C&O 355: Lecture 22 Notes

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1 Integral Polyhedra

We've observed the following fact on several occasions.

Fact 1.1. Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron that does not contain a line. For all vectors c such that the LP $\max \{ c^\mathsf{T} x : x \in P \}$ has finite optimal value, there is an optimal solution that is an extreme point of P.

We are interested in the case when the optimal values of this LP are integral. The following theorem was proved by Alan Hoffman in 1974.

Theorem 1.2. Let $P = \{ x : Ax \leq b \} \subset \mathbb{R}^n$ be a non-empty polyhedron that does not contain a line, and such that the entries of A are integers. Then the following are equivalent.

- **(1):** For all vectors $c \in \mathbb{Z}^n$, if the LP $\max \{ c^\mathsf{T} x : x \in P \}$ has finite optimal value, then the optimal value is an integer.
- **(2):** Every extreme point of the polyhedron P is an integral vector.

Actually, our hypotheses of this theorem are unnecessarily strong. This theorem also holds if the entries of A are arbitrary real numbers. Furthermore, it can be generalized to polyhedra that do contain a line.

Proof. (2) $\Rightarrow$ (1): Let $c \in \mathbb{Z}^n$ be arbitrary. Suppose that the LP has finite optimal value. By Fact 1.1, there is an extreme point x that is an optimal solution. By (2), x is an integral vector, so $c^\mathsf{T} x$ is an integer.

(1) $\Rightarrow$ (2): Let x be an arbitrary extreme point of P. Since extreme points of polyhedra are also vertices, there exists a vector $c \in \mathbb{R}^n$ such that x is the unique optimal solution of $\max \{ c^\mathsf{T} x : x \in P \}$. In fact, we may also assume that c is an integral vector; this follows from our proof in Lecture 3 that extreme points are vertices, since it defines c to be a sum of several rows of A, and A has integer entries.

We will show that x_1 must be an integer. Since P has finitely many extreme points, there exist $\epsilon > 0$ and $\beta > 0$ such that every other extreme point x' of P satisfies both $c^\mathsf{T} x > c^\mathsf{T} x' + \epsilon$ and $x'_1 - x_1 < \beta$. Let M be any integer with $M > \beta/\epsilon$. Then we have

$$Mc^\mathsf{T} x - Mc^\mathsf{T} x' > M\epsilon > \beta$$

for all other extreme points x'. Let d be the vector $d = (Mc_1 + 1, Mc_2, Mc_3, ..., Mc_n)$. Then, for all other extreme points x', we have

$$d^\mathsf{T} x - d^\mathsf{T} x' \;=\; Mc^\mathsf{T} x - Mc^\mathsf{T} x' + x_1 - x'_1 \;>\; \beta - \beta \;=\; 0.$$

This shows that x is the unique optimal solution to the LP $\max\left\{ d^{\mathsf{T}}x : x \in P \right\}$. So we have

$$\begin{aligned} \max\left\{ d^{\mathsf{T}}x : x \in P \right\} &= Mc^{\mathsf{T}}x + x_1 \\ \max\left\{ Mc^{\mathsf{T}}x : x \in P \right\} &= Mc^{\mathsf{T}}x \end{aligned}$$

Since we assume that (1) holds, both of these values are integers, and thus x_1 is an integer.

The same argument applies to every coordinate x_j, and thus x is an integral vector. ■