C&O 355: Mathematical Programming
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Lecture 11 Notes

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1 Faces of Polyhedra

Recall from Lecture 10 the following definition.

Definition 1.1. Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A ***face*** of P is any set F of the form

$$F \;=\; P \cap \left\{ \mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{a}^\mathsf{T}\mathbf{x} = b \right\}, \tag{1.1}$$

where $\mathbf{a}^\mathsf{T}\mathbf{x} \leq b$ is a valid inequality for P.

We remark that unbounded polyhedra might not have any vertices, or edges, or even facets. For example, the polyhedron could be an affine space. On the other hand, a non-empty polytope always has at least one vertex. This follows from our theorem that feasible and bounded LPs always have an optimal solution at an extreme point.

Fact 1.2. Let P be a polyhedron with $\dim P = d$. Let F be a face of P. Then

$$\left\{ F' \;:\; F' \text{ is a face of } F \right\} \;=\; \left\{ F' \;:\; F' \text{ is a face of } P, \text{ and } F' \subseteq F \right\}.$$

Furthermore, each facet of F can be obtained by intersecting F with another facet of P.

Fact 1.3. Let $P = \left\{ \mathbf{x} \,:\, \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \;\forall i \right\}$ be a polyhedron in $\mathbb{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct vertices. Recall our notation $\mathcal{I}_\mathbf{x} = \left\{ i \,:\, \mathbf{a}_i^\mathsf{T}\mathbf{x} = b_i \right\}$. Suppose $\operatorname{rank}\{ \mathbf{a}_i \,:\, i \in \mathcal{I}_\mathbf{x} \cap \mathcal{I}_\mathbf{y} \} = n-1$. Then the line segment

$$L_{\mathbf{x},\mathbf{y}} \;=\; \{ \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \,:\, \lambda \in [0,1] \} \tag{1.2}$$

is an edge of P. Moreover, every bounded edge arises in this way.

Definition 1.4. Let $P = \left\{ \mathbf{x} \,:\, \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \;\forall i \right\}$ be a polyhedron in $\mathbb{R}^n$. An inequality $\mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i$ is called ***facet-defining*** if the face

$$P \cap \left\{ \mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{a}^\mathsf{T}\mathbf{x} = b \right\}$$

is a facet.

Fact 1.5. Let $P = \left\{ \mathbf{x} \,:\, \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \;\forall i \right\}$ be a full-dimensional polyhedron in $\mathbb{R}^n$. Let

$$\mathcal{I} \;=\; \left\{ i \,:\, \text{the inequality ``}\mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i\text{'' is facet-defining} \right\}.$$

Then

$$P \;=\; \left\{ \mathbf{x} \,:\, \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \;\forall i \in \mathcal{I} \right\}.$$

2 Polyhedra and Graphs

Recall from Lecture 10 that every polyhedron has finitely many vertices. By Fact 1.3, every bounded edge of a polyhedron can be described as the line segment $L_{\mathbf{x},\mathbf{y}}$ connecting two particular vertices $\mathbf{x}$ and $\mathbf{y}$. Thus the vertices and bounded edges of polyhedra naturally form a ***graph***.

Definition 2.1. Let P be a polyhedron and let V be the set of its vertices. Define the graph $G(P) = (V, E)$, where

$$E \;=\; \{\, \{u, v\} \,:\, L_{u,v} \text{ is a bounded edge of } P \,\}\,.$$

This graph is called the ***1-skeleton*** of P.

Definition 2.2. For any finite graph $G = (V, E)$, the ***distance*** between two vertices $u, v \in V$, denoted $\operatorname{dist}(u, v)$, is defined to be the minimum number of edges in any path from u to v. The ***diameter*** of G is

$$\operatorname{diam} G \;=\; \max_{u,v \in V} \operatorname{dist}(u, v).$$

Alternatively, $\operatorname{diam} G$ is the smallest number p such that any two vertices can be connected by a path with p edges.

Let P be a polyhedron with m facets, at least one vertex, and $\dim P = n$. We would like to know how large the quantity $\operatorname{diam} G(P)$ can be. Define

$$\Delta(n, m) \;=\; \max_{P} \operatorname{diam} G(P),$$

where the maximum is taken over all n-dimensional polyhedra with m facets. As an example, it is easy to see that $\Delta(2, m)$ is precisely $m - 1$. (If we restricted to bounded, 2-dimensional polyhedra, it would be $\lfloor m/2 \rfloor$.)

The following notorious conjecture dates back to 1957.

Conjecture 2.3 (The Hirsch Conjecture). $\Delta(n, m) \le m - n$.

This conjecture was disproven for unbounded polyhedra by Klee and Walkup in 1967, but it remained open for polytopes. In early 2010, Santos disproved the conjecture for polytopes. Some people now conjecture that $\Delta(n, m) \le \operatorname{poly}(n, m)$, where $\operatorname{poly}(n, m)$ denotes some polynomial function in m and n. The following theorem gives (nearly) the best-known upper bound on $\Delta(n, m)$.

Theorem 2.4 (Kalai 1991 & Kalai-Kleitman 1992). $\Delta(n, m) \le n^{4 \ln m}$.

2.1 Notation

Before proving this theorem, we must introduce some notation. Consider any n-dimensional polyhedron P. Let V denote the collection of vertices of P, and let $\mathcal{F}$ denote the collection of facets of P.

- For any $v \in V$, let $\mathcal{F}(v)$ denote the collection of facets which contain the point v.
- For any two vertices $v, w \in V$, let $\operatorname{dist}(v, w)$ denote the length of the shortest path from v to w in $G(P)$.

- For any vertex v and integer $t \geq 0$, let $B(v,t) = \{\, w \in V \,:\, \mathrm{dist}(v,w) \leq t \,\}$. This can be thought of as the ball of radius t around vertex v in $G(P)$.
- For any vertex v and integer $t \geq 0$, let $\mathcal{F}(v,t) = \bigcup_{w \in B(v,t)} \mathcal{F}(w)$. This is the set of all facets that can be "touched" by walking from v at most t steps between the vertices of P.

3 Proof of Kalai-Kleitman

Consider any n-dimensional polyhedron P whose collection of facets is $\mathcal{F}$ and $|\mathcal{F}| = m$. The distance between any two vertices x and y in $G(P)$ is denoted $\mathrm{dist}_P(x,y)$. Fix any two vertices u and v of P. Define

$$\begin{aligned} k_u &= \max\{\, t \,:\, |\mathcal{F}(u,t)| \leq m/2 \,\} \\ k_v &= \max\{\, t \,:\, |\mathcal{F}(v,t)| \leq m/2 \,\} \end{aligned}$$

By the pigeonhole principle, $\mathcal{F}(u, k_u+1) \cap \mathcal{F}(v, k_v+1)$ is non-empty. So there exists a facet f and two vertices $u', v' \in f$ such that

$$\begin{aligned} \mathrm{dist}_P(u,u') &\leq k_u + 1 \\ \mathrm{dist}_P(v,v') &\leq k_v + 1. \end{aligned} \tag{3.1}$$

Claim 3.1. $\mathrm{dist}_P(u',v') \leq \Delta(n-1, m-1)$.

Proof. By definition, f is an $(n-1)$-dimensional polyhedron. By Fact 1.2, each facet of f is the intersection of f with some other facet of P. So f has at most $m-1$ facets. Since every vertex (or edge) of f is also a vertex (or edge) of P, any path in $G(f)$ is also a path in $G(P)$. Thus $\mathrm{dist}_P(u',v') \leq \mathrm{dist}_f(u',v') \leq \Delta(n-1,m-1)$. ∎

Claim 3.2. $k_v \leq \Delta(n, \lfloor m/2 \rfloor)$.

We prove Claim 3.2 below; this is the heart of the theorem. Claim 3.1 and Claim 3.2 lead to the following recursion.

$$\begin{aligned} \mathrm{dist}_P(u,v) &\leq \mathrm{dist}_P(u,u') + \mathrm{dist}_P(u',v') + \mathrm{dist}_P(v',v) \\ &\leq (k_u+1) + \Delta(n-1,m-1) + (k_v+1) \\ &\leq \Delta(n-1,m-1) + 2\Delta(n,\lfloor m/2 \rfloor) + 2 \end{aligned}$$

Since u and v are arbitrary, we have

$$\Delta(n,m) \leq \Delta(n-1,m-1) + 2\Delta(n,\lfloor m/2 \rfloor) + 2. \tag{3.2}$$

The theorem follows by analyzing this recurrence, which we do below in Claim 3.3.

Proof (of Claim 3.2). Consider any vertex w with $\mathrm{dist}_P(v,w) \leq k_v$. We will obtain a recursive bound on this distance by defining a new polyhedron with fewer facets. Let Q be the polyhedron obtained from P by deleting all inequalities corresponding to facets in $\mathcal{F} \setminus \mathcal{F}(v,k_v)$. In other words, let Q be the polyhedron defined by the intersection of all half-spaces induced by the facets in $\mathcal{F}(v,k_v)$. By choice of k_v, Q has at most $\lfloor m/2 \rfloor$ facets. Note that w is a vertex of Q.

The key step of the proof is to prove that

$$\mathrm{dist}_Q(v,w) \geq \mathrm{dist}_P(v,w). \tag{3.3}$$

Once this is proven, we have $\text{dist}_P(v,w) \leq \text{dist}_Q(v,w) \leq \Delta(n, \lfloor m/2 \rfloor)$, which is the desired inequality.

So suppose to the contrary that $\text{dist}_Q(v,w) < \text{dist}_P(v,w)$. Consider any shortest path p from v to w in $G(Q)$. Then there must be some edge on path p that is not an edge in $G(P)$ (otherwise path p would be a v-w path in $G(P)$ of length less that $\text{dist}_P(v,w)$). Let $L_{\mathbf{x},\mathbf{y}}$ be the first such edge, i.e., the edge closest to v. Then $\mathbf{x}$ must be a vertex of P (since it is a face of the previous edge). However $\mathbf{y}$ cannot be a vertex of P, otherwise $L_{\mathbf{x},\mathbf{y}}$ would be an edge of P. In fact, the reason that $\mathbf{y}$ is not a vertex of P is that it is not even feasible. To see this, note that the tight constraints of Q at $\mathbf{y}$ have dimension n, and these are a subset of P's constraints. So $\mathbf{y}$ has enough tight constraints to be a vertex of P, so only reason it cannot be a vertex is that it is infeasible.

The line segment $L_{\mathbf{x},\mathbf{y}}$ is feasible for P at $\mathbf{x}$, but infeasible at $\mathbf{y}$, so it must intersect one of the facets of P that is not a facet of Q. Call this facet f and this intersection point z, so we have $f \notin \mathcal{F}(v, k_v)$. Then z is a vertex of P and $f \in \mathcal{F}(z)$. Furthermore, since the portion of path p from v to x is a path in $G(P)$, we have

$$\text{dist}_P(v,z) \;\leq\; \text{dist}_Q(v,y) \;\leq\; \text{dist}_Q(v,w) \;<\; \text{dist}_P(v,w) \;\leq\; k_v.$$

Thus $z \in B(v, k_v)$ and $f \in \mathcal{F}(v, k_v)$, which is a contradiction. Thus Eq. (3.3) holds. ■

The final step is to analyze the recurrence in Eq. (3.2).

Claim 3.3. $\Delta(n,m) \leq \exp(4\ln(n)\ln(m))$.

Proof. By induction on m, and also using our earlier observation $\Delta(2,m) \leq m-1$. We have:

$$\begin{aligned}
\Delta(n,m) &\leq \Delta(n-1, m-1) + 2\Delta(n, \lfloor m/2 \rfloor) + 2 \\
&\leq \Delta(n-1, m) + 2\Delta(n, \lfloor m/2 \rfloor) + 2
\end{aligned}$$

Let's unroll the recurrence by expanding $\Delta(n-1,m)$.

$$\leq \Big(\Delta(n-2,m) + 2\Delta(n-1, \lfloor m/2 \rfloor) + 2\Big) + 2\Delta(n, \lfloor m/2 \rfloor) + 2$$

Now repeatedly unrolling the recurrence until $n = 2$, we obtain

$$\begin{aligned}
&\leq \Delta(2,m) + 2\sum_{i=3}^{n} \big(\Delta(i, \lfloor m/2 \rfloor) + 1\big) \\
&\leq m + 2\sum_{i=3}^{n} \big(e \cdot \Delta(n, \lfloor m/2 \rfloor)\big) \\
&\leq m + e^2(n-2)\Delta(n, \lfloor m/2 \rfloor) \\
&\leq m + e^2(n-2)\exp(4\ln(n)\ln(m/2))
\end{aligned}$$

One may check that $m \leq e^2 \exp(4\ln(n)\ln(m/2))$ holds for all $n \geq 2$ and $m \geq 2$.

$$\begin{aligned}
&\leq e^2 n \exp(4\ln(n)\ln(m/2)) \\
&\leq \exp(4\ln(n)\ln(m/2) + \ln(n) + 2) \\
&\leq \exp\Big(4\ln(n)\big(\ln(m) - 1\big) + \ln(n) + 2\Big) \\
&= \exp\big(4\ln(n)\ln(m) - 3\ln(n) + 2\big) \\
&\leq \exp\big(4\ln(n)\ln(m)\big)
\end{aligned}$$

This completes the inductive proof. ∎

Claim 3.3 shows that

$$\Delta(n, m) \;\leq\; \exp(4 \ln(n) \ln(m)) \;=\; n^{4 \ln m}.$$

This proves Theorem 2.4.