C&O 355
Lecture 9

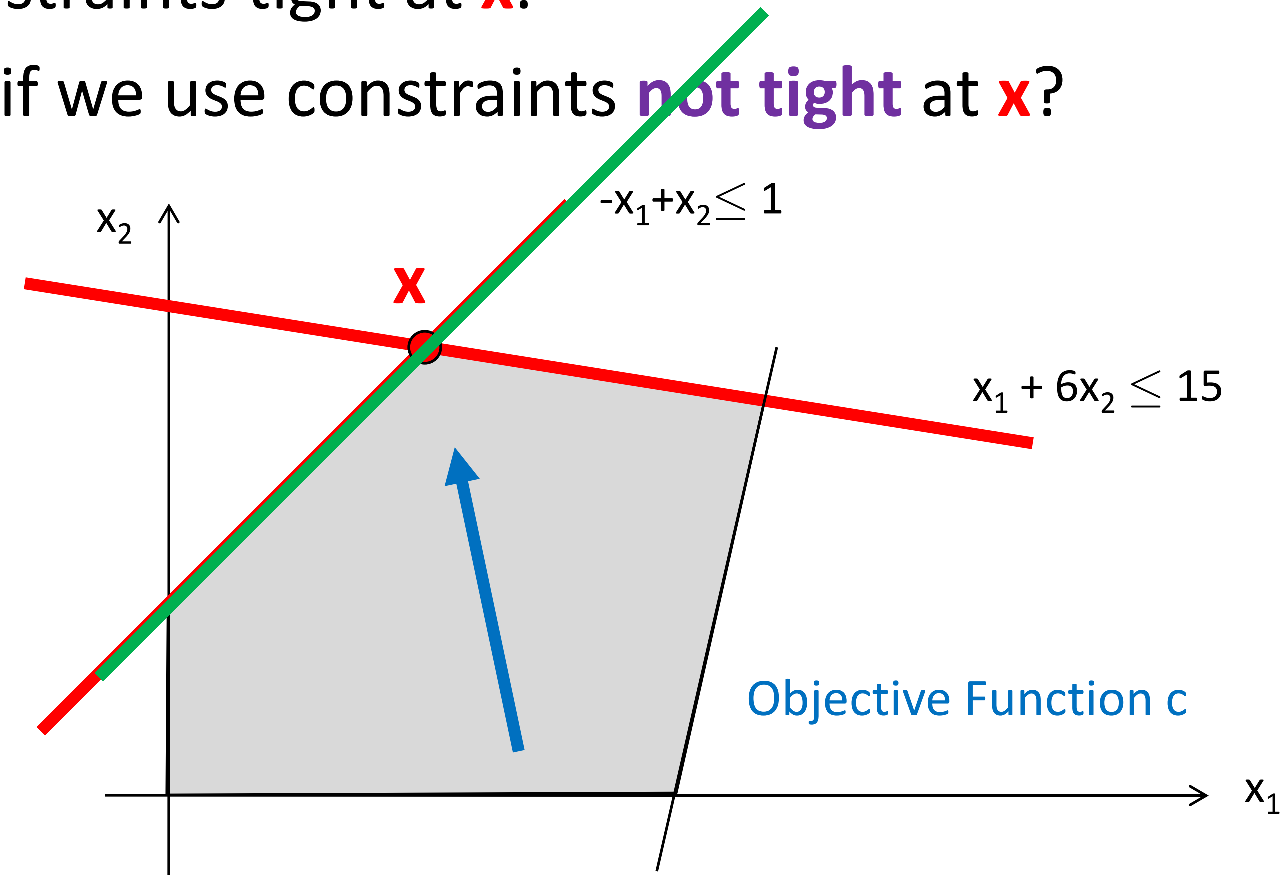
N. Harvey
http://www.math.uwaterloo.ca/~harvey/

Outline

- Complementary Slackness
- “Crash Course” in Computational Complexity
- Review of Geometry & Linear Algebra
- Ellipsoids

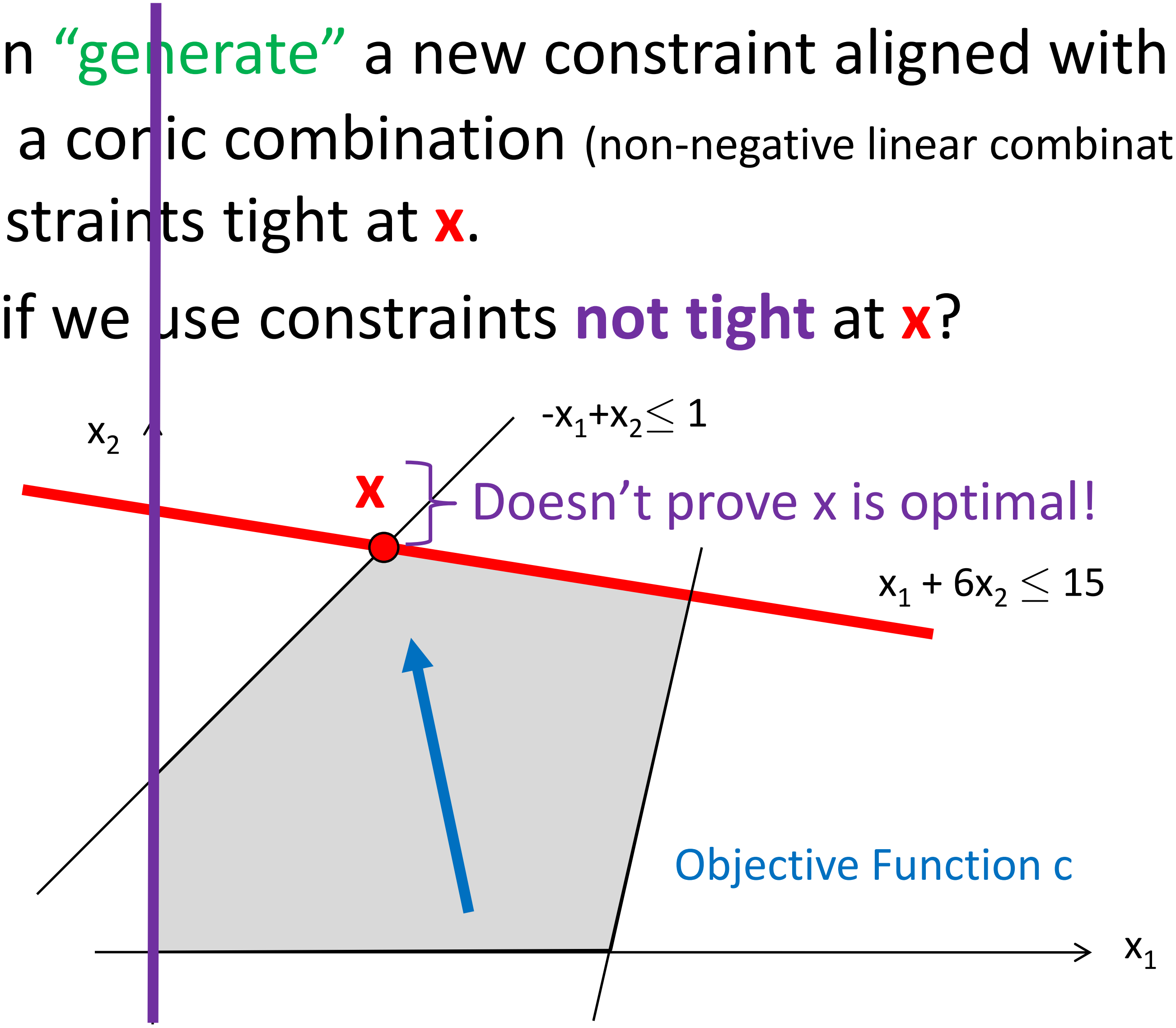
Duality: Geometric View

- We can "generate" a new constraint aligned with **c** by taking a conic combination (non-negative linear combination) of constraints tight at **x**.
- What if we use constraints **not tight** at **x**?



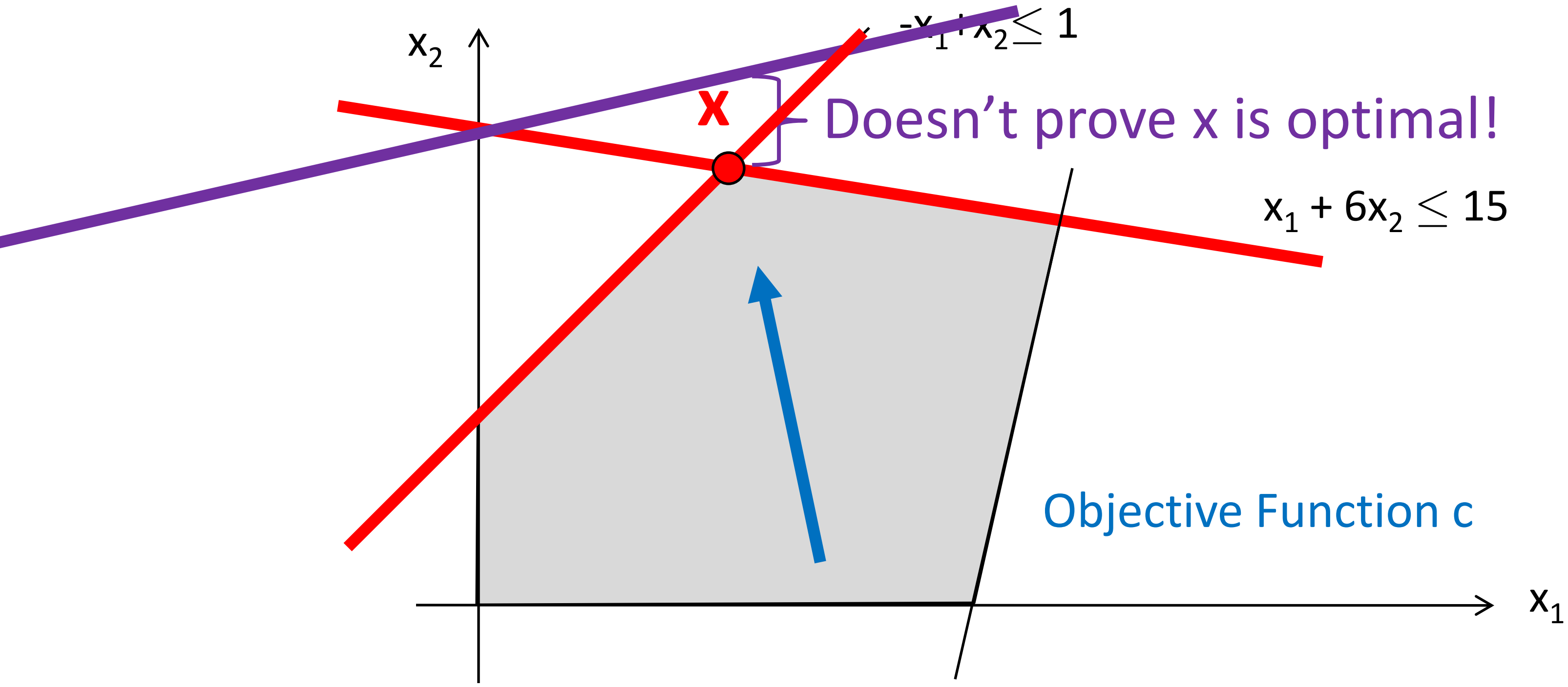
Duality: Geometric View

- We can "generate" a new constraint aligned with c by taking a conic combination (non-negative linear combination) of constraints tight at **x**.
- What if we use constraints **not tight** at **x**?



Duality: Geometric View

- What if we use constraints **not tight** at **x**?
- This linear combination is a **feasible** dual solution, but **not an optimal** dual solution
- **Complementary Slackness:** To get an **optimal** dual solution, must only use constraints tight at **x**.



Weak Duality

Primal LP

$$\begin{aligned} \max \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^{\mathsf{T}} y \\ \text{s.t.} \quad & A^{\mathsf{T}} y = c \\ & y \geq 0 \end{aligned}$$

Theorem: **"Weak Duality Theorem"**

If x feasible for Primal and y feasible for Dual then $c^T x \leq b^T y$.

Proof: $c^T x = (A^T y)^T x = y^T A x \leq y^T b$. ■

Since $y \geq 0$ and $Ax \leq b$

Weak Duality

Primal LP

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} A_{i,j} x_j \le b_i \ \forall i = 1, \ldots, m$$

Dual LP

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \sum_{i=1}^{m} A_{i,j} y_i = c_j \ \forall j = 1, \ldots, n$$
$$y \ge 0$$

Theorem: "Weak Duality Theorem"

If x feasible for Primal and y feasible for Dual then $c^T x \le b^T y$.

Proof:

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} A_{i,j} y_i \Big) x_j = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} A_{i,j} x_j \Big) y_i \le \sum_{i=1}^{m} b_i y_i$$

When does equality hold here?

Corollary:

If x and y both feasible and $c^T x = b^T y$ then x and y are both optimal.

Weak Duality

Primal LP

$$\max \quad \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \quad \sum_{j=1}^{n} A_{i,j} x_j \quad \leq b_i \;\; \forall i = 1, ..., m$$

Dual LP

$$\min \quad \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \quad \sum_{i=1}^{m} A_{i,j} y_i \quad = c_j \;\; \forall j = 1, ..., n$$
$$y \geq 0$$

Theorem: "Weak Duality Theorem"

If x feasible for Primal and y feasible for Dual then $c^T x \leq b^T y$.

Proof:

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} A_{i,j} y_i \Big) x_j = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} A_{i,j} x_j \Big) y_i \leq \sum_{i=1}^{m} b_i y_i$$

When does equality hold here?

Equality holds for i^{th} term if either $y_i=0$ or $\sum_{j=1}^{n} A_{i,j} x_j = b_i$

Weak Duality

Primal LP

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} A_{i,j} x_j \leq b_i \;\; \forall i = 1, \ldots, m$$

Dual LP

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \sum_{i=1}^{m} A_{i,j} y_i = c_j \;\; \forall j = 1, \ldots, n$$
$$y \geq 0$$

Theorem: **"Weak Duality Theorem"**

If x feasible for Primal and y feasible for Dual then $c^T x \leq b^T y$.

Proof:

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} A_{i,j} y_i \Big) x_j = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} A_{i,j} x_j \Big) y_i \leq \sum_{i=1}^{m} b_i y_i$$

Theorem: **"Complementary Slackness"**

Suppose x feasible for Primal, y feasible for dual, and

for every i, either $y_i=0$ or $\sum_{j=1}^{n} A_{i,j} x_j = b_i$.

Then x and y are both optimal.

Proof: Equality holds here. ■

General Complementary Slackness Conditions

Let x be feasible for primal and y be feasible for dual.

	Primal	Dual
Objective	max $c^T x$	min $b^T y$
Variables	$x_1, \ldots, x_n$	$y_1, \ldots, y_m$
Constraint matrix	A	A^T
Right-hand vector	b	c
Constraints versus Variables	i^{th} constraint: $\leq$ i^{th} constraint: $\geq$ i^{th} constraint: $=$ $x_j \geq 0$ $x_j \leq 0$ x_j unrestricted	$y_i \geq 0$ $y_i \leq 0$ y_i unrestricted j^{th} constraint: $\geq$ j^{th} constraint: $\leq$ j^{th} constraint: $=$

for all i,
equality holds either
for primal or dual

and

for all j,
equality holds either
for primal or dual

$\Leftrightarrow$

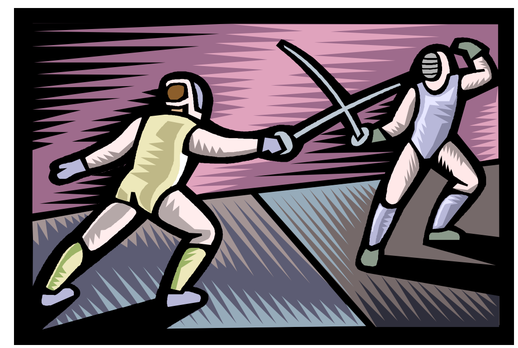
x and y are
both optimal

Example

- Primal LP

$$\begin{aligned} \min \quad & 5x_1 + 6x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 && = 3 \\ & x_1 + 2x_2 + 3x_3 && \leq 8 \\ & 4x_2 + 5x_3 && \geq 2 \\ & x_2, x_3 && \geq 0 \end{aligned}$$

- **Challenge:** What is the dual?
- What are CS conditions?



- **Claim:** Optimal primal solution is (3,0,5/3). Can you prove it?

Example

Primal LP

$$\begin{aligned} \min \quad & 5x_1 + 6x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 = 3 \\ & x_1 + 2x_2 + 3x_3 \le 8 \\ & 4x_2 + 5x_3 \ge 2 \\ & x_2, x_3 \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 3y_1 + 8y_2 + 2y_3 \\ \text{s.t.} \quad & y_1 + y_2 = 5 \\ & y_1 + 2y_2 + 4y_3 \le 6 \\ & 3y_2 + 5y_3 \le -1 \\ & y_2 \le 0 \\ & y_3 \ge 0 \end{aligned}$$

- CS conditions:
  - Either $x_1+2x_2+3x_3=8$ or $y_2=0$
  - Either $4x_2+5x_3=2$ or $y_3=0$
  - Either $y_1+2y+2+4y_3=6$ or $x_2=0$
  - Either $3y_2+5y_3=-1$ or $x_3=0$

- $x=(3,0,5/3) \Rightarrow$ y must satisfy:
  - $y_1+y_2=5$ $\quad y_3=0$ $\quad y_2+5y_3=-1$

  $\Rightarrow y = (16/3, -1/3, 0)$

Complementary Slackness Summary

- Gives "optimality conditions" that must be satisfied by optimal primal and dual solutions
- (Sometimes) gives useful way to compute optimum dual from optimum primal
  - More about this in Assignment 3
- Extremely useful in "primal-dual algorithms". Much more of this in
  - C&O 351: Network Flows
  - C&O 450/650: Combinatorial Optimization
  - C&O 754: Approximation Algorithms

We've now finished C&O 350!

- Actually, they will cover 2 topics that we won't
  - **Revised Simplex Method:** A faster implementation of the algorithm we described
  - **Sensitivity Analysis:** If we change c or b, how does optimum solution change?

Computational Complexity

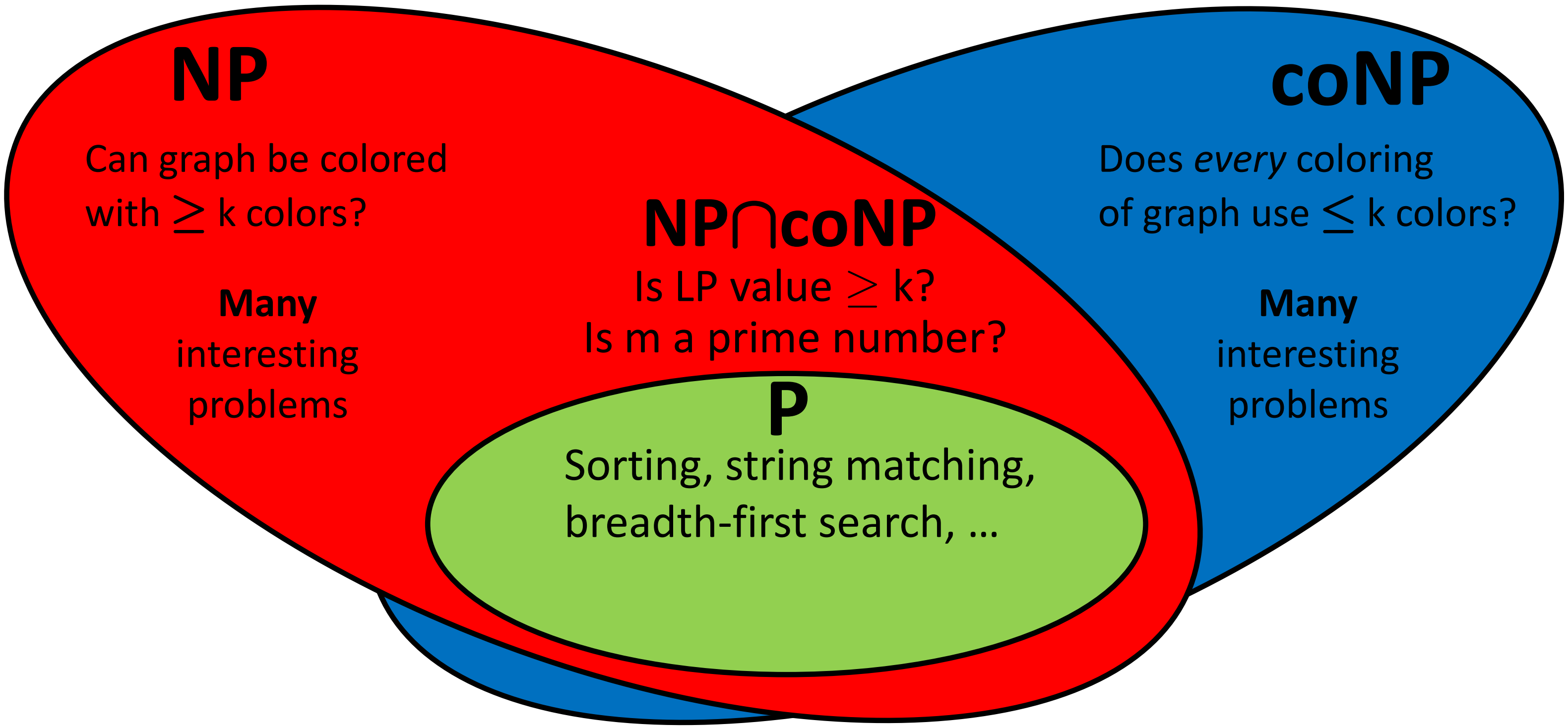
- Field that seeks to understand how efficiently computational problems can be solved
  - See CS 360, 365, 466, 764…
- What does "efficiently" mean?
  - If problem input has size **n**, how much time to compute solution? (as a function of **n**)
  - Problem can be solved "efficiently" if it can be solved in time $\leq \mathbf{n^c}$, for some constant **c**.
  - P = class of problems that can be solved efficiently.

Computational Complexity

- P = class of problems that can be solved efficiently
  i.e., solved in time $\leq n^c$, for some constant **c**, where **n**=input size.
- Related topic: **certificates**
  - Instead of studying efficiency, study **how easily can you certify the answer?**
  - NP = class of problems for which you can efficiently certify that “answer is yes”
  - coNP = class of problems for which you can efficiently certify that “answer is no”

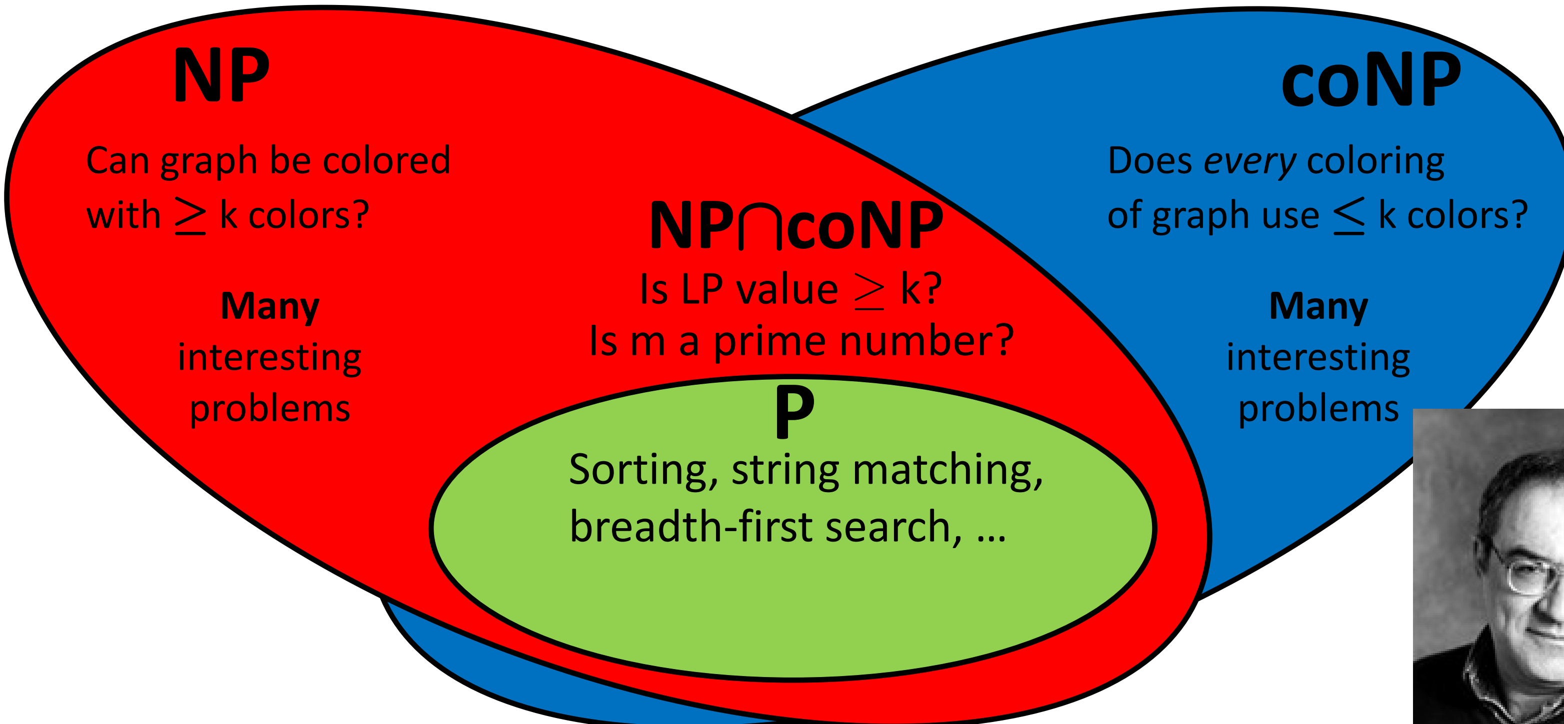
– Linear Programs
  - Can certify that optimal value is large (by giving primal solution x)
  - Can certify that optimal value is small (by giving dual solution y)

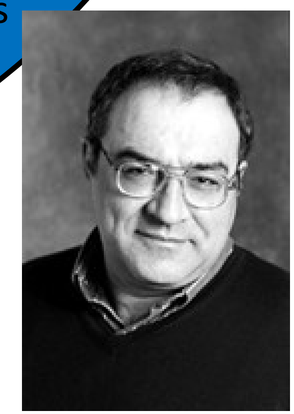
Computational Complexity



- **Open Problem:** Is P=NP?
  - Probably not
  - One of the 7 most important problems in mathematics
  - You win $1,000,000 if you solve it.

Computational Complexity




Leonid Khachiyan

- **Open Problem:** Is P=NP∩coNP?
  - Maybe… for most problems in NP∩coNP, they are also in P.
  - "Is m a prime number?". Proven in P in 2002. "Primes in P"
  - "Is LP value ≥ k?". Proven in P in 1979. "Ellipsoid method"

Review

- Basics of Euclidean geometry
  - Euclidean norm, unit ball, affine maps, volume
- Positive Semi-Definite Matrices
  - Square roots
- Ellipsoids
- Rank-1 Updates
- Covering Hemispheres by Ellipsoids

2D Example

Define $T(x) = Ax + b$ where $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

```
implicitplot([x^2+y^2=1,(x-1)^2-4*(x-1)*(y-1)+13*(y-1)^2=9], x=-5..5, y=-5..5,
numpoints=10000, color=[red,blue] );
```