C&O 355
Mathematical Programming
Fall 2010
Lecture 19

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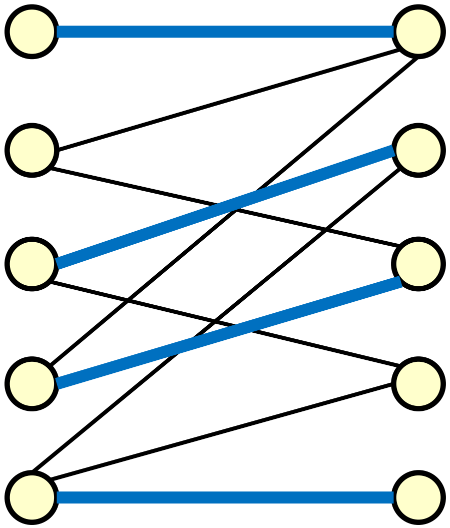
Topics

- Vertex Covers in Bipartite Graphs
- Konig’s Theorem
- Vertex Covers in Non-bipartite Graphs

Maximum Bipartite Matching

- Let G=(V, E) be a bipartite graph.
- We're interested in **maximum size matchings**.
- How do I know M has maximum size? Is there a 5-edge matching?
- Is there a **certificate** that a matching has maximum size?

Blue edges are a maximum-size matching M



Vertex covers

- Let G=(V, E) be a graph.
- A set $C \subseteq V$ is called a **vertex cover** if every edge $e \in E$ has at least one endpoint in C.
- **Claim:** If M is a matching and C is a vertex cover then $|M| \leq |C|$.
- **Proof:** Every edge in M has at least one endpoint in C.

  Since M is a matching, its edges have distinct endpoints.

  So C must contain at least $|M|$ vertices. □

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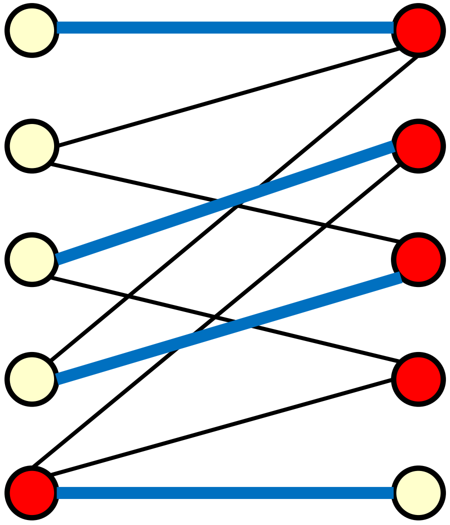

Red vertices form a vertex cover C

Vertex covers

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- Suppose we find a matching M and vertex cover C s.t. $|M|=|C|$.
- Then M must be a maximum cardinality matching: every other matching M' satisfies $|M'| \leq |C| = |M|$.
- And C must be a minimum cardinality vertex cover: every other vertex cover C' satisfies $|C'| \geq |M| = |C|$.
- Then M certifies optimality of C and vice-versa.

Vertex covers & matchings

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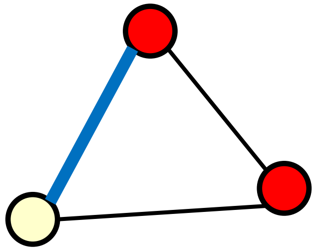
- Suppose we find a matching M and vertex cover C s.t. $|M|=|C|$.
- Then M certifies optimality of C and vice-versa.
- Do such M and C always exist?
- No...



Maximum size of a matching = 1

Minimum size of a vertex cover = 2

Vertex covers & matchings

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- Suppose we find a matching M and vertex cover C s.t. $|M|=|C|$.
- Then M certifies optimality of C and vice-versa.
- Do such M and C always exist?
- No… unless G is bipartite!

- **Theorem** (Konig's Theorem): If G is bipartite then there exists a matching M and a vertex cover C s.t. $|M|=|C|$.

Earlier Example

- Let G=(V, E) be a bipartite graph.
- We're interested in **maximum size matchings**.
- How do I know M has maximum size? Is there a 5-edge matching?
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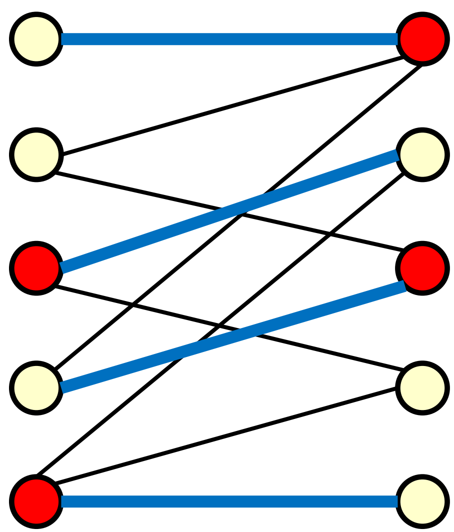


Red vertices form a vertex cover C

- Since |M|=|C|=4, both M and C are optimal!

LPs for Bipartite Matching

- Let G=(V, E) be a bipartite graph.
- Recall our IP and LP formulations for maximum-size matching.

(IP)

$$\begin{array}{llll} \max & \sum_{e \in E} x_e & & \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\ & x_e & \in \{0,1\} & \forall e \in E \end{array}$$

(LP)

$$\begin{array}{llll} \max & \sum_{e \in E} x_e & & \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\ & x_e & \geq 0 & \forall e \in E \end{array}$$

- **Theorem**: Every BFS of (LP) is actually an (IP) solution.
- What is the dual of (LP)?

(LP-Dual)

$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array}$$

Dual of Bipartite Matching LP

- What is the dual LP?

(LP-Dual)
$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array}$$

- Note that any optimal solution must satisfy $y_v \leq 1 \ \forall v \in V$
- Suppose we impose integrality constraints:

(IP-Dual)
$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \in \{0,1\} & \forall v \in V \end{array}$$

- **Claim:** If y is feasible for IP-dual then $C = \{ v : y_v = 1 \}$ is a vertex cover. Furthermore, the objective value is $|C|$.
- So IP-Dual is precisely the minimum vertex cover problem.
- **Theorem**: Every optimal BFS of (LP-Dual) is an (IP-Dual) solution (in the case of bipartite graphs).

- Let G=(U∪V, E) be a bipartite graph. Define A by

$$A_{v,e} = \begin{cases} 1 & \text{if vertex v is an endpoint of edge e} \\ 0 & \text{otherwise} \end{cases}$$

- **Lemma:** A is TUM.
- **Claim:** If A is TUM then A^T is TUM.
- **Proof:** Exercise?
- **Corollary:** Every **BFS** of P = { x : $A^T y \geq \mathbf{1}$, $y \geq 0$ } is integral.
- But LP-Dual is

$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array} \quad = \quad \begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \text{s.t.} & A^\mathsf{T} y & \geq \mathbf{1} \\ & y & \geq 0 \end{array}$$

- So our Corollary implies every BFS of LP-dual is integral
- Every optimal solution must have $y_v \leq 1 \ \forall v \in V$

  $\Rightarrow$ every optimal BFS has $y_v \in \{0,1\} \ \forall v \in V$, and hence it is a feasible solution for IP-Dual. ■

Proof of Konig's Theorem

- **Theorem** (Konig's Theorem): If G is bipartite then there exists a matching M and a vertex cover C s.t. |M|=|C|.
- **Proof:**

  Let x be an optimal BFS for (LP).

  Let y be an optimal BFS for (LP-Dual).

  Let $M = \{ e : x_e=1 \}$.

  M is a matching with |M| = objective value of x. (By earlier theorem)

  Let $C = \{ v : y_v = 1 \}$.

  C is a vertex cover with |C| = objective value of y. (By earlier theorem)

  By Strong LP Duality:

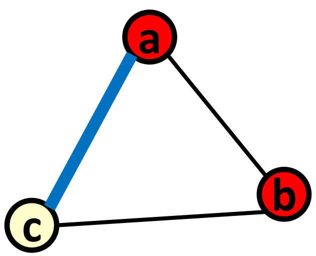
  |M| = LP optimal value = LP-Dual optimal value = |C|. ∎

Beyond Bipartite Graphs

- Given a bipartite graph, we can efficiently find a minimum-size vertex cover. Just compute a BFS of

(LP-Dual)
$$\begin{array}{llll} \min & \sum_{v\in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array}$$

- For non-bipartite graphs, this doesn't work:



Maximum size of a matching = 1

Minimum size of a vertex cover = 2

  - Setting $y_a = y_b = y_c = 0.5$ gives a feasible solution to LP-Dual with objective value 1.5
  - So optimal BFS has value $\leq$ 1.5.
    But no vertex cover has size < 2.

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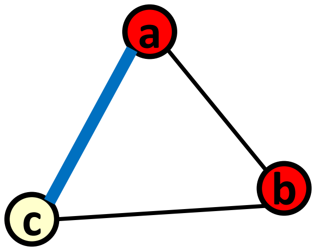

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- **Key point:** (IP) captures vertex cover problem, but (LP) does not. We have no efficient way to solve (IP).

(IP)

$$\begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \text{s.t.} & y_u + y_v & \geq 1 \quad \forall \{u,v\} \in E \\ & y_v & \in \{0,1\} \quad \forall v \in V \end{array}$$

(LP)

$$\begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \text{s.t.} & y_u + y_v & \geq 1 \quad \forall \{u,v\} \in E \\ & y_v & \geq 0 \quad \forall v \in V \end{array}$$

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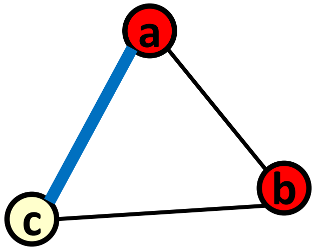

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(LP)

$$\begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \text{s.t.} & y_u + y_v & \geq 1 \quad \forall \{u, v\} \in E \\ & y_v & \geq 0 \quad \forall v \in V \end{array}$$

- What's the problem? The constraint matrix A is **not** totally unimodular:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \implies \det A = 2$$

Beyond Bipartite Graphs

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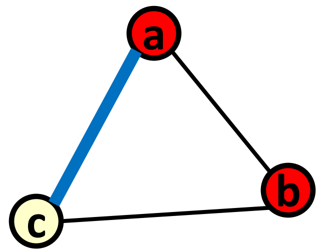


Maximum size of a matching = 1

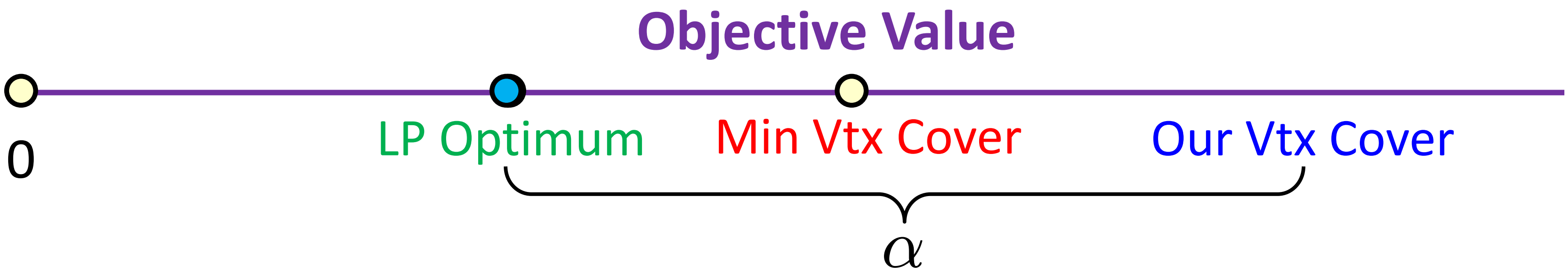
Minimum size of a vertex cover = 2

  - Setting $y_a = y_b = y_c = 0.5$ gives a feasible solution to LP-Dual with objective value 1.5.
  - So optimal BFS has value $\leq$1.5. But no vertex cover has size < 2.

- **Theorem:** There is **no** efficient algorithm to find a min size vertex cover in general graphs. (Assuming P$\neq$NP)

- **Theorem:** There **is** an efficient algorithm to find a vertex cover whose size is at most **twice** the minimum.

Our Game Plan



- Solve the vtx cover LP: $$\begin{aligned} \min \quad & \textstyle\sum_{v \in V} y_v & & \\ \text{s.t.} \quad & y_u + y_v & \geq 1 & \quad \forall \{u, v\} \in E \\ & y_v & \geq 0 & \quad \forall v \in V \end{aligned}$$
- **Rounding:** Extract Our Vtx Cover from LP optimum solution
- Prove that Our Vtx Cover is close to LP Optimum, i.e. $\alpha = \dfrac{\text{Size of Our Vtx Cover}}{\text{LP Opt Value}}$ is as **small** as possible.

  $\Rightarrow$ Our Vtx Cover is close to Min Vtx Cover, i.e., $\dfrac{\text{Size of Our Vtx Cover}}{\text{Size of Min Vtx Cover}} \leq \alpha$
- So Our Vtx Cover is within a factor α of the minimum

Our Algorithm

- **Theorem:** [Folklore]
  There exists an algorithm to extract a vertex cover from the LP optimum such that

$$\alpha = \frac{\text{Size of Vtx Cover}}{\text{Size of Min Vtx Cover}} \leq 2$$

- Astonishingly, this seems to be optimal:
- **Theorem:** [Khot, Regev 2003]
  No efficient algorithm can approximate the min vtx cover with factor better than 2, assuming a certain conjecture in complexity theory. (Similar to P≠NP)

Our Algorithm

- Solve the vertex cover LP

$$\begin{aligned} \min \quad & \textstyle\sum_{v \in V} y_v & & \\ \text{s.t.} \quad & y_u + y_v & \geq 1 \quad & \forall \{u,v\} \in E \\ & y_v & \geq 0 \quad & \forall v \in V \end{aligned}$$

- Return C = { $v \in V$: $y_v \geq ½$ }

- **Claim 1:** C is a vertex cover.

- **Claim 2:** |C| is at most twice the size of the minimum vertex cover.

Our Algorithm

- Solve the vertex cover LP

$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array}$$

- Return C = { v∈V : $y_v \geq ½$ }

- **Claim 1:** C is a vertex cover.
- **Proof:** Consider any edge {u,v}.

  Since $y_u + y_v \geq 1$, either $y_u \geq ½$ or $y_v \geq ½$.

  So either u∈C or v∈C. ■

Our Algorithm

- Solve the vertex cover LP

$$\begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \text{s.t.} & y_u + y_v \geq 1 & \forall \{u,v\} \in E \\ & y_v \geq 0 & \forall v \in V \end{array}$$

- Return C = { v∈V : $y_v \geq ½$ }

- **Claim 2:** |C| $\leq$ 2·|minimum vertex cover|.
- **Proof:** |C| = |{ v∈V : $y_v \geq ½$ }|

$$\leq 2 \cdot \sum_{y_v \geq 1/2} y_v$$

$$\leq 2 \cdot \sum_{v \in V} y_v$$

= 2 · LP optimum value

$\leq$ 2·|minimum vertex cover| ■

Summary

- Bipartite graphs
  - Vertex cover problem is dual of matching problem
  - Vertex Cover (IP) and (LP) are equivalent (by Total Unimodularity)
  - Can efficiently find minimum vertex cover
- Non-bipartite Graphs
  - Vertex cover not related to matching problem
  - Vertex Cover (IP) and (LP) are not equivalent
  - Cannot efficiently find minimum vertex cover
  - Can find a vertex cover of size at most twice minimum