C&O 355
Mathematical Programming
Fall 2010
Lecture 6

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Polyhedra

- **Definition:** For any $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, define

$$H_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x = b \}$$

Hyperplane

$$H^+_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x \geq b \}$$

$$H^-_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x \leq b \}$$

Halfspaces

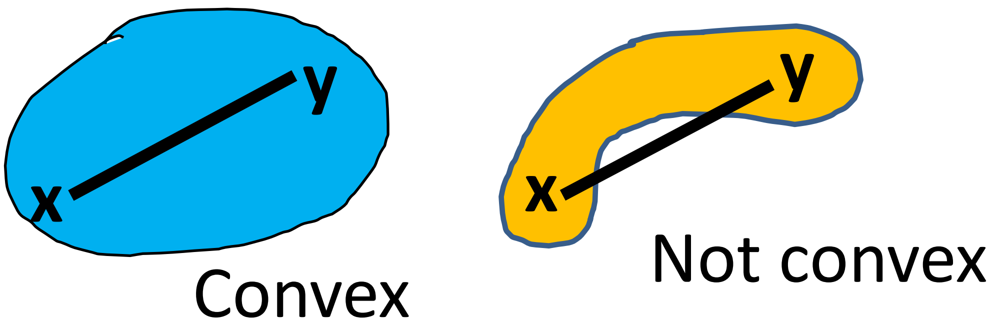
- **Def:** Intersection of finitely many halfspaces is a **polyhedron**
- **Def:** A **bounded** polyhedron is a **polytope**, i.e., $P \subseteq \{ x : -M \leq x_i \leq M \;\; \forall i \}$ for some scalar M

- So the feasible region of LP is polyhedron $P = \bigcap_{i=1}^{m} H^-_{a_i,b_i}$

$$\begin{array}{lll} \max & c^\mathsf{T} x & \\ \text{s.t.} & a_i^\mathsf{T} x & \leq b_i \quad \forall i = 1 \ldots m \end{array}$$

Convex Sets

- **Def:** Let $x_1,\ldots,x_k \in \mathbb{R}^n$.
  Let $\alpha_1,\ldots,\alpha_k$ satisfy $\alpha_i \geq 0$ for all i and $\sum_i \alpha_i = 1$.
  The point $\sum_i \alpha_i x_i$ is a **convex combination** of the x_i's.
- **Def:** A set $C \subseteq \mathbb{R}^n$ is **convex** if for every $x, y \in C$ and $\forall \alpha \in [0,1]$, the convex combination $\alpha x+(1-\alpha)y$ is in C.



- **Claim 1**: Any halfspace is convex.
- **Claim 2**: The intersection of any number of convex sets is convex.
- **Corollary**: Every polyhedron is convex.

Convex Functions

- Let $C \subseteq \mathbb{R}^n$ be convex.
- **Def:** $f : C \to \mathbb{R}$ is a **convex function** if

  $$f(\alpha x+(1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \qquad \forall x,y \in C$$
- **Claim:** Let $f : C \to \mathbb{R}$ be a convex function, and let $a \in \mathbb{R}$. Then

  $$\{ x \in C : f(x) \leq a \}$$ (the "sub-level set")

  is convex.
- **Example:** Let $f(x) = \|x\| = \sqrt{x^\mathsf{T} x}$. Then f is convex.
- **Corollary:** Let $B = \{ x : \|x\| \leq 1 \}$. (The Euclidean Ball) Then B is convex.

Affine Maps

- **Def:** A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called an **affine map** if $f(x) = Ax + b$ for some matrix A and vector b.
- **Fact:** Let $C \subseteq \mathbb{R}^n$ have defined volume. Let $f(x)=Ax + b$. Then $\text{vol } f(C) = |\det A| \cdot \text{vol } C$.
- **Claim 1:** Let $C \subseteq \mathbb{R}^n$ be convex and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map. Then $f(C) = \{ f(x) : x \text{ in } C \}$ is convex.
- **Question:** If $P \subseteq \mathbb{R}^n$ is a polyhedron and f is an affine map, is it true that f(P) is a polyhedron?
- **Answer:** Yes, but it's not so easy to prove…

Fourier-Motzkin Elimination

Joseph Fourier

Theodore Motzkin

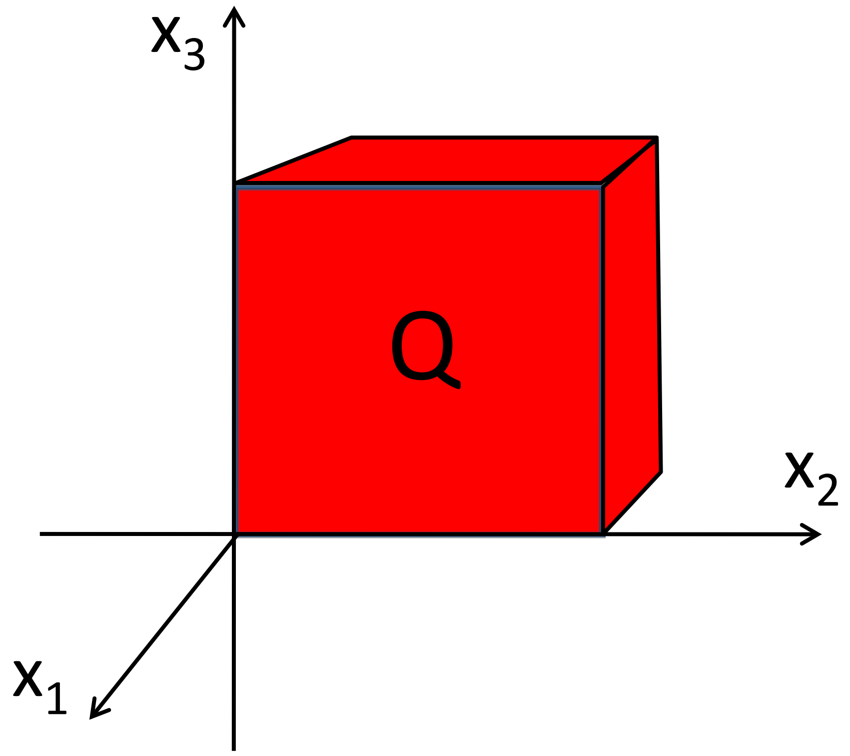
- Given a polyhedron $Q \subseteq \mathbb{R}^n$,
  we want to find the set $Q' \subseteq \mathbb{R}^{n-1}$ satisfying

$$(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n \text{ s.t. } (x_1,\ldots,x_{n-1},x_n) \in Q$$

- Q' is called the **projection** of Q onto first $n-1$ coordinates
- Fourier-Motzkin Elimination constructs Q' by generating (finitely many) constraints from the constraints of Q.

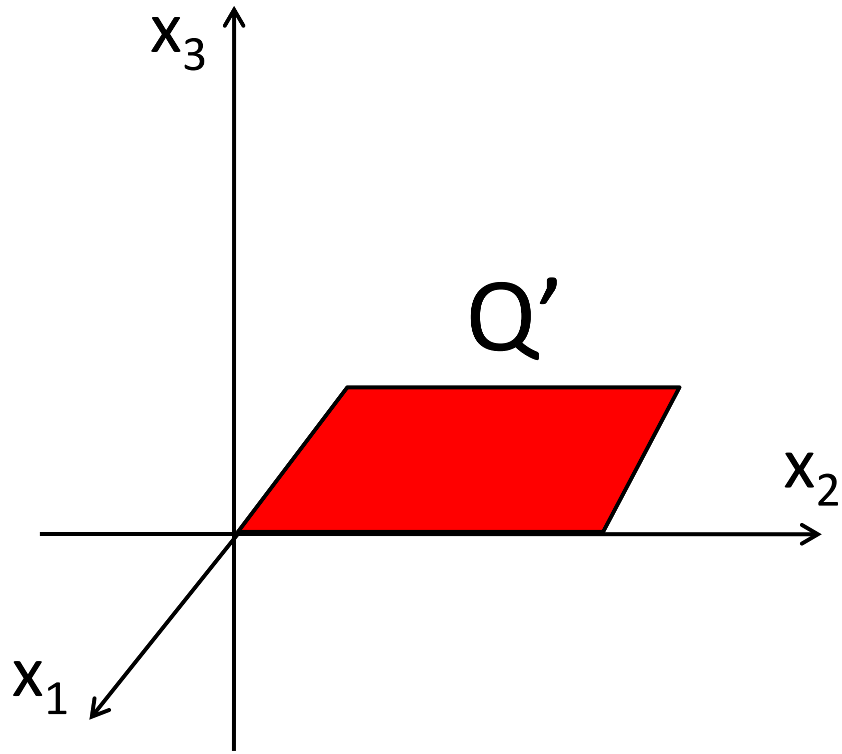
- **Corollary:** Q' is a polyhedron.

Elimination Example



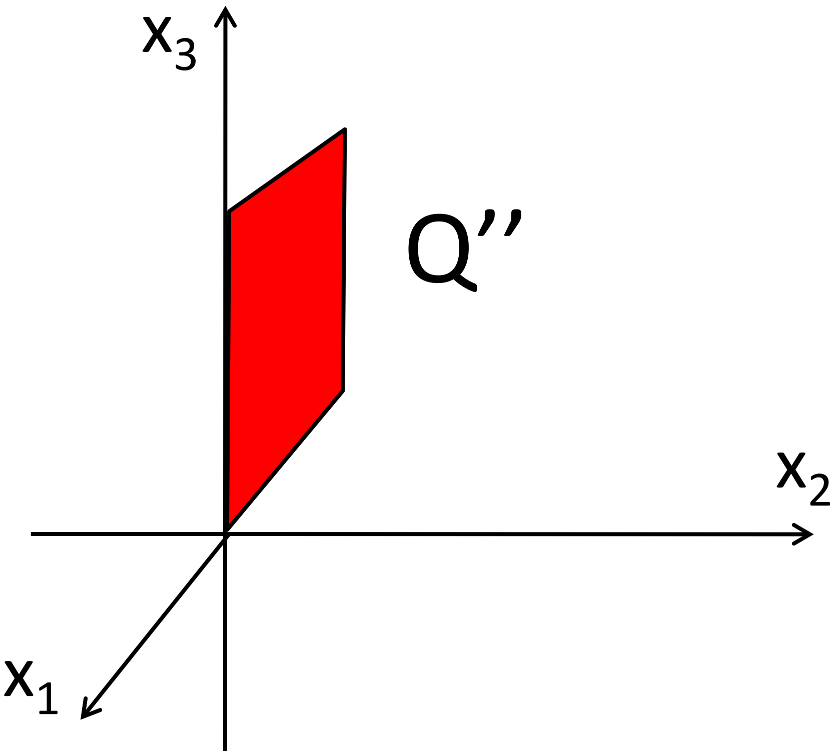
- Project Q onto coordinates $\{x_1, x_2\}$...

Elimination Example



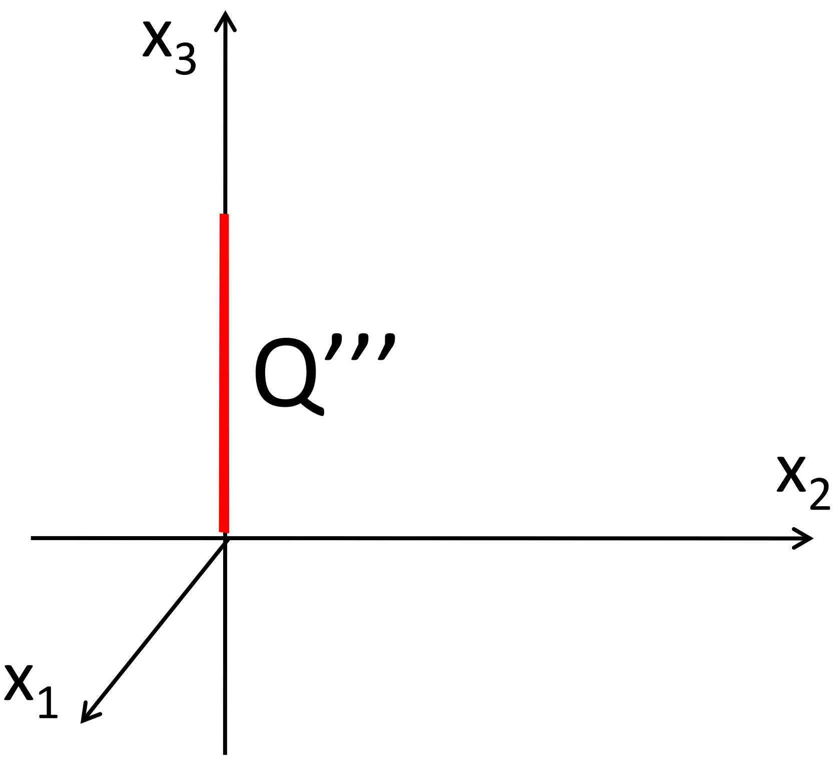
- Project Q onto coordinates $\{x_1, x_2\}$...
- **Fourier-Motzkin:** Q’ is a polyhedron.
- Of course, the ordering of coordinates is irrevelant.

Elimination Example



- Of course, the ordering of coordinates is irrevelant.
- **Fourier-Motzkin:** Q’’ is also a polyhedron.
- I can also apply Elimination twice…

Elimination Example



- **Fourier-Motzkin:** Q''' is also a polyhedron.

Projecting a Polyhedron Onto Some of its Coordinates

- **Lemma:** Given a polyhedron $Q \subseteq \mathbb{R}^n$.
  Let $S=\{s_1,\ldots,s_k\}\subseteq\{1,\ldots,n\}$ be any subset of the coordinates.
  Let $Q_S = \{ (x_{s1},\ldots,x_{sk}) : x \in Q \} \subseteq \mathbb{R}^k$.

  In other words, Q_S is projection of Q onto coordinates in S.

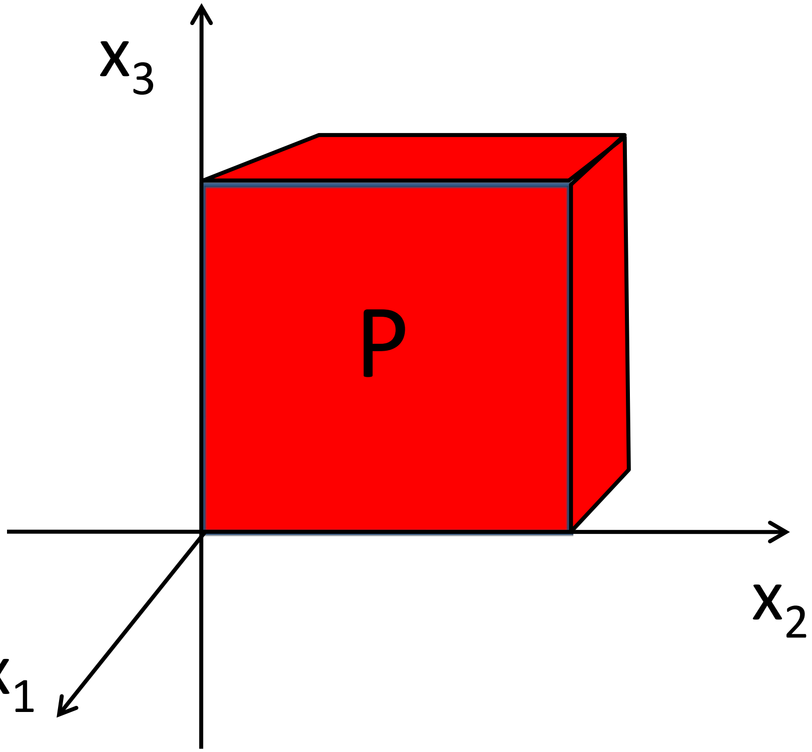
  Then Q_S is a polyhedron.
- **Proof:**
  Direct from Fourier-Motzkin Elimination.
  Just eliminate all coordinates not in S. ■

Linear Transformations of Polyhedra

- **Lemma:** Let P = $\{ x : Ax \leq b \} \subseteq \mathbb{R}^n$ be a polyhedron.
  Let M be any matrix of size $p \times n$.
  Let Q = $\{ Mx : x \in P \} \subseteq \mathbb{R}^p$. Then Q is a polyhedron.

Let M =

1	0	0
-1	1	0
0	0	1



Linear Transformations of Polyhedra

- **Lemma:** Let P = { x : $Ax \leq b$ } $\subseteq \mathbb{R}^n$ be a polyhedron.
  Let M be any matrix of size p$\times$n.
  Let Q = { Mx : $x \in P$ } $\subseteq \mathbb{R}^p$. Then Q is a polyhedron.

  Geometrically obvious, but not easy to prove...

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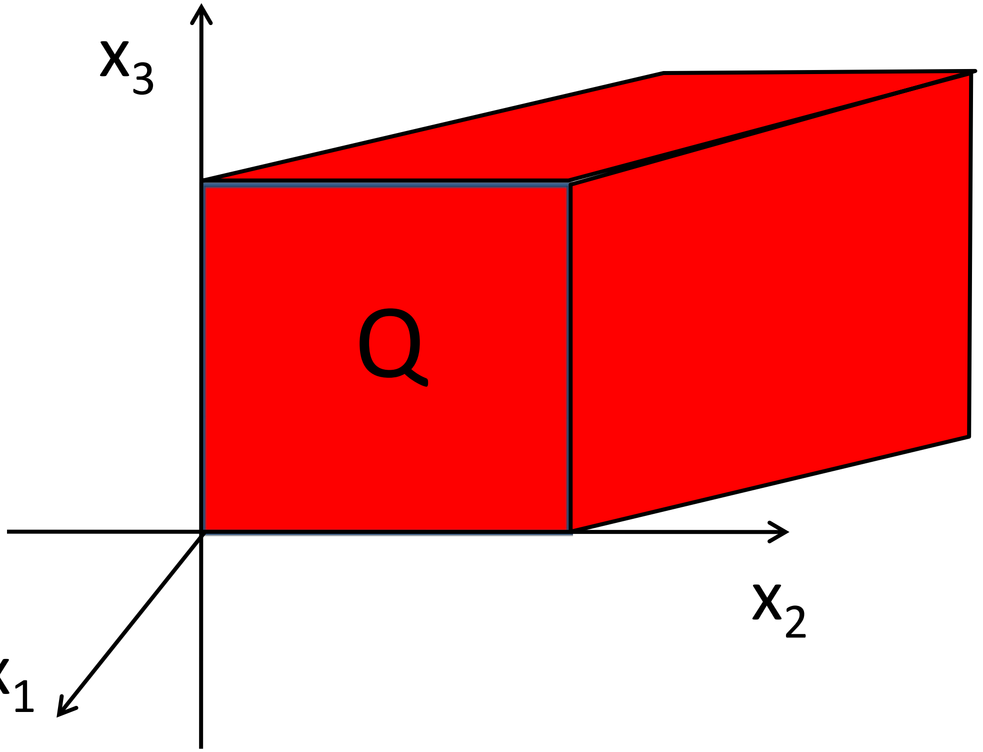


Linear Transformations of Polyhedra

- **Lemma:** Let P = $\{ x : Ax \leq b \} \subseteq \mathbb{R}^n$ be a polyhedron.
  Let M be any matrix of size p×n.
  Let Q = $\{ Mx : x \in P \} \subseteq \mathbb{R}^p$. Then Q is a polyhedron.

  Geometrically obvious, but not easy to prove...

  ...unless you know Fourier-Motzkin Elimination!

- **Proof:**

  Let $P' = \{ (x,y) : Mx=y, Ax \leq b \} \subseteq \mathbb{R}^{n+p}$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$.

  P' is obviously a polyhedron.

  Note that Q is projection of P' onto y-coordinates.

  By previous lemma, Q is a polyhedron. ■

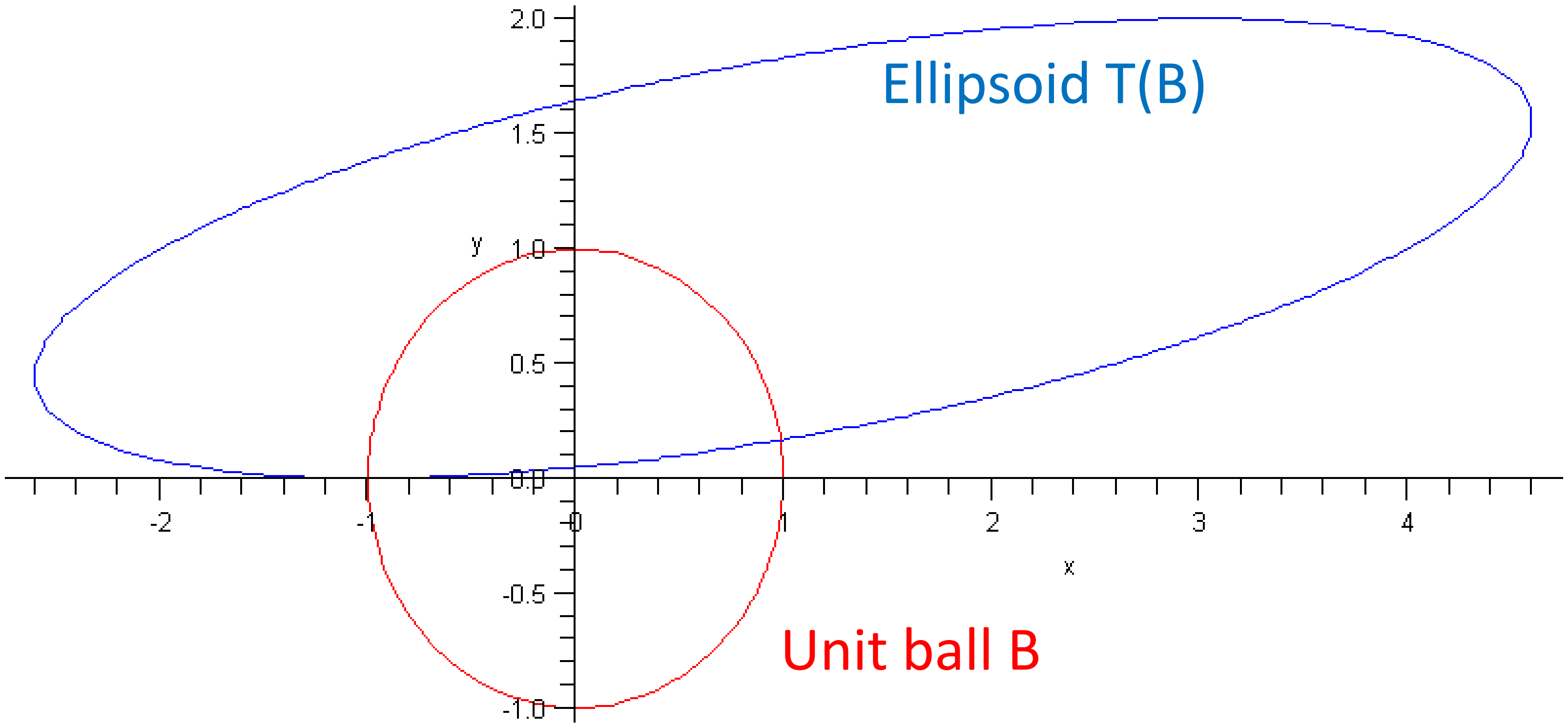
Ellipsoids

- **Def:** Let B = { x : $\|x\| \leq 1$ }. Let f : $\mathbb{R}^n \to \mathbb{R}^m$ be an affine map. Then f(B) is an **ellipsoid**.
- We restrict to the case n=m and f invertible.
  (i.e., f(x) = Ax+b where A is square and non-singular)
- **Claim 2:** f(B) = { $x \in \mathbb{R}^n$: $(x-b)^T A^{-T} A^{-1} (x-b) \leq 1$ }.

2D Example

Define $T(x) = Ax + b$ where $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

```
implicitplot([x^2+y^2=1,(x-1)^2-4*(x-1)*(y-1)+13*(y-1)^2=9], x=-5..5, y=-5..5,
numpoints=10000, color=[red,blue] );
```



Positive Semi-Definite Matrices

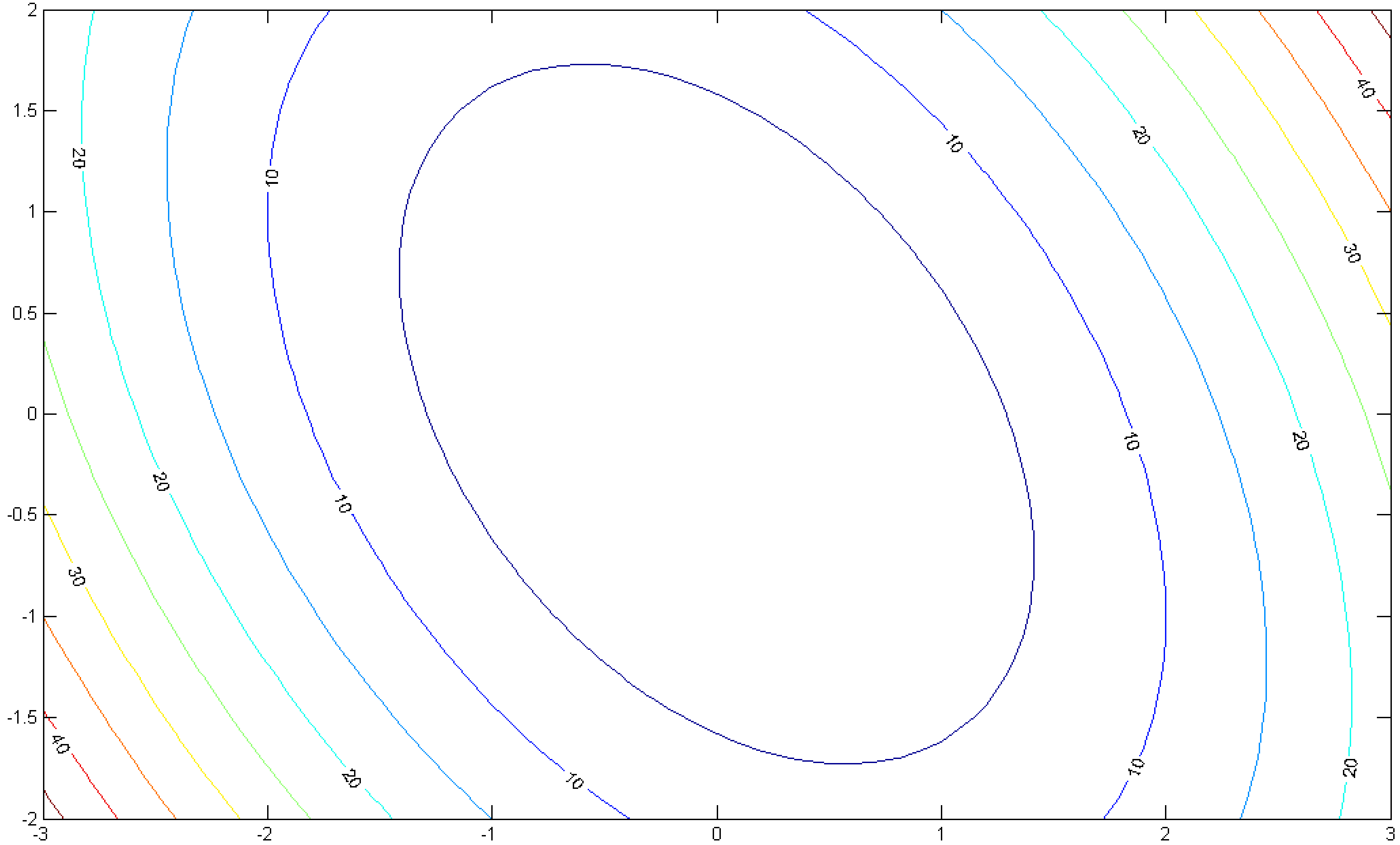
- **Def:** A real, symmetric matrix M is called **positive semi-definite** if

  $M = V^T V$, for some matrix V (not necessarily square).

  If M is also non-singular it is called **positive definite**.
- From now on "positive definite" means "real, symmetric, positive definite".
- **Fact:** If M is positive definite, then M^{-1} is also positive definite.
- **Fact:** If M is positive definite, there is a unique positive definite matrix $M^{1/2}$ such that $M = M^{1/2} M^{1/2}$. $M^{1/2}$ is called the **square root** of M.
- **Claim:** If M is positive definite then $(M^{1/2})^{-1} = (M^{-1})^{1/2}$.

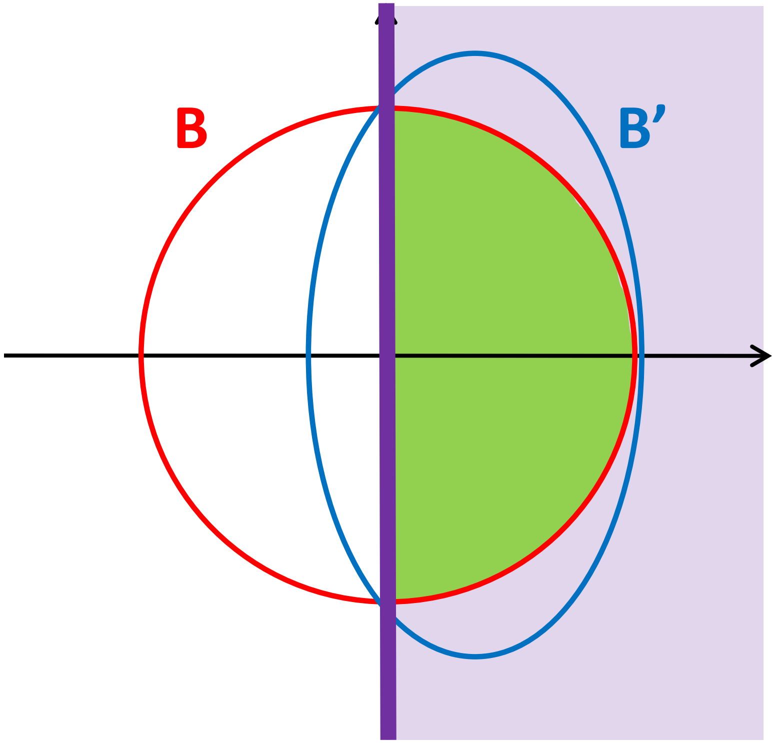
More on Ellipsoids

- We're studying ellipsoids of the form
  $$\{ x \in \mathbb{R}^n : (x-b)^T A^{-T} A^{-1} (x-b) \leq 1 \},$$
  for some (non-singular) matrix A and vector b.
- Equivalently, this is ellipsoids of the form
  $$E(M,b) = \{ x \in \mathbb{R}^n : (x-b)^T M^{-1} (x-b) \leq 1 \},$$
  for some positive definite matrix M and vector b.
- This helps us understand positive definite matrices.
  Consider $f(x) = x^T M x$, where M is positive definite.
  Its sub-level sets are $\{ x \in \mathbb{R}^n : x^T M x \leq a \} = E(aM^{-1}, 0)$.
- **Note:** $E(M,b) = f(B)$ where $f(x) = M^{1/2}x + b$.
  So $\mathrm{vol}\ E(M,b) = |\det M^{1/2}| \cdot \mathrm{vol}\ B = |\det M|^{1/2} \cdot \mathrm{vol}\ B$.



- Let $M = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$
- Plot of level sets of $x^T M x$.

Covering Hemispheres by Ellipsoids



- Let B = { unit ball }.
- Let H = $\{ x : x^T e_1 \geq 0 \} = \{ x : x_1 \geq 0 \}$.
- Find a small ellipsoid B' that covers $B \cap H$.

Rank-1 Updates

- **Def:** Let z be a column vector and α a scalar. A matrix of the form $I + \alpha z z^{\mathsf{T}}$ is called a **rank-1 update matrix**.
- **Claim 1:** Suppose $\alpha \neq -1/z^{\mathsf{T}}z$. Then $(I + \alpha z z^{\mathsf{T}})^{-1} = I + \beta z z^{\mathsf{T}}$ where $\beta = -\alpha/(1+\alpha z^{\mathsf{T}}z)$.
- **Claim 2:** If $\alpha \geq -1/z^{\mathsf{T}} z$ then $I + \alpha z z^{\mathsf{T}}$ is PSD.
  If $\alpha > -1/z^{\mathsf{T}} z$ then $I + \alpha z z^{\mathsf{T}}$ is PD.
- **Claim 3:** $\det(I + \alpha z z^{\mathsf{T}}) = 1 + \alpha z^{\mathsf{T}} z$