C&O 355
Mathematical Programming
Fall 2010
Lecture 18

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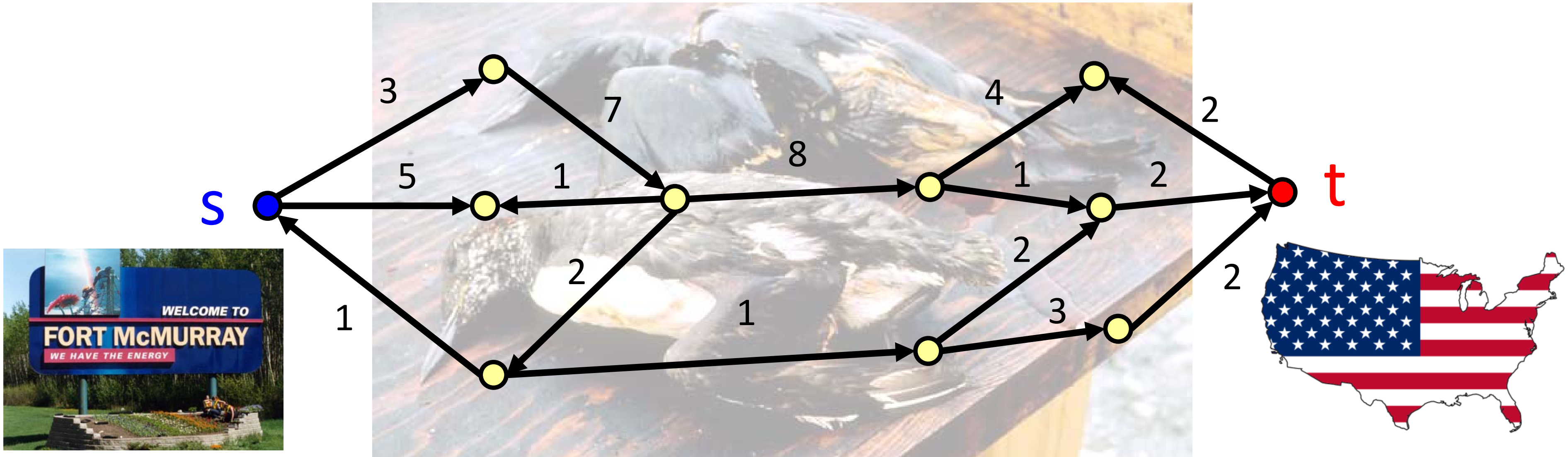
Topics

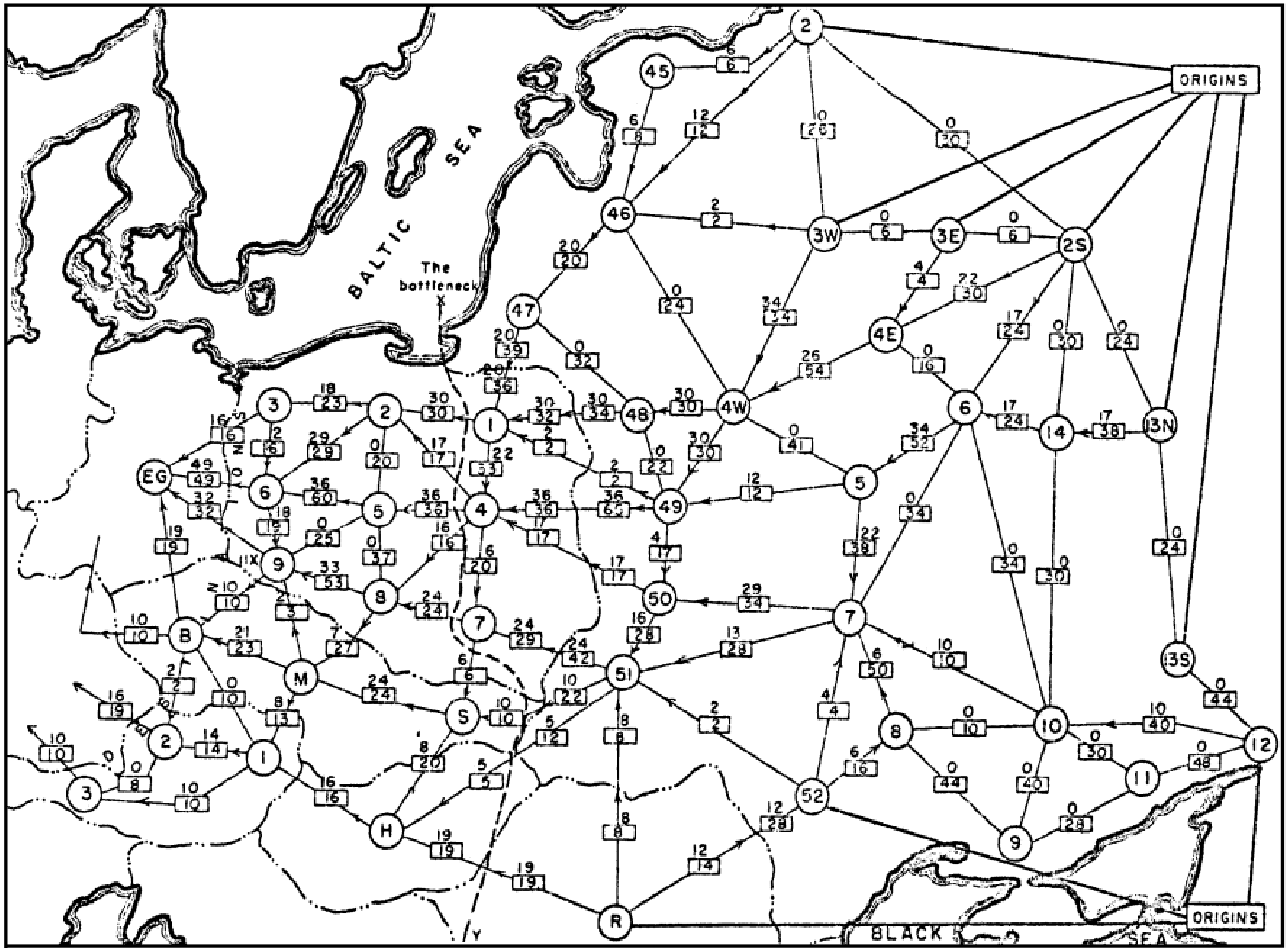
- Network Flow
  - Max Flow / Min Cut Theorem
- Total Unimodularity
- Directed Graphs & Incidence Matrices
- Proof of Max Flow / Min Cut Theorem

Network Flow

- Let $D=(N,A)$ be a directed graph.
- Every arc a has a “capacity” $c_a \geq 0$. (Think of it as an oil pipeline)
- Want to send oil from node s to node t through pipelines
- Oil must not leak at any node, except s and t: flow in = flow out.
- How much oil can we send?
- For simplicity, assume no arc enters s and no arc leaves t.



Max Flow & Min Cut



Harris and Ross [1955]

Schematic diagram of the railway network of the Western Soviet Union and Eastern European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe, and a cut of capacity 163,000 tons indicated as ‘The bottleneck’. [Schrijver, 2005]

Max Flow & Min Cut

- Let D=(N,A) be a digraph, where arc a has capacity c_a.
- **Definition:** For any U⊆N, the **cut** δ^+(U) is:

$$\delta^+(U) = \{\ uv\ :\ u \in U,\ v \notin U,\ uv \in A\ \}$$

The **capacity** of the cut is:

$$c(\delta^+(U)) = \sum_{a \in \delta^+(U)} c_a$$

Delbert Ray Fulkerson

- **Theorem:** [Ford & Fulkerson 1956]
  The maximum amount of flow from s to t equals
  the minimum capacity of a cut δ^+(U), where s∈U and t∉U
- Furthermore, if c is integral then there is an integral flow that achieves the maximum flow.

LP Formulation of Max Flow

- **Variables:** x_a = amount of flow to send on arc a
- **Constraints:**
  For every node except s & t, flow in = flow out.
  Flow through each arc can not exceed its capacity.
- **Objective value:** Total amount of flow sent by s.
- **Notation:** $\delta^+(v)$ = arcs with tail at v
  $\delta^-(v)$ = arcs with head at v
- The LP is:

$$\begin{array}{lll} \max & \sum_{a \in \delta+(s)} x_a & \\ \text{s.t.} & \sum_{a \in \delta-(v)} x_a - \sum_{a \in \delta+(v)} x_a \quad = 0 & \forall v \in N \setminus \{s,t\} \\ & 0 \leq x_a \leq c_a & \forall a \in A \end{array}$$

Incidence Matrix of a Directed Graph

$$
\begin{array}{lllll}
\max & \sum_{a \in \delta^+(s)} x_a & & & \\
\text{s.t.} & \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_a & = 0 & & \forall v \in N \setminus \{s,t\} \\
 & 0 \leq x_a \leq c_a & & & \forall a \in A
\end{array}
$$

- What is the matrix M defining the constraints of this LP?
  - Row for every node (except s or t)
  - Column for every arc

$$
M_{v,a} = \begin{cases} +1 & \text{if node v is the head of arc a} \\ -1 & \text{if node v is the tail of arc a} \\ 0 & \text{otherwise} \end{cases}
$$

- **Goal:** Analyze extreme points of this LP.
- **Plan:** Show M is totally unimodular.

Total Unimodularity

- Let M be a real m×n matrix
- **Definition:** Suppose that every square submatrix of M has determinant in {0, +1, -1}. Then M is **totally unimodular (TUM)**.
  - In particular, every entry of M must be in {0, +1, -1}

- **Key point:** Polytopes defined by TUM matrices have integral extreme points.

- For example, last time we showed:

Lemma: Suppose M is TUM. Let b be any integer vector. Then every basic feasible solution of $P = \{ x : Mx \leq b \}$ is integral.

Total Unimodularity

- Let M be a real m×n matrix
- **Definition:** Suppose that every square submatrix of M has determinant in {0, +1, -1}. Then M is **totally unimodular (TUM)**.

- **Lemma:** Suppose A is TUM. Let b be any integer vector. Then every extreme point of P = { x : Mx$\leq$b } is integral.

- **Claim:** Suppose M is TUM. Then $\begin{pmatrix} M \\ -M \\ I \\ -I \end{pmatrix}$ is also TUM.
- **Proof:** Exercise?

- **Corollary:** Suppose M is TUM. Let b and c be integer vectors. Then every extreme point of P = { x : Mx=b, 0$\leq$x$\leq$c } is integral.

Incidence Matrices are TUM

- Let D=(N, A) be a directed graph. Define M by:

$$M_{u,a} = \begin{cases} +1 & \text{if node u is the head of arc a} \\ -1 & \text{if node u is the tail of arc a} \\ 0 & \text{otherwise} \end{cases}$$

- **Lemma:** M is TUM.
- **Proof:** Let Q be a kxk submatrix of M. Argue by induction on k. If k=1 then Q is a single entry of M, so det(Q) is either 0 or ± 1. So assume k>1.

- **Lemma:** M is TUM.
- **Proof:** Let Q be a kxk submatrix of M. Assume k>1.

Case 1:

If some column of Q has **no** non-zero entries, then det(Q)=0.

Case 2:

Suppose j^th column of Q has **exactly one** non-zero entry, say $Q_{t,j} \neq 0$

Use "Column Expansion" of determinant:

$$\det Q \;=\; \sum_i (-1)^{i+j} Q_{i,j} \cdot \det Q_{\mathrm{del}(i,j)} \;=\; (-1)^{t+j} Q_{t,j} \cdot \det Q_{\mathrm{del}(t,j)} \,,$$

where t is the unique non-zero entry in column j.

By induction, det $Q_{\mathrm{del}(t,j)}$ in {0,+1,-1} $\Rightarrow$ det Q in {0,+1,-1}.

Case 3:

Suppose **every** column of Q has **exactly two** non-zero entries.

- For each column, one non-zero is a +1 and the other is a -1.

So summing all rows in Q gives the vector [0,0,…,0].

Thus Q is singular, and det Q = 0. ■

The Max Flow LP

$$\begin{array}{lll} \max & \sum_{a \in \delta^+(s)} x_a & \\ \text{s.t.} & \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_a = 0 & \forall v \in N \setminus \{s,t\} \\ & 0 \leq x_a \leq c_a & \forall a \in A \end{array}$$

- **Observations:**
  - The LP is feasible (assume the capacities are all non-negative)
  - The LP is bounded (because the feasible region is bounded)
  - It has an optimal solution, i.e., a maximum flow. (by FTLP)
- The feasible region is

$$P = \left\{ x : \begin{pmatrix} M \\ -M \\ I \\ -I \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \end{pmatrix} \right\}$$

  where M is TUM.
- **Corollary:** If c is integral, then every extreme point is integral, and so there is a maximum flow that is integral.
- **Q:** Why does P have any extreme points? **A:** It contains no line.

Max Flow LP & Its Dual

$$\begin{array}{lll} \max & \sum_{a\in\delta^+(s)} x_a & \\ \text{s.t.} & \sum_{a\in\delta^-(v)} x_a - \sum_{a\in\delta^+(v)} x_a \quad = 0 & \forall v \in N \setminus \{s,t\} \\ & 0 \le x_a \le c_a & \forall a \in A \end{array}$$

- **Dual variables:**
  - A variable y_v for every $v \in N \setminus \{s,t\}$
  - A variable z_{uv} for every arc uv
- The dual is

$$\begin{array}{llll} \min & \sum_{a\in A} c_a \, z_a & & \\ \text{s.t.} & -y_u + y_v + z_{uv} & \ge 0 & \forall uv \in A, v, w \in N \setminus \{s,t\} \\ & y_v + z_{sv} & \ge 1 & \forall sv \in A \\ & -y_u + z_{ut} & \ge 0 & \forall ut \in A \\ & z & \ge 0 & \end{array}$$

- **Let's simplify:** Set $y_s = 1$ and $y_t = 0$

The Dual

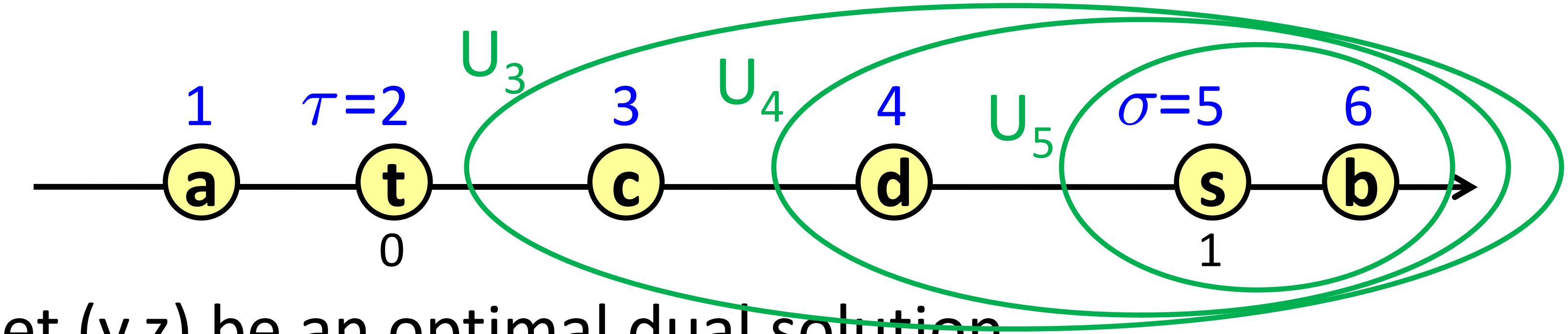
$$\begin{array}{llll} \min & \sum_{a \in A} c_a \, z_a & & \\ \text{s.t.} & -y_u + y_v + z_{uv} & \geq 0 & \forall uv \in A \\ & z & \geq 0 & \end{array}$$

where y_s and y_t are **not** variables: $y_s = 1$ and $y_t = 0$

- **We will show:** Given an optimal solution (y,z), we can construct a cut $\delta^+(U)$ such that
$$c(\delta^+(U)) = \sum_{a \in A} c_a z_a$$

“Rounding”

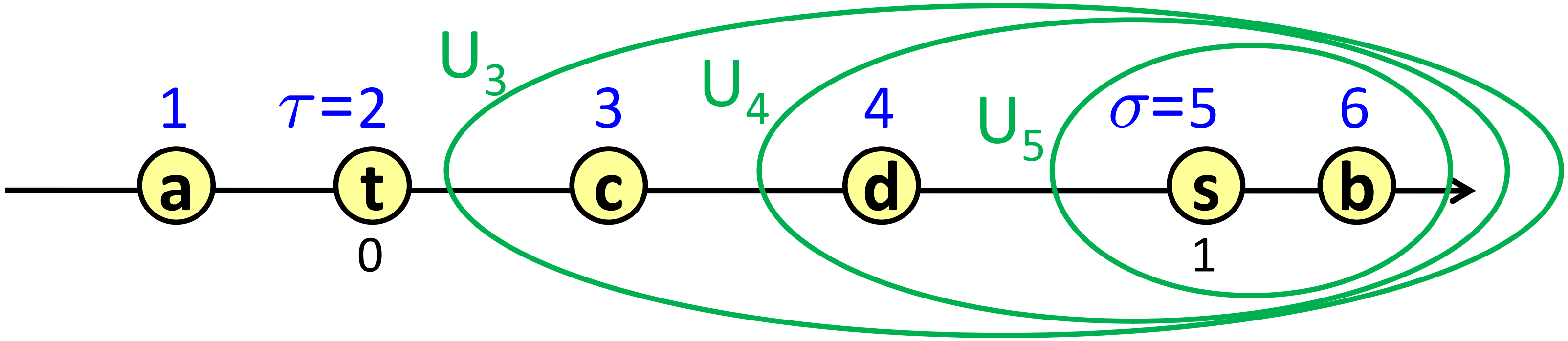
- In other words, the capacity of the cut $\delta^+(U)$ equals the optimal value of the dual LP.
- By strong LP duality, this equals the optimal value of the primal LP, which is the maximum flow value.
- Every cut has capacity at least the max flow value, so this must be a minimum cut.



- Let (y,z) be an optimal dual solution
- Every node u lies at some position y_u on the real line
- Number the nodes 1,…,n so that $y_1 \leq \cdots \leq y_n$
- Suppose node t numbered τ and node s numbered σ
- Let U_i = { i, i+1, …, n }, for $\tau < i \leq \sigma$ (Here n=|N|, the total # nodes)
- Pick cut U_i with probability y_i - y_{i-1}

$$\mathrm{Ex}[\,c(\delta^+(U_i))\,] \;=\; \sum_{i=\tau+1}^{\sigma} (y_i - y_{i-1}) \cdot c(\delta^+(U_i))$$

- Arc jk contributes 0 if j<k, and at most $(y_j - y_k)c_{jk}$ if j>k
- So $$\mathrm{Ex}[\,c(\delta^+(U_i))\,] \leq \sum_{jk \in A\,:\,j>k} (y_j - y_k)\,c_{jk}$$



- Let U_i = { i, i+1, …, n }, for $\tau < i \leq \sigma$
- Pick cut U_i with probability $y_i - y_{i-1}$

$$\left.\begin{aligned} \mathrm{Ex}[\, c(\delta^+(U_i)\,)\,] &\leq \sum_{jk \in A \,:\, j>k} (y_j - y_k)\, c_{jk} \\ &\leq \sum_{jk \in A \,:\, j>k} z_{jk}\, c_{jk} \end{aligned}\right\} \text{By dual feasibility}$$

$$\leq \sum_{a \in A} z_a\, c_a \quad = \text{ Optimum value of Dual LP}$$

- So the **average capacity** of the U_i's is $\leq$ Dual Opt. Value

$\Rightarrow$ **minimum capacity** of a U_i is $\leq$ Dual Opt. Value, and so it is a minimum cut.

Summary

- We have proven:

- **Theorem:** [Ford & Fulkerson 1956]
The maximum amount of flow from s to t equals
the minimum capacity of a cut $\delta^+(U)$, where $s \in U$ and $t \notin U$
Furthermore, if c is integral then there is an integral flow
that achieves the maximum flow.

- We also get an algorithm for finding max flow & min cut
  - Solve Max Flow LP by the ellipsoid method.
  - Get an extreme point solution. It is an integral max flow.
  - Solve Dual LP by the ellipsoid method.
  - Use rounding method to get a min cut.
- This algorithm runs in polynomial time