means, the question can be restated: show that

$$\sup \{\, x_2 \,:\, x_1 \geq 1 \text{ and } x_1 x_2 \leq -1 \,\}$$

is finite but not achieved.)

Question 3: (15 points)

There are n students in a classroom. Every pair of students are either enemies or friends. (The relationship is mutual — it is never the case that student A thinks student B is a friend, but person B thinks otherwise.) The teacher would like to remove as few students as possible from the classroom so that, for every pair of enemies, at least one of them is removed.

- **(a):** Formulate this as an **integer** program.
- **(b):** Now obtain a **linear** program by removing the restriction that the variables are integers. Do your integer program and linear program always have the same optimal value, regardless the problem input (i.e., regardless of which students are friends or enemies)?

Question 4: (12 points)

Are the following sets $C \subset \mathbb{R}^2$ convex?

- **(a):** $C = \{ (x, y) : y \geq 0 \} \setminus \{(0, 0)\}$.
- **(b):** $C = \{ (x, y) : x \geq 0 \text{ and } y \geq 0 \} \setminus \{(0, 0)\}$.
- **(c):** $C = \{ (x, y) : x > 0 \text{ and } y > 0 \} \cup \{(0, 0)\}$.
- **(d):** $C = \left\{ (x, y) : y \geq ax^2 + bx + c \right\}$, where $a, b, c \in \mathbb{R}$ and $a > 0$.

Question 5: (15 points)
Give an example of a linear program in inequality form (i.e., $\max \left\{ c^\mathsf{T} x : Ax \leq b \right\}$) such that it is infeasible **and** its dual is also infeasible. Prove your claims of infeasibility algebraically.

Question 6: (20 points)
Consider an inequality form LP $\max \left\{ c^\mathsf{T} x : Ax \leq b \right\}$. Let x be a feasible point. Prove, using (any variant of) Farkas' lemma, that exactly one of the following holds.

- c is a non-negative linear combination of the constraints that are tight at x.
- x is not an optimal solution.

You may **not** use the Strong LP Duality Theorem as part of your proof.