C&O 355
Lecture 4

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Outline

- Equational form of LPs
- Basic Feasible Solutions for Equational form LPs
- Bases and Feasible Bases
- Brute-Force Algorithm
- Neighboring Bases

Local-Search Algorithm: Pitfalls & Details

Algorithm

Let x be any corner point

For each corner point y that is a neighbor of x

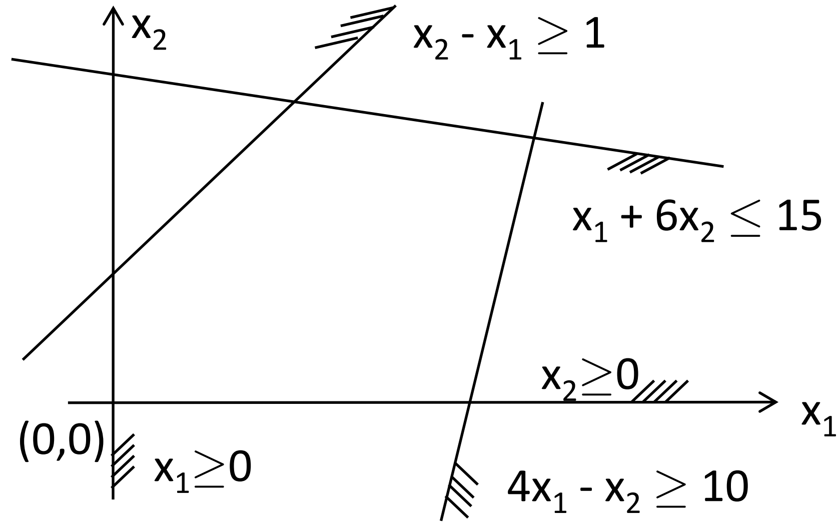
  If $c^T y > c^T x$ then set $x=y$

Halt

1. What is a corner point?
2. What if there are no corner points?
3. What are the “neighboring” corner points?
4. What if there are no neighboring corner points?
5. How can I find a starting corner point?
6. Does the algorithm terminate?
7. Does it produce the right answer?

Pitfall #2: No corner points?

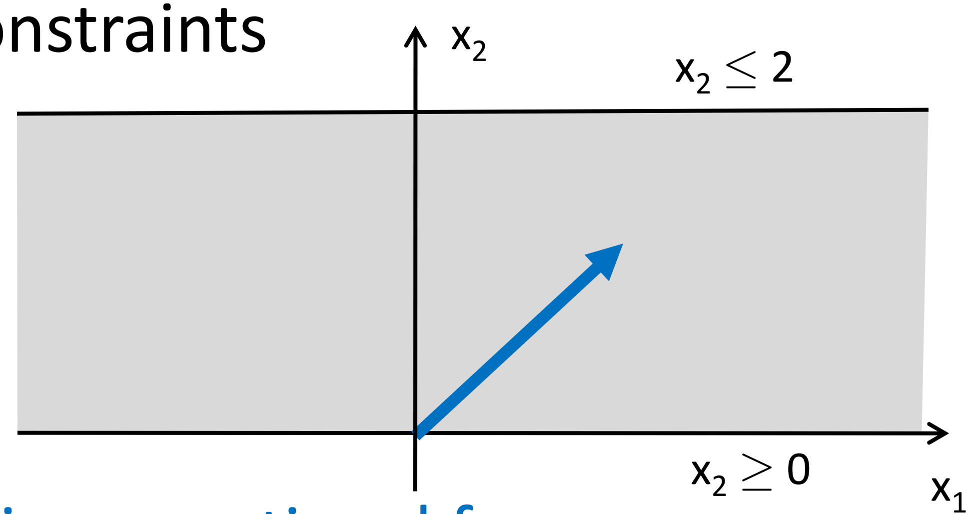
- This is possible
  - Case 1: LP infeasible



This is unavoidable.
Algorithm must detect this case.

  - Case 2: Not enough constraints

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_2 \leq 2 \\
& x_2 \geq 0
\end{aligned}
$$



A Fix!

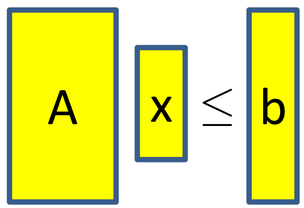
We avoid this case by using equational form.

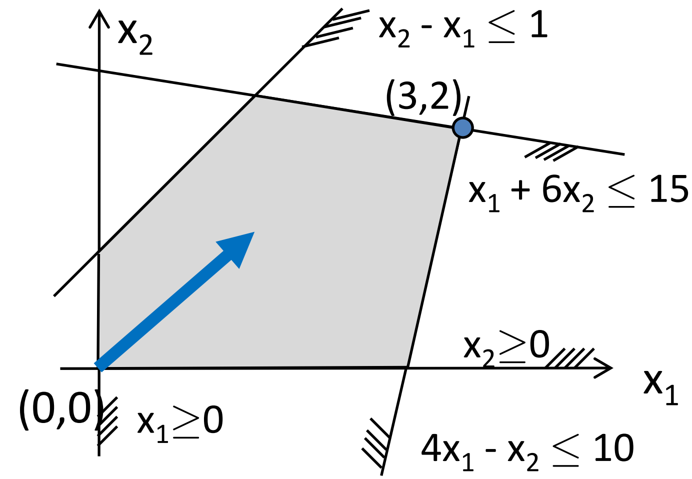
Converting to Equational Form

- "Inequality form"

$$\begin{aligned} \max \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

Tall, skinny A

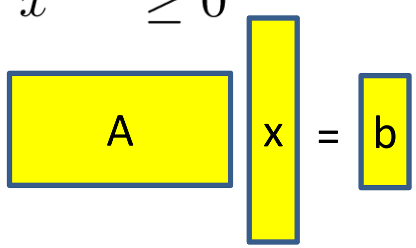


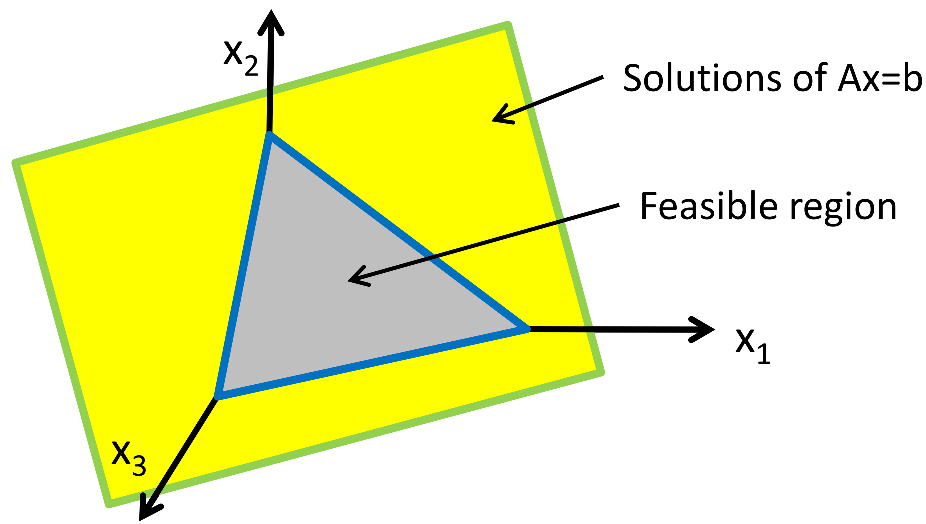


- "Equational form"

$$\begin{aligned} \max \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Short, wide A





- **Claim**: These two forms of LPs are equivalent.

“Inequality form”

$$\begin{aligned} \min \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

$\leftarrow$

“Equational form”

$$\begin{aligned} \min \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Easy: Just use “Simple LP Manipulations” from Lecture 2

Trick 1: “$\geq$” instead of “$\leq$”

Trick 2: “=” instead of “$\leq$”

This shows P={ x : Ax=b, x$\geq$0 } **is a polyhedron**.

"Inequality form" → **"Equational form"**

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \end{aligned} \quad \rightarrow \quad \begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = b \\ & x \ge 0 \end{aligned}$$

Trick 1: $x \in \mathbb{R}$ can be written $x = y - z$ where $y, z \ge 0$

So
$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \end{aligned} \quad \equiv \quad \begin{aligned} \min \quad & c^\mathsf{T}(y-z) \\ \text{s.t.} \quad & A(y-z) \le b \\ & y, z \ge 0 \end{aligned}$$

"slack variable"

Trick 2: For $u, v \in \mathbb{R}$, $u \le v \Leftrightarrow \exists w \ge 0$ s.t. $u + w = v$

So
$$\begin{aligned} \min \quad & c^\mathsf{T}(y-z) \\ \text{s.t.} \quad & A(y-z) \le b \\ & y, z \ge 0 \end{aligned} \quad \equiv \quad \begin{aligned} \min \quad & c^\mathsf{T}(y-z) \\ \text{s.t.} \quad & A(y-z) + w = b \\ & y, z, w \ge 0 \end{aligned}$$

This is already in equational form!

Rewrite it: $\tilde{A} = [A, -A, I] \quad \tilde{c} = [c, -c, 0] \quad \tilde{x} = [y, z, w]$

Then
$$\begin{aligned} \min \quad & c^\mathsf{T}(y-z) \\ \text{s.t.} \quad & A(y-z) + w = b \\ & y, z, w \ge 0 \end{aligned} \quad \equiv \quad \begin{aligned} \min \quad & \tilde{c}^\mathsf{T} \tilde{x} \\ \text{s.t.} \quad & \tilde{A}\tilde{x} = b \\ & \tilde{x} \ge 0 \end{aligned}$$

Pitfall #2: No corner points?

Lemma: Consider the polyhedron P = { Ax=b, $x \geq 0$ }.
If P is non-empty, it has **at least one corner point**.

Proof:

Claim: P contains no line.

Proof: Let $L = \{ r+\lambda s : \lambda \in \mathbb{R} \}$ be a line, where $r,s \in \mathbb{R}^n$ and $s \neq 0$.

WLOG $s_i > 0$. Let $x = r+\lambda s$, where $\lambda = -r_i/s_i - 1$.

Then $x_i = r_i + (-r_i/s_i - 1)s_i = -s_i < 0 \quad \Rightarrow \quad x \notin P$. □

Lemma from Lecture 3: "Any non-empty polyhedron containing no line must have a corner point."

So P has a corner point. ■

Local-Search Algorithm: Pitfalls & Details

Algorithm

Let x be any corner point
For each corner point y that is a neighbor of x
    If $c^T y > c^T x$ then set x=y
Halt

1. What is a corner point?
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Let's revisit #1

BFS for Equational Form LPs

- **Recall definition:** $x \in P$ is a BFS $\Leftrightarrow$ rank $\mathcal{A}_x = n$

  "x has n linearly indep. tight constraints"

- Equational form LPs have another formulation of BFS

- Let $P = \{ x : Ax=b,\ x \geq 0 \} \subseteq \mathbb{R}^n$, where A has size $m \times n$

  (Assume rank A=m, i.e., no redundant constraints)

- **Notation:** For $B \subseteq \{1,\ldots,n\}$ define

  A_B = submatrix of A containing columns indexed by B

- **Lemma:** Fix $x \in P$. x is a BFS $\Leftrightarrow$ $\exists B \subseteq \{1,\ldots,n\}$ with $|B|=m$ s.t.
  - A_B has full rank
  - $x_i = 0 \ \ \forall i \notin B$

Useful Linear Algebra Trick

- **Lemma:** Let $M = \begin{bmatrix} W & X \\ 0 & Y \end{bmatrix}$, where W and Y square.

  Then det M = det W · det Y.

- **Corollary:** M non-singular $\Leftrightarrow$ W, Y both non-singular.

Let P = { x : Ax=b, x$\geq$0 }. Assume rank A=m. Fix x$\in$P.

Lemma: x is a BFS $\Leftrightarrow$ $\exists$B$\subseteq$\{1,…,n\} with |B|=m s.t.

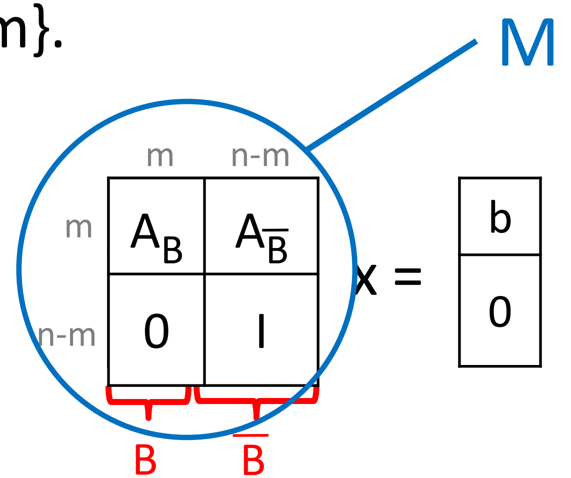
- A_B has full rank
- $x_i = 0\ \ \forall i \notin B$

Proof: $\Leftarrow$ direction. WLOG B=\{1,…,m\}.

x satisfies the constraints:

$$Ax = b$$
$$x_i = 0 \ \ \forall i \notin B$$

$\equiv$



Using trick: Since A_B and I are non-singular, M is non-singular.

So x satisfies n constraints of P with equality, and these constraints are linearly independent.

$\Rightarrow$ x is a BFS. ∎

Let $P = \{ x : Ax=b, x \geq 0 \}$. Assume rank $A=m$. Fix $x \in P$.

Lemma: x is a BFS $\Leftrightarrow$ $\exists B \subseteq \{1,\ldots,n\}$ with $|B|=m$ s.t.

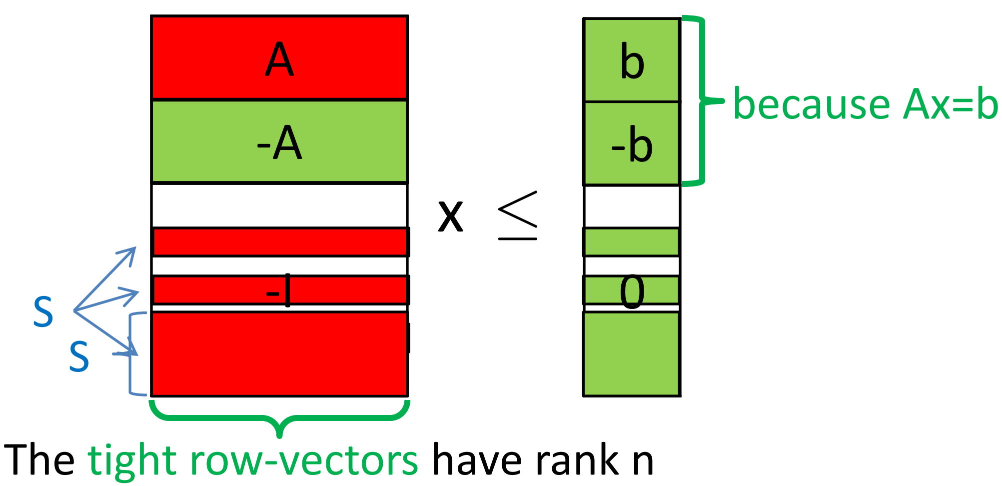
- A_B has full rank
- $x_i = 0 \;\; \forall i \notin B$

Proof: $\Rightarrow$ direction.

x a BFS $\Rightarrow$ rank $\mathcal{A}_x = n$

The constraints:

The **tight** constraints:



The rows of A are linearly independent (m of them)

Can augment rows of A to a **basis**, using only tight row-vectors

(i.e., add n-m more tight rows, preserving linear independence)

Let $S = \{ i :$ constraint "$-x_i \leq 0$" was added to **basis** $\}$. So $|S|=n-m$.

WLOG, $S=\{m+1,\ldots,n\}$. Note $x_i=0 \;\forall i \in S$.

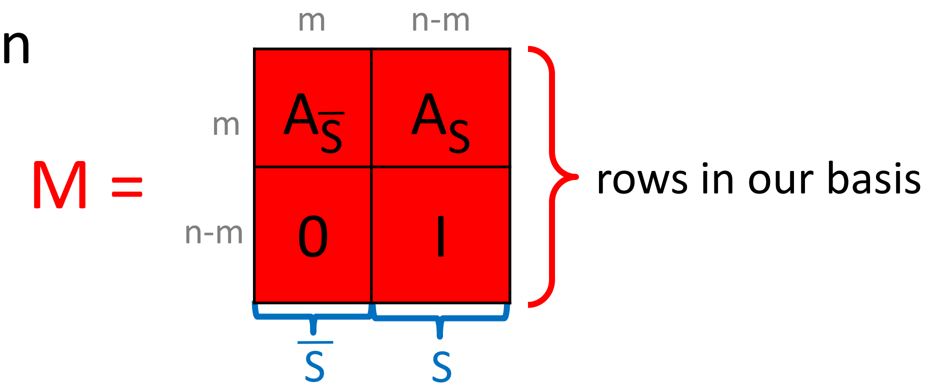
Let P = { x : Ax=b, x$\geq$0 }. Assume rank A=m. Fix x$\in$P.

Lemma: x is a BFS $\Leftrightarrow$ $\exists B \subseteq \{1,\ldots,n\}$ with $|B|=m$ s.t.

- A_B has full rank
- $x_i = 0 \ \forall i \notin B$

Proof: $\Rightarrow$ direction.

x a BFS $\Rightarrow$ rank $\mathcal{A}_x$=n



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Let S = { i : constraint "$-x_i \leq 0$" was added to **basis** }. So $|S|$=n-m.

WLOG, S={m+1,…,n}. Note $x_i=0 \ \forall i \in S$.

Rows are basis $\Rightarrow$ M non-sing. $\Rightarrow$ $A_{\bar{S}}$ non-sing. Take B=$\bar{S}$. ■

Bases

Let $P = \{ x : Ax=b, x \geq 0 \}$. Assume rank $A=m$. Fix $x \in P$.

Lemma: x is a BFS $\Leftrightarrow$ $\exists B \subseteq \{1,\ldots,n\}$ with $|B|=m$ s.t.

- A_B has full rank
- $x_i = 0 \quad \forall i \notin B$

B is called a "**basis**"

Let's use B to define a vector x. Set:

$$x_B = A_B^{-1} b \qquad \text{and} \qquad x_i = 0 \quad \forall i \notin B$$

Restrict x to components in B

Say this x is **defined** by B

Note: $Ax = b$

Is $x \geq 0$? Maybe not...

If $x \geq 0$, B is called a **feasible basis**.

Above lemma can be restated:

- If x is a BFS, then there is (at least one) feasible basis that defines it
- If basis B defines x and $x \geq 0$ then x is a BFS

Our definitions:

- $B \subseteq \{1,\ldots,n\}$ is a basis if $|B|=m$ and A_B has full rank
- Basis B defines x if $A_B x_B = b$ and $x_i = 0 \ \forall i \notin B$

Our Lemma:

- If x is a BFS, then there is (at least one) basisthat defines it
- If basis B defines x and $x \geq 0$ then x is a BFS

Corollary: $P = \{ x : Ax=b, x \geq 0 \}$ has at most $\binom{n}{m}$ extreme points

Gives simple algorithm for solving LP $\max \{ c^T x : x \in P \}$

Brute-Force Algorithm

$z = -\infty$

For every $B \subseteq \{1,\ldots,n\}$ with $|B|=m$

  If A_B has full rank (B is a basis)

    Solve $A_B\, x_B = b$ for x; set $x_{\overline{B}}=0$

    If $x \geq 0$ and $c^T x > z$ (x is a BFS)

      $z = c^T x$

Gives optimal value, assuming LP is not unbounded

Two Observations

1) It is natural to iterate over bases instead of BFS
2) Multiple bases can define same BFS $\Rightarrow$ algorithm can revisit same x

Revised Algorithm

Let B be a feasible basis (if none, infeasible)

For each feasible basis B’ that is a neighbor of B

    Compute BFS y defined by B’

    If $c^T y > c^T x$ then set $x=y$

Halt

- Revised Local-Search algorithm that gives bases a more prominent role than corner points
- This will help us define “neighbors”

Local-Search Algorithm

Let B be a feasible basis (if none, infeasible)

For each feasible basis B’ that is a neighbor of B

  Compute BFS y defined by B’

  If $c^T y > c^T x$ then set x=y

Halt

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Neighboring Bases

- **Notation:** A_k = k^{th} column of A
- Suppose we have a feasible basis B (|B|=m, A_B full rank)
  - It defines BFS x where $x_B = A_B^{-1}b$ and $x_{\overline{B}} = 0$
- Can we find a basis "similar" to B but containing some $k \notin B$?
- …next time…