# C&O 355: Mathematical Programming
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# Lecture 20 Notes

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## 1 Finding Neighbouring Vertices in the Simplex Method

Let $x$ be a basic feasible solution of the polyhedron $P = \{\, x : Ax \leq b \,\}$. Let $B$ be the subset of the constraints that are tight at $x$. Let $A_B$ denote the submatrix of $A$ corresponding to these constraints. Similarly, let $b_B$ denote the portion of $b$ corresponding to these constraints. So $A_B x = b_B$ holds.

Assume that we have perturbed the matrix $A$ such that each vertex of $P$ has exactly $n$ tight constraints. Then $|B| = n$, so $A_B$ is square. Since $x$ is a basic feasible solution, rank $A_B = n$, and so $A_B$ is invertible.

Since $A_B$ is invertible, we can express the objective function $c$ as a linear combination of the tight constraints. That is, there exists a vector $u$ such that $c^{\mathsf{T}} = u^{\mathsf{T}} A_B$.

**Case 1: $u \geq 0$.** In this case, we have expressed the objective function as a non-negative linear combination of the constraints that are tight at $x$. By Question 6 on Assignment 1, this implies that $x$ is an optimal solution of the LP.

**Case 2: $u \not\geq 0$.** Let $k \in B$ be such that $u_k < 0$. Let $a_i$ denote the $i^{\text{th}}$ row of $A$. Since $A_B$ is invertible, there exists a vector $d$ such that

$$a_k^{\mathsf{T}} d = -1 \qquad \text{and} \qquad a_i^{\mathsf{T}} d = 0 \quad \forall i \in B \setminus \{k\}\,.$$

The idea is to "move" from $x$ in the direction $d$, either finding a new vertex or moving off to infinity. More formally, we consider points of the form $x + \lambda d$ for $\lambda > 0$. Any such point has a strictly better objective value because

$$\begin{aligned} c^{\mathsf{T}}(x + \lambda d) &= c^{\mathsf{T}} x + \lambda c^{\mathsf{T}} d \\ &= c^{\mathsf{T}} x + \lambda u^{\mathsf{T}} A_B d \\ &= c^{\mathsf{T}} x - \lambda u_k \\ &> c^{\mathsf{T}} x. \end{aligned}$$

**Case 2a:** $a_i^{\mathsf{T}} d \leq 0$ for all rows $i$. In this case $a_i^{\mathsf{T}}(x + \lambda d) \leq a_i^{\mathsf{T}} x \leq b_i$ for every row $i$, and so $x + \lambda d$ is feasible for all $\lambda > 0$. Since the LP objective value strictly increases with $\lambda$, this means that the LP must be unbounded.

**Case 2b:** $a_i^{\mathsf{T}} d > 0$ for some row $i$. In this case, we wish to find the maximum value of $\lambda$ such that $x + \lambda d$ is still feasible. As we saw above, we do not need to worry about rows such that $a_i^{\mathsf{T}} d \leq 0$. So

$x + \lambda d$ is feasible if and only if

$$\begin{aligned} a_i^\mathsf{T}(x+\lambda d) &\leq b_i && \forall i \text{ s.t. } a_i^\mathsf{T} d > 0 \\ \iff \qquad \lambda a_i^\mathsf{T} d &\leq b_i - a_i^\mathsf{T} x && \forall i \text{ s.t. } a_i^\mathsf{T} d > 0 \\ \iff \qquad \lambda &\leq \frac{b_i - a_i^\mathsf{T} x}{a_i^\mathsf{T} d} && \forall i \text{ s.t. } a_i^\mathsf{T} d > 0 \end{aligned}$$

So the largest value of $\lambda$ we can take is simply

$$\lambda^* \;=\; \min \left\{ \frac{b_i - a_i^\mathsf{T} x}{a_i^\mathsf{T} d} \;:\; a_i^\mathsf{T} d > 0 \right\}.$$

Let $i^*$ be a row achieving this minimum.

At every point $x + \lambda d$ the $i^{\text{th}}$ constraint remains tight for every $i \in B \setminus \{k\}$. This is because

$$a_i^\mathsf{T}(x + \lambda d) \;=\; a_i^\mathsf{T} x \;=\; b_i.$$

Since those constraints have rank $n-1$, this moving through the points $x + \lambda d$ traverses an edge of the polyhedron. The point $x + \lambda^* d$ is an endpoint of this edge, so it is a vertex of the polyhedron. So the edge that we traversed is

$$\{\; x + \lambda d \;:\; 0 \leq \lambda \leq \lambda^* \;\}$$

and the neighboring vertex is $x + \lambda^* d$.

We remark that $\lambda^* > 0$ because $i^* \notin B$, so constraint $i^*$ is not tight at $x$.