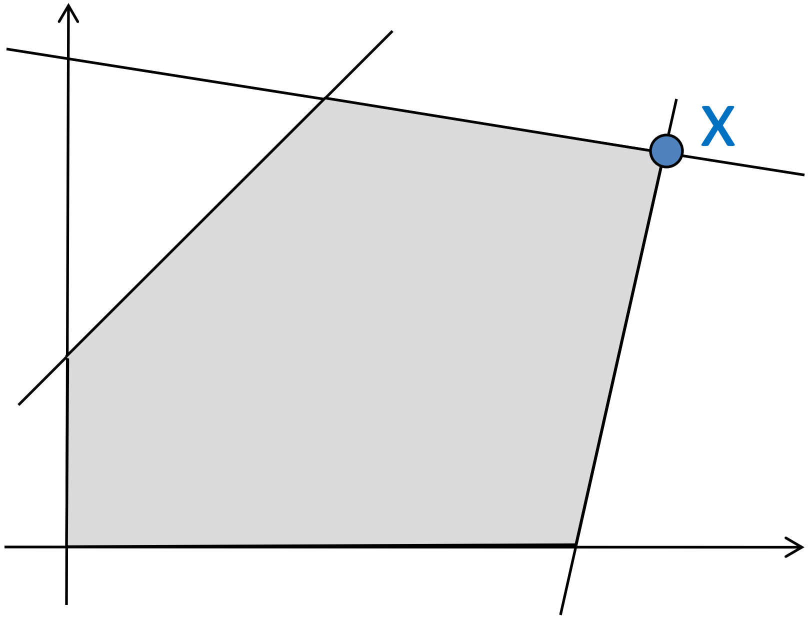
C&O 355
Mathematical Programming
Fall 2010
Lecture 10

N. Harvey

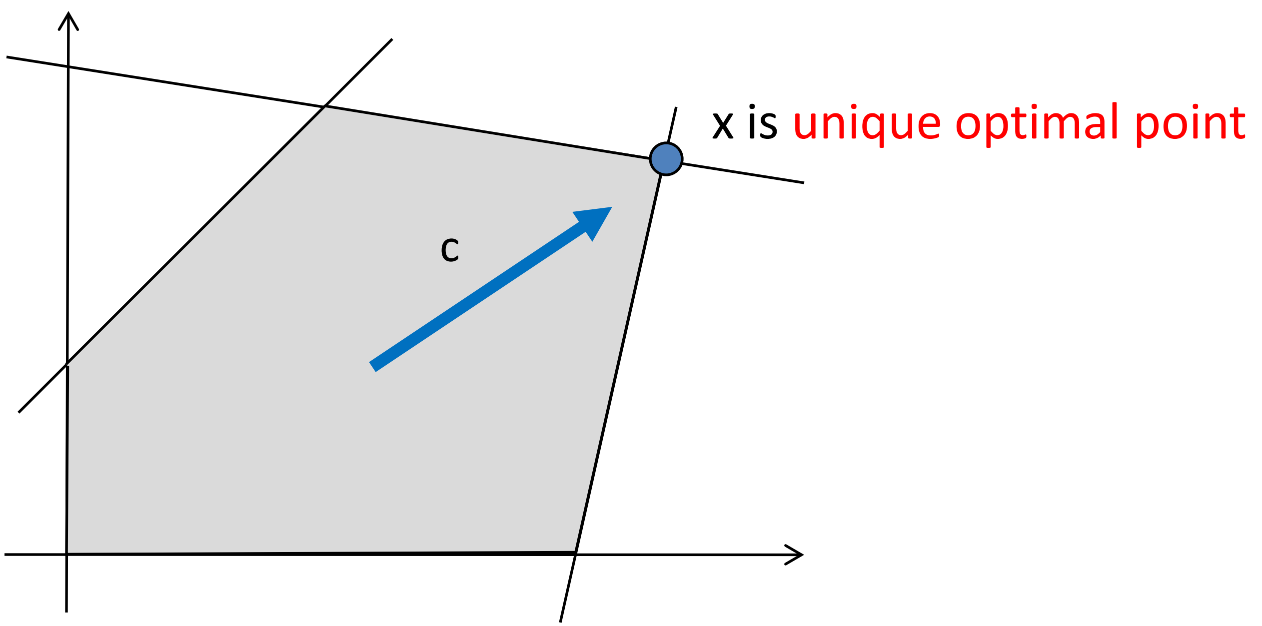
What is a corner point?

- How should we define corner points?
- Under any reasonable definition, point x should be considered a corner point



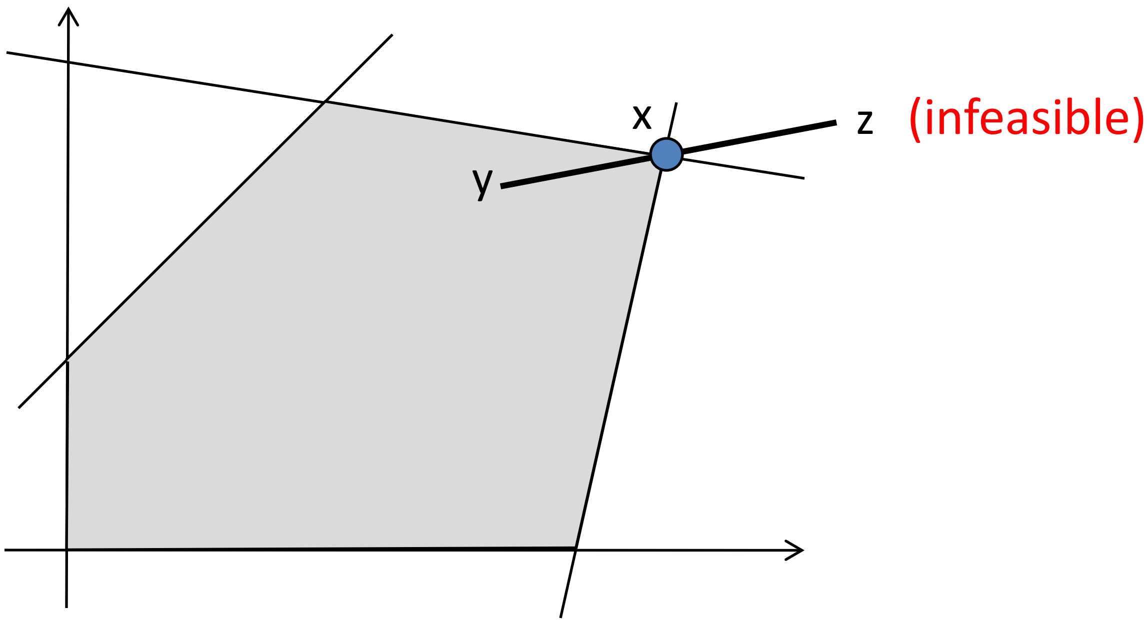
What is a corner point?

- Attempt #1: "x is the 'farthest point' in some direction"
- Let P = { feasible region }
- There exists $c \in \mathbb{R}^n$ s.t. $c^T x > c^T y$ for all $y \in P \setminus \{x\}$
- "For some objective function, x is the unique optimal point when maximizing over P"
- Such a point x is called a "**vertex**"



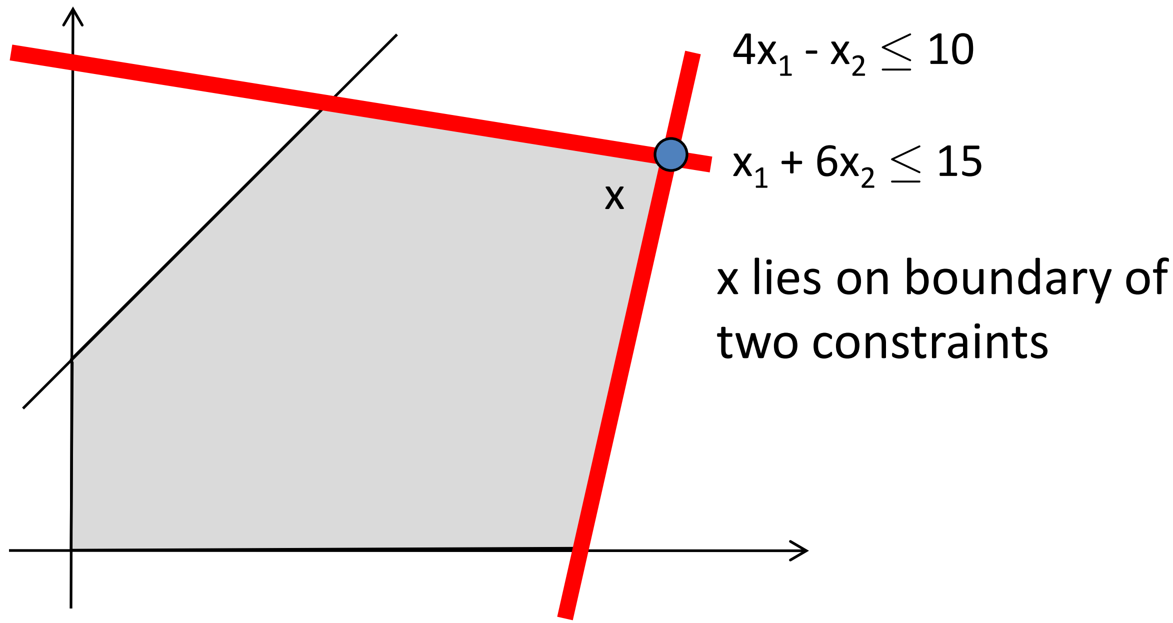
What is a corner point?

- Attempt #2: "There is no feasible line-segment that goes through x in both directions"
- Whenever $x=\alpha y+(1-\alpha)z$ with $y,z\neq x$ and $\alpha\in(0,1)$, then either y or z must be infeasible.
- "If you write x as a convex combination of two feasible points y and z, the only possibility is x=y=z"
- Such a point x is called an "**extreme point**"



What is a corner point?

- Attempt #3: "x lies on the boundary of many constraints"



What is a corner point?

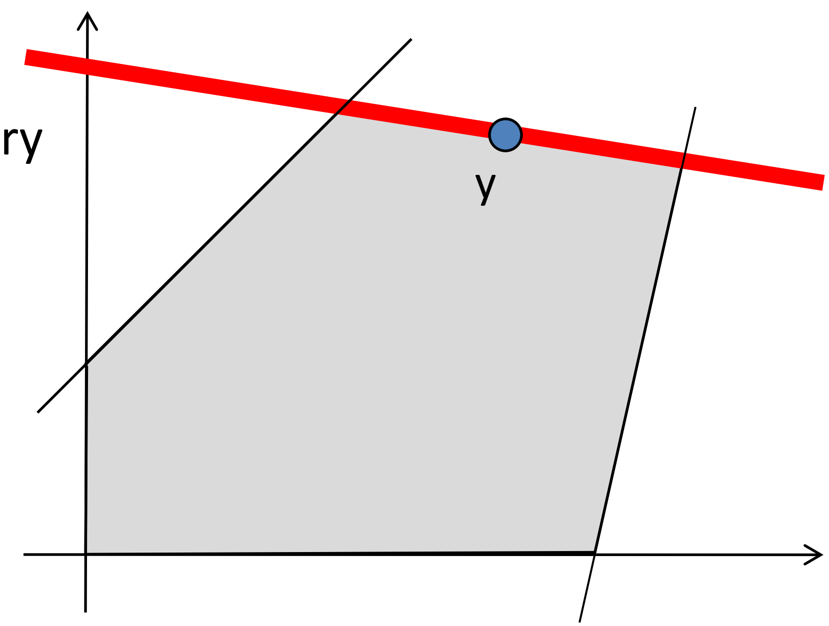
- Attempt #3: "x lies on the boundary of many constraints"
- What if I introduce **redundant** constraints?

Not the right condition

$x_1 + 6x_2 \leq 15$

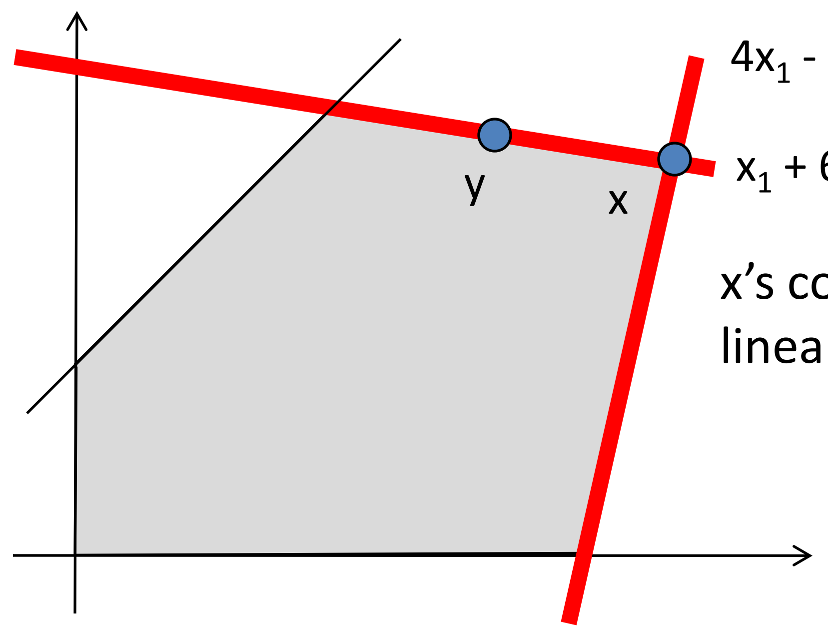
$2x_1 + 12x_2 \leq 30$

y also lies on boundary of two constraints



What is a corner point?

- Revised Attempt #3: "x lies on the boundary of many **linearly independent** constraints"
- Feasible region: $P = \{ x : a_i^T x \leq b_i \; \forall \; i \} \subset \mathbb{R}^n$
- Let $\mathcal{I}_x = \{ i : a_i^T x = b_i \}$ and $\mathcal{A}_x = \{ a_i : i \in \mathcal{I}_x \}$. **("Tight constraints")**
- x is a **"basic feasible solution (BFS)"** if rank $\mathcal{A}_x = n$



Lemma: Let P be a polyhedron. The following are equivalent.

i. x is a vertex (unique maximizer)

ii. x is an extreme point (not convex combination of other points)

iii. x is a basic feasible solution (BFS) (tight constraints have rank n)

Proof of (i)$\Rightarrow$(ii):

x is a vertex $\Rightarrow$ $\exists$ c s.t. x is unique maximizer of $c^T x$ over P

Suppose $x = \alpha y + (1-\alpha) z$ where $y,z \in P$ and $\alpha \in (0,1)$.

Suppose $y \neq x$. Then

$$c^T x = \alpha \underbrace{c^T y}_{< c^T x} + (1-\alpha) \underbrace{c^T z}_{\leq c^T x}$$

$\leq c^T x$ (since $c^T x$ is optimal value)

$< c^T x$ (since x is **unique** optimizer)

$\Rightarrow$ $c^T x < \alpha c^T x + (1-\alpha) c^T x = c^T x$ **Contradiction!**

So y=x. Symmetrically, z=x.

So x is an extreme point of P. ■

Lemma: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

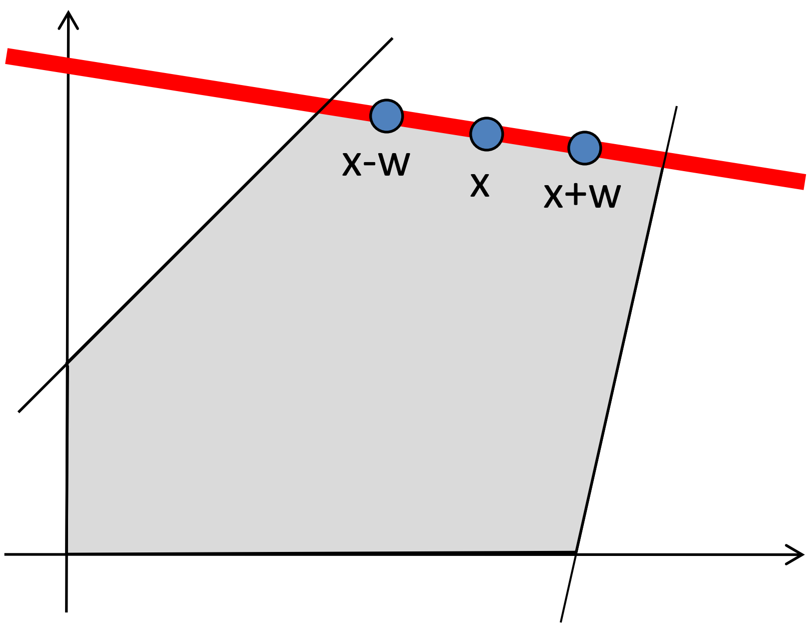
i. x is a vertex (unique maximizer)

ii. x is an extreme point (not convex combination of other points)

iii. x is a basic feasible solution (BFS) (tight constraints have rank n)

Proof Idea of (ii)$\Rightarrow$(iii):

x **not** a BFS $\Rightarrow$ rank $\mathcal{A}_x \leq$ n-1



- Each tight constraint removes one degree of freedom
- At least one degree of freedom remains
- So x can "wiggle" while staying on all the tight constraints
- Then x is a convex combination of two points obtained by "wiggling".
- So x is not an extreme point.

Lemma: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex (unique maximizer)

ii. x is an extreme point (not convex combination of other points)

iii. x is a basic feasible solution (BFS) (tight constraints have rank n)

Proof of (ii)$\Rightarrow$(iii): x **not** a BFS $\Rightarrow$ rank $\mathcal{A}_x < n$ (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

Claim: $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \ \forall a_i \in \mathcal{A}_x$ (w orthogonal to all of $\mathcal{A}_x$)

Proof: Let M be matrix whose rows are the a_i's in $\mathcal{A}_x$.

dim row-space(M) + dim null-space(M) = n

But dim row-space(M)<n $\Rightarrow$ $\exists w \neq 0$ in the null space. ☐

Lemma: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex (unique maximizer)

ii. x is an extreme point (not convex combination of other points)

iii. x is a basic feasible solution (BFS) (tight constraints have rank n)

Proof of (ii)$\Rightarrow$(iii): x **not** a BFS $\Rightarrow$ rank $\mathcal{A}_x$<n (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

Claim: $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \ \forall a_i \in \mathcal{A}_x$ (w orthogonal to all of $\mathcal{A}_x$)

Let $y=x+\epsilon w$ and $z=x-\epsilon w$, where $\epsilon>0$.

Claim: If ϵ very small then $y,z \in P$.

Proof: First consider tight constraints at x. (i.e., those in $\mathcal{I}_x$)

$$a_i^T y = a_i^T x + \epsilon a_i^T w = b_i + 0$$

So y satisfies this constraint. Similarly for z.

Next consider the loose constraints at x. (i.e., those not in $\mathcal{I}_x$)

$$b_i - a_i^T y = \underbrace{b_i - a_i^T x}_{\text{Positive}} - \underbrace{\epsilon a_i^T w}_{\text{As small as we like}} \geq 0$$

So y satisfies these constraints. Similarly for z. □

Then $x=\alpha y+(1-\alpha)z$, where $y,z \in P$, $y,z \neq x$, and $\alpha=1/2$.

So x is **not** an extreme point. ■

Lemma: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex (unique maximizer)

ii. x is an extreme point (not convex combination of other points)

iii. x is a basic feasible solution (BFS) (tight constraints have rank n)

Proof of (iii)$\Rightarrow$(i): Let x be a BFS $\Rightarrow$ rank $\mathcal{A}_x$=n (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

Let $c = \sum_{i \in \mathcal{I}_x} a_i$.

Claim: $c^T x = \sum_{i \in \mathcal{I}_x} b_i$

Proof: $c^T x = \sum_{i \in \mathcal{I}_x} a_i^T x = \sum_{i \in \mathcal{I}_x} b_i$. □

Claim: x is an optimal point of $\max \{ c^T x : x \in P \}$.

Proof: $y \in P \Rightarrow a_i^T y \leq b_i$ for all i

$\Rightarrow c^T y = \sum_{i \in \mathcal{I}_x} a_i^T y \leq \sum_{i \in \mathcal{I}_x} b_i = c^T x$. □

If one of these is strict, then this is strict.

Claim: x is the **unique** optimal point of $\max \{ c^T x : x \in P \}$.

Proof: If for any $i \in \mathcal{I}_x$ we have $a_i^T y < b_i$ then $c^T y < c^T x$.

So every optimal point y has $a_i^T y = b_i$ for all $i \in \mathcal{I}_x$.

Since rank $\mathcal{A}_x$=n, there is only one solution: y=x! □

So x is a vertex. ■

Lemma: Let P={ x : $a_i^T x \leq b_i$ $\forall$i }$\subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex (unique maximizer)

ii. x is an extreme point (not convex combination of other points)

iii. x is a basic feasible solution (BFS) (tight constraints have rank n)

Interesting Corollary

Corollary: Any polyhedron has finitely many extreme points.

Proof: Suppose the polyhedron is defined by m inequalities.

Each extreme point is a BFS, so it corresponds to a choice of n linearly independent tight constraints.

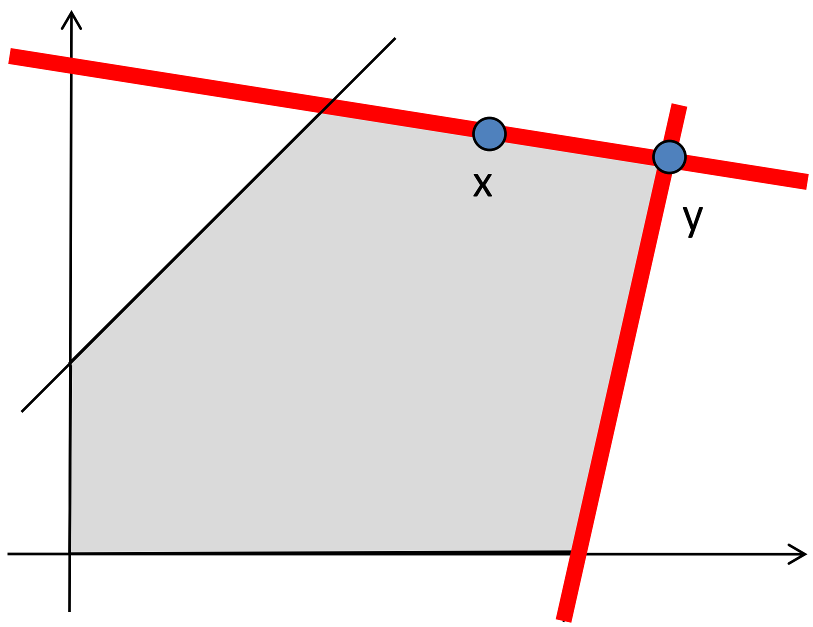
There are $\leq \binom{m}{n}$ ways to choose these tight constraints. ■

Optimal solutions at extreme points

Definition: A line is a set $L=\{ r+\lambda s : \lambda \in \mathbb{R} \}$ where $r,s \in \mathbb{R}^n$ and $s \neq 0$.

Lemma: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \}$. Suppose P does not contain any line.
Suppose the LP $\max \{ c^T x : x \in P \}$ has an optimal solution.
Then some extreme point is an optimal solution.

Proof Idea: Let x be optimal. Suppose x not a BFS.



- At least one degree of freedom remains at x
- So x can "wiggle" while staying on all the tight constraints
- x cannot wiggle off to infinity in both directions because P contains no line
- So when x wiggles, it hits a constraint
- When it hits first constraint, it is still feasible.
- So we have found a point y which has a new tight constraint.
- Repeat until we get a BFS.

Lemma: Let $P=\{ x : a_i^T x \leq b_i \; \forall i \}$. Suppose P does not contain any line.
Suppose the LP $\max \{ c^T x : x \in P \}$ has an optimal solution.
Then some extreme point is an optimal solution.

Proof: Let x be optimal, with maximal number of tight constraints.
Suppose x not a BFS.

Claim: $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \;\; \forall i \in \mathcal{I}_x$ (We saw this before)

Let $y(\epsilon) = x + \epsilon w$. Suppose $c^T w = 0$.

Claim: $\exists \delta$ s.t. $y(\delta) \notin P$. WLOG $\delta > 0$. (Otherwise P contains a line)

Set $\delta = 0$ and gradually increase δ. What is largest δ s.t. $y(\delta) \in P$?

$$y(\delta) \in P \;\Leftrightarrow\; a_i^T y(\delta) \leq b_i \;\; \forall i$$

$$\Leftrightarrow\; a_i^T x + \delta a_i^T w \leq b_i \;\; \forall i \qquad \text{(Always satisfied if } a_i^T w \leq 0\text{)}$$

$$\Leftrightarrow\; \delta \leq (b_i - a_i^T x)/a_i^T w \;\; \forall i \text{ s.t. } a_i^T w > 0$$

Let h be the i that minimizes this. So $\delta = (b_h - a_h^T x)/a_h^T w$.

$y(\delta)$ is also optimal because $c^T y(\delta) = c^T(x + \delta w) = c^T x$.

But $y(\delta)$ has one more tight constraint than x. Contradiction!

Lemma: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \}$. Suppose P does not contain any line.
Suppose the LP $\max \{ c^T x : x \in P \}$ has an optimal solution.
Then some extreme point is an optimal solution.

Proof: Let x be optimal, with maximal number of tight constraints.
Suppose x not a BFS.

Claim: $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \ \ \forall i \in \mathcal{I}_x$ (We saw this before)

Let $y(\epsilon) = x + \epsilon w$. Suppose $c^T w$ **> 0**.

Claim: $\exists \delta > 0$ s.t. $y(\delta) \in P$. (Same argument as before)

But then $c^T y(\delta) = c^T (x + \delta w) > c^T x$.
This contradicts optimality of x. ■

Lemma: Let $P=\{ x : a_i^T x \leq b_i \; \forall i \}$. Suppose P does not contain any line.
Suppose the LP $\max \{ c^T x : x \in P \}$ has an optimal solution.
Then some extreme point is an optimal solution.

Interesting Consequence

A simple but finite algorithm for solving LPs

Input: An LP $\max \{ c^T x : x \in P \}$ where $P=\{ x : a_i^T x \leq b_i \; \forall i=1 \ldots m \}$.

Caveat: We assume P contains no line, and the LP has an optimal solution.

Output: An optimal solution.

For every choice of n of the constraints
  If these constraints are linearly independent
    Find the unique point x for which these constraints are tight
    If x is feasible, add it to a list of all extreme points.
  End
End
Output the extreme point that maximizes $c^T x$

Dimension of Sets

- **Def:** An **affine space** A is a set A = { x+z : x$\in$L }, where L is a linear space and z is any vector.
  The **dimension** of A is dim L.

- Let's say dim $\emptyset$ = -1.

- **Def:** Let C $\subseteq$ $\mathbb{R}^n$ be arbitrary. The **dimension** of C is
  min { dim A : A is an affine space with C$\subseteq$A }.

Faces

- **Def:** Let $C \subseteq \mathbb{R}^n$ be any convex set. A halfspace $H = \{ x : a^T x \leq b \}$ is called **valid** if $C \subseteq H$.
- **Def:** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A **face** of P is a set
$$F = P \cap \{ x : a^T x = b \}$$
where $H = \{ x : a^T x \leq b \}$ is a valid halfspace.
- Clearly every face of P is also a polyhedron.
- **Claim:** P is a face of P.
- **Proof:** Take a=0 and b=0.
- **Claim:** $\emptyset$ is a face of P.
- **Proof:** Take a=0 and b=1.

k-Faces

- **Def:** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A **face** of P is a set
$$F = P \cap \{ x : a^T x = b \}$$
where $H = \{ x : a^T x \leq b \}$ is a valid halfspace.
- **Def:** A face F with dim F = k is called a **k-face**.
- Suppose dim P = d
  - A (d-1)-face is called a **facet**.
  - A (d-2)-face is called a **ridge**.
  - A 1-face is called an **edge**.
  - A 0-face F has the form $F = \{v\}$ where $v \in P$.
- **Claim:** If $F = \{v\}$ is a 0-face then v is a vertex of P.

k-Faces

- **Def:** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A **face** of P is a set
$$F = P \cap \{ x : a^T x = b \}$$
where $H = \{ x : a^T x \leq b \}$ is a valid halfspace.
- **Def:** A face F with dim F = k is called a **k-face**.

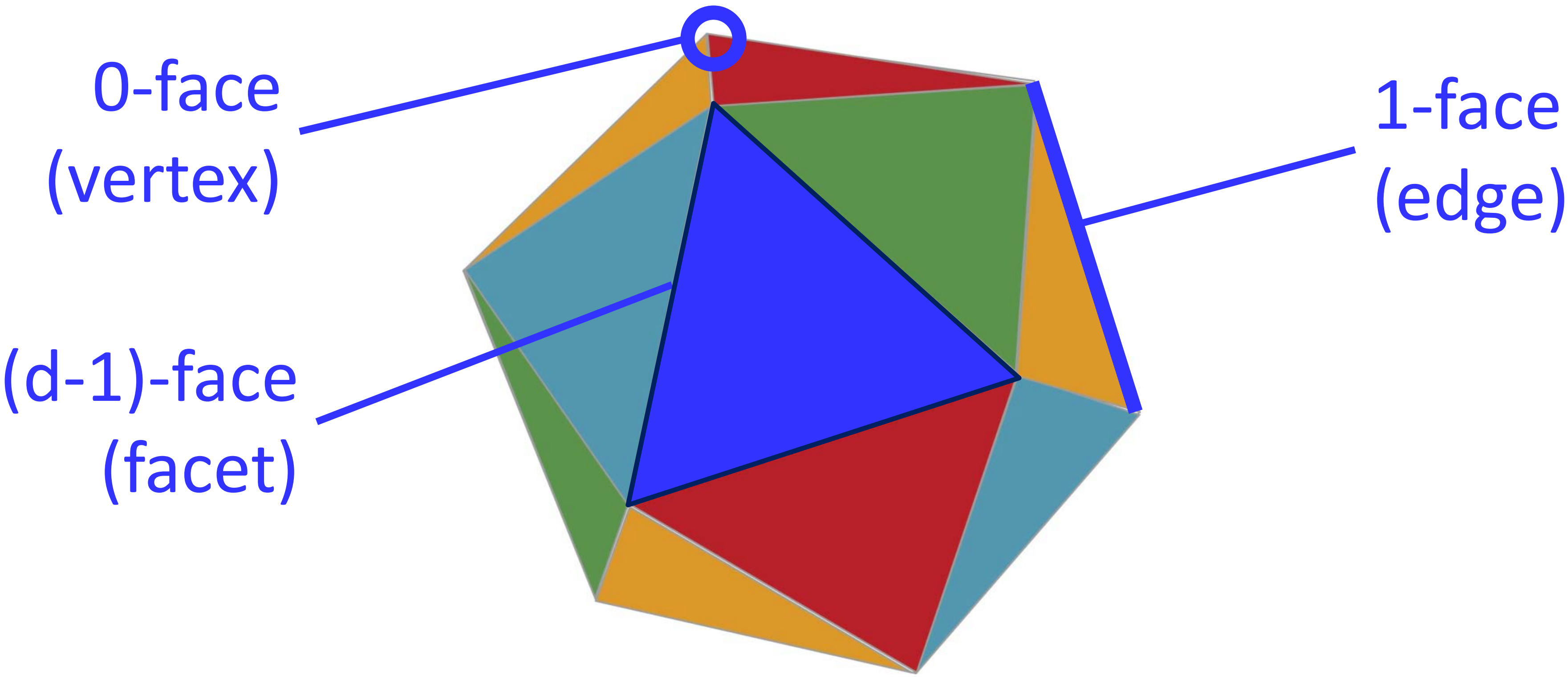


Image: http://torantula.blogspot.com/

The Simplex Method

- “The obvious idea of moving along edges from one vertex of a convex polygon to the next” [Dantzig, 1963]

Algorithm

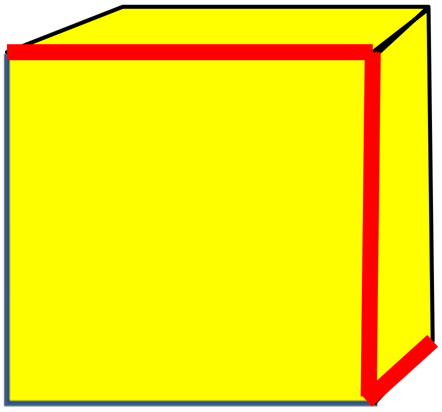
Let x be any vertex (we assume LP is feasible)
For each edge containing x
    If moving along the edge increases the objective function
        If the edge is infinitely long,
            Halt: LP is unbounded
        Else
            Set x to be other vertex in the edge
            Restart loop
Halt: x is optimal

- In practice, very efficient.
- In theory, very hard to analyze.
- How many edges must we traverse in the worst case?

Why is analyzing the simplex method hard?

- For any polyhedron, and for any two vertices, are they connected by a path of few edges?
- **The Hirsch Conjecture** (1957)
  Let $P = \{ x : Ax \leq b \}$ where A has size m x n. Then any two vertices are connected by a path of $\leq$ m-n edges.

Example: A cube.
Dimension n=3.
constraints m=6.
Connected by a length-3 path?



Yes!

Why is analyzing the simplex method hard?

- For any polyhedron, and for any two vertices, are they connected by a path of few edges?
- **The Hirsch Conjecture** (1957)
  Let P = { x : Ax$\leq$b } where A has size m x n. Then any two vertices are connected by a path of $\leq$ m-n edges.
- We have no idea how to prove this.
- **Disproved!** There is a polytope with n=43, m=86, and two vertices with no path of length $\leq$ 43 [Santos, 2010].
- **Theorem:** [Kalai-Kleitman 1992] There is always a path with $\leq$ $m^{\log n+2}$ edges.
- **Think you can do better?** A group of (very eminent) mathematicians have a blog organizing a massively collaborative project to do just that.