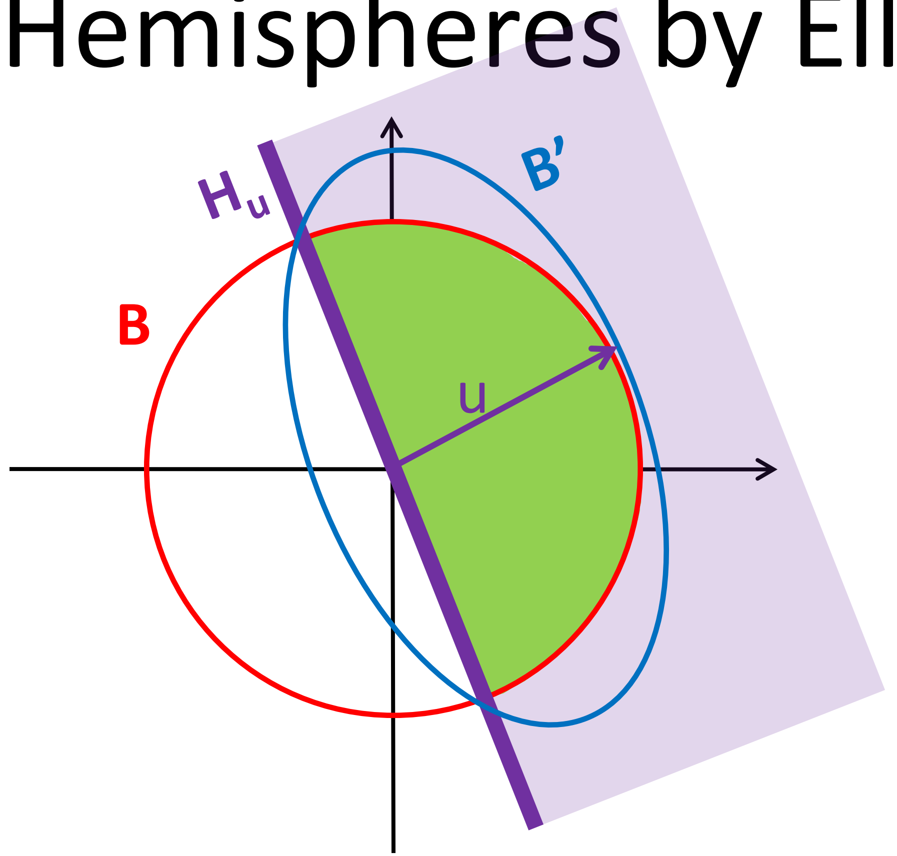
C&O 355
Mathematical Programming
Fall 2010
Lecture 7

N. Harvey

Covering Hemispheres by Ellipsoids



- Let B = { unit ball }.
- Let H_u = { $x : x^T u \geq 0$ }, where $\|u\|=1$.
- Find a small ellipsoid B' that covers B∩H.

Rank-1 Updates

- **Def:** Let z be a column vector and α a scalar. A matrix of the form $I + \alpha z z^\mathsf{T}$ is called a **rank-1 update matrix**.
- **Claim 1:** Suppose $\alpha \neq -1/z^\mathsf{T}z$. Then $(I + \alpha z z^\mathsf{T})^{-1} = I + \beta z z^\mathsf{T}$ where $\beta = -\alpha/(1+\alpha z^\mathsf{T}z)$.
- **Claim 2:** If $\alpha \geq -1/z^\mathsf{T} z$ then $I + \alpha z z^\mathsf{T}$ is PSD.
  If $\alpha > -1/z^\mathsf{T} z$ then $I + \alpha z z^\mathsf{T}$ is PD.
- **Claim 3:** $\det(I + \alpha z z^\mathsf{T}) = 1 + \alpha z^\mathsf{T} z$

Main Theorem:

Let B = { x : $\|x\| \leq 1$ } and H_u = { x : $x^T u \geq 0$ }, where $\|u\|=1$.

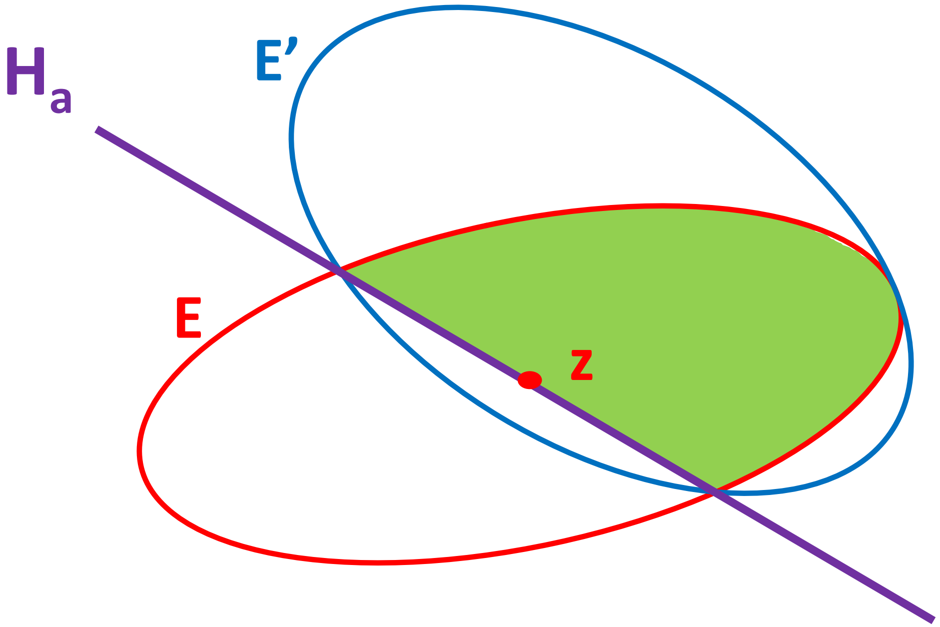
Let $M = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} u u^{\mathsf{T}}\Big)$ and $b = \frac{u}{n+1}$.

Let B' = E(M, b). Then:

1) $B \cap H_u \subseteq B'$.
2) $\frac{\mathrm{vol}(B')}{\mathrm{vol}(B)} \leq e^{-1/4(n+1)} \leq 1 - \frac{1}{8(n+1)}$

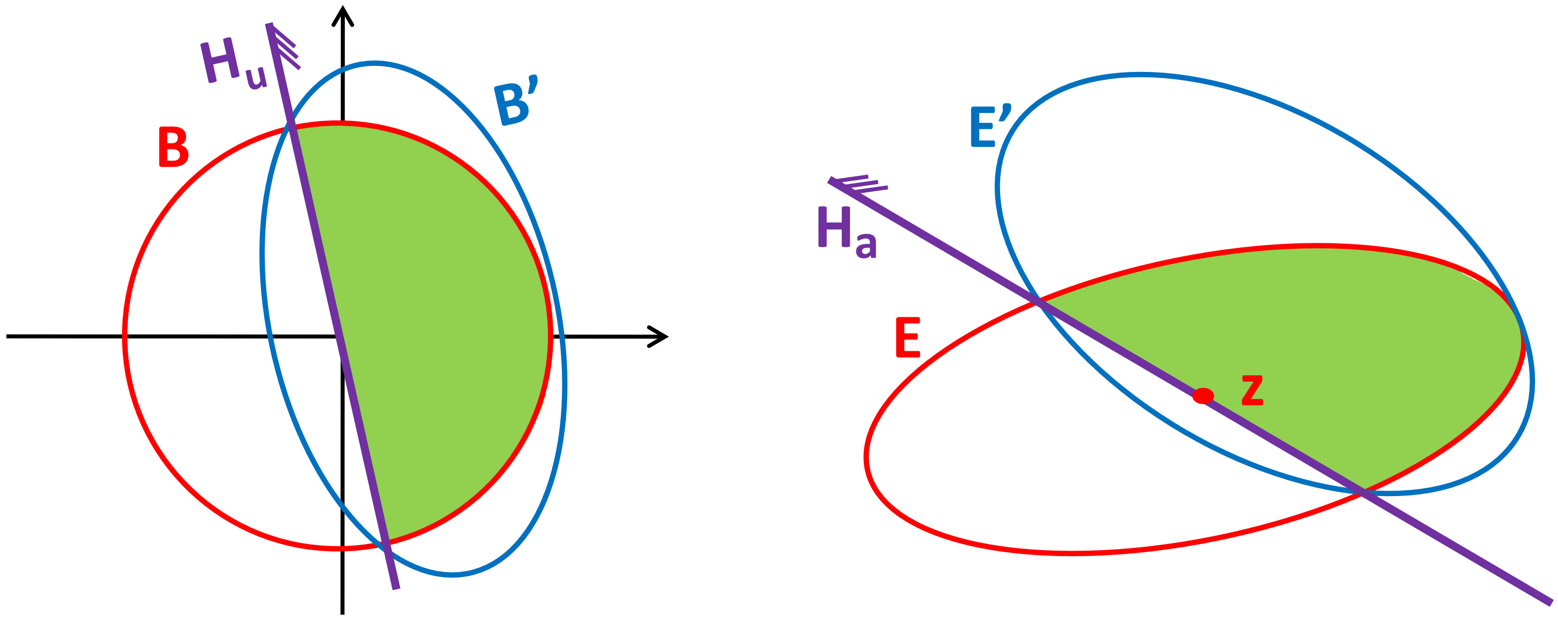
Remark: This notation only makes sense if M is positive definite.
Claim 2 on rank-1 updates shows that it is, assuming $n \geq 2$.

Covering Half-ellipsoids by Ellipsoids



- Let E be an ellipsoid centered at z
- Let $H_a = \{ x : a^T x \geq a^T z \}$
- Find a small ellipsoid E' that covers $E \cap H_a$

Use our solution for hemispheres!



Goal

Find an affine map f and choose u such that:

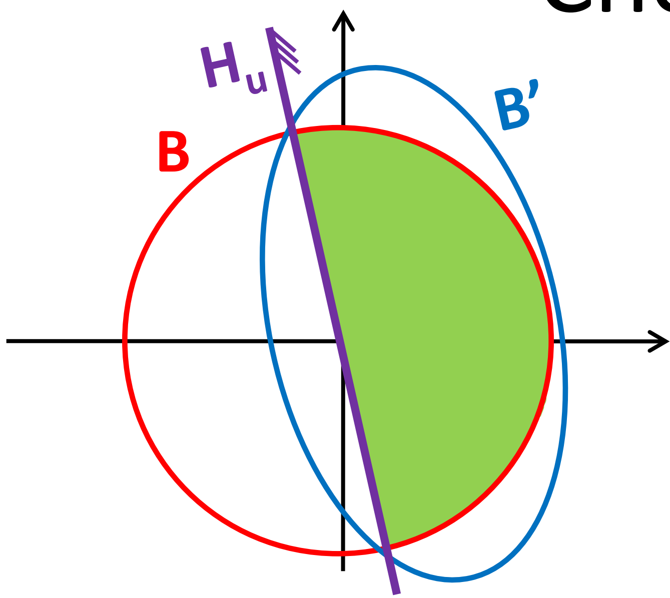
$$f(B) = E \qquad \text{and} \qquad f(H_u) = H_a$$

Define E’ = f(B’).

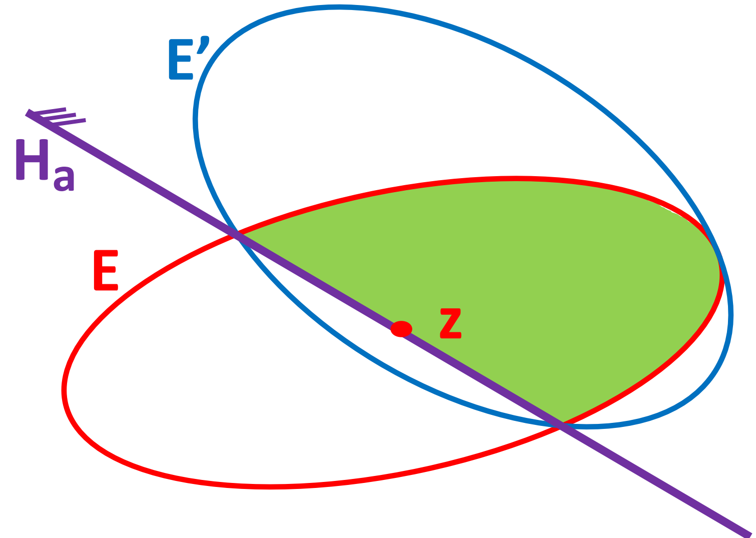
Claim: E’ is an ellipsoid.

Claim: $E \cap H_a \subseteq E'$.

Choosing u



$H_u = \{ x : u^T x \geq 0 \}$



$H_a = \{ x : a^T (x-z) \geq 0 \}$

- Assume E=E(N,z) and consider the map f(x) = $N^{1/2} x + z$. In Lecture 6 we showed that E = f(B).
- Now choose u such that f(H_u) = H_a.

$$f(H_u) = \left\{ N^{1/2}x + z \;:\; u^{\mathsf{T}} x \geq 0 \right\}$$
$$= \left\{ x \;:\; u^{\mathsf{T}} N^{-1/2}(x-z) \geq 0 \right\}$$

$\Rightarrow$ take u = $N^{1/2}$a