Lecture 9: Euclidean Geometry in R^n

Monday, October 12, 2009
2:21 PM

We will study Euclidean geometry in $\mathbb{R}^n$.

The norm of a vector x is $\|x\| := \sqrt{x^T x}$

The unit ball is

$$B := \{ x : \|x\| \le 1 \} = \{ x : x^T x \le 1 \}$$

We'll be interested in studying certain linear transformations of B

Recall a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a linear map (or affine map) if T is of the form

$$T(x) = Ax + b$$

where A is an $n \times n$ matrix and $b \in \mathbb{R}^n$.

One can define the volume of a set $S \subseteq \mathbb{R}^n$ using vector calculus. We'll just use your intuition.
The important property for us is:

$$\frac{\mathrm{vol}(T(S))}{\mathrm{vol}(S)} = |\det A|$$

Example

Define $T(x) = Ax + b$ where

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given $y = T(x)$, x is $A^{-1}(y - b)$

So
$$\begin{aligned} T(B) &= \{ T(x) : x \in B \} \\ &= \{ T(x) : x^T x \leq 1 \} \\ &= \{ y : (A^{-1}(y-b))^T (A^{-1}(y-b)) \leq 1 \} \\ &= \{ y : (y-b)^T (A^{-1})^T A^{-1} (y-b) \leq 1 \} \end{aligned}$$

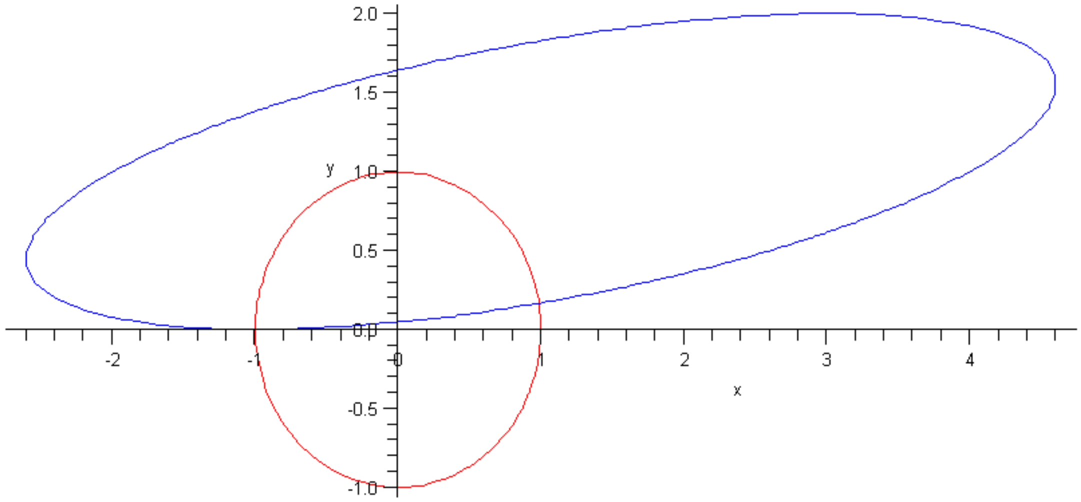
Note $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ and

$$(A^{-1})^T \cdot A^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & -2 \\ -2 & 13 \end{bmatrix}.$$

$$T(B) = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : [y_1 - 1, y_2 - 1] \, \frac{1}{9} \begin{bmatrix} 1 & -2 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} y_1 - 1 \\ y_2 - 1 \end{bmatrix} \leq 1 \right\}$$

$$= \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : (y_1^2 - 2y_1 + 1) - 4(y_1 y_2 - y_1 - y_2 + 1) + 13(y_2^2 - 2y_2 + 1) \leq 9 \right\}$$

```
implicitplot([x^2+y^2=1,(x-1)^2-4*(x-1)*(y-1)+13*(y-1)^2=9], x=-5..5, y=-5..5,
numpoints=10000, color=[red,blue] );
```



Note: center of $T(B)$ is $b=(1,1)$

We have $\dfrac{\text{vol } T(B)}{\text{vol } B} = |\det A| = 3$