# C&O 355
# Lecture 20

N. Harvey

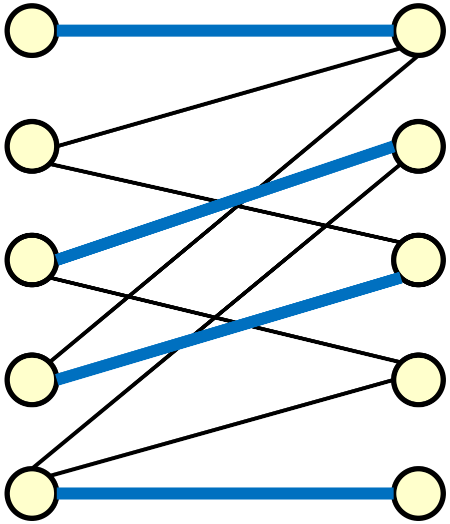
# Topics

- Vertex Covers
- Konig’s Theorem
- Hall’s Theorem
- Minimum s-t Cuts

# Maximum Bipartite Matching

- Let G=(V, E) be a bipartite graph.
- We're interested in **maximum size matchings**.
- How do I know M has maximum size? Is there a 5-edge matching?
- Is there a **certificate** that a matching has maximum size?

Blue edges are a maximum-size matching M



# Vertex covers

- Let G=(V, E) be a graph.
- A set $C \subseteq V$ is called a **vertex cover** if every edge $e \in E$ has at least one endpoint in C.

- **Claim:** If M is a matching and C is a vertex cover then $|M| \leq |C|$.
- **Proof:** Every edge in M has at least one endpoint in C.

  Since M is a matching, its edges have distinct endpoints.

  So C must contain at least $|M|$ vertices. □

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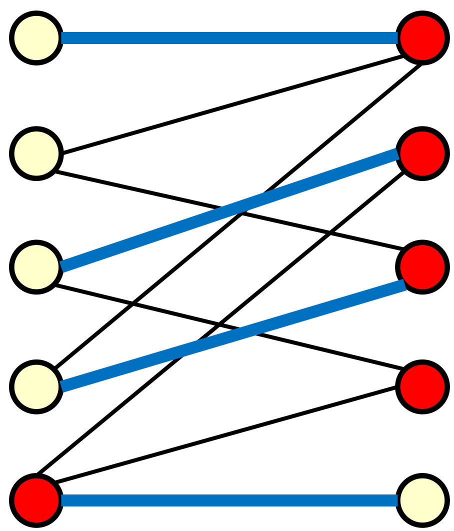

Red vertices form a vertex cover C

# Vertex covers

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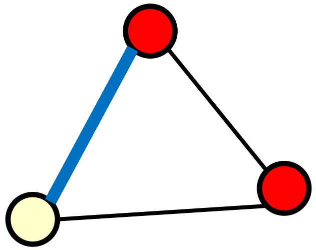
- **Claim:** If M is a matching and C is a vertex cover then $|M| \leq |C|$.
- **Proof:** Every edge in M has at least one endpoint in C.

  Since M is a matching, its edges have distinct endpoints.

  So C must contain at least $|M|$ vertices. □
- Suppose we find a matching M and vertex cover C s.t. $|M|=|C|$.
- Then M must be a maximum cardinality matching: every other matching M’ satisfies $|M'| \leq |C| = |M|$.
- And C must be a minimum cardinality vertex cover: every other vertex cover C’ satisfies $|C'| \geq |M| = |C|$.
- Then M certifies optimality of C and vice-versa.

# Vertex covers & matchings

- Let G=(V, E) be a graph.
- A set $C \subseteq V$ is called a **vertex cover** if every edge $e \in E$ has at least one endpoint in C.
- **Claim:** If M is a matching and C is a vertex cover then $|M| \leq |C|$.
- Suppose we find a matching M and vertex cover C s.t. $|M|=|C|$.
- Then M certifies optimality of C and vice-versa.
- Do such M and C always exist?
- No…



Maximum size of a matching = 1

Minimum size of a vertex cover = 2

# Vertex covers & matchings

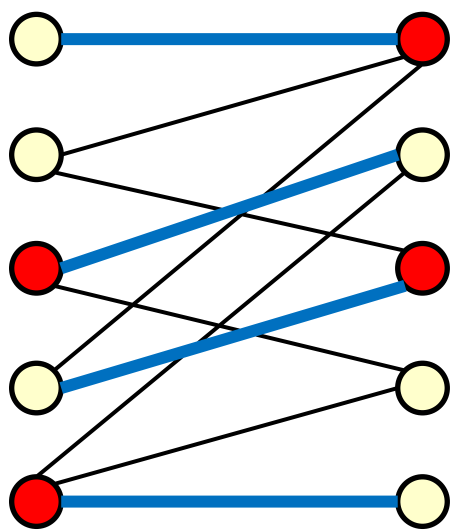
- Let G=(V, E) be a graph.
- A set C$\subseteq$V is called a **vertex cover** if every edge e$\in$E has at least one endpoint in C.
- **Claim:** If M is a matching and C is a vertex cover then $|M| \leq |C|$.
- Suppose we find a matching M and vertex cover C s.t. $|M|=|C|$.
- Then M certifies optimality of C and vice-versa.
- Do such M and C always exist?
- No… unless G is bipartite!

- **Theorem** (Konig's Theorem): If G is bipartite then there exists a matching M and a vertex cover C s.t. $|M|=|C|$.

# Earlier Example

- Let G=(V, E) be a bipartite graph.
- We're interested in **maximum size matchings**.
- How do I know M has maximum size? Is there a 5-edge matching?
- Is there a **certificate** that a matching has maximum size?

Blue edges are a maximum-size matching M



Red vertices form a vertex cover C

- Since |M|=|C|=4, both M and C are optimal!

# LPs for Bipartite Matching

- Let G=(V, E) be a bipartite graph.
- Recall our IP and LP formulations for maximum-size matching.

(IP)
$$\begin{array}{llll} \max & \sum_{e\in E} x_e & & \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\ & x_e & \in \{0,1\} & \forall e \in E \end{array}$$

(LP)
$$\begin{array}{llll} \max & \sum_{e\in E} x_e & & \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\ & x_e & \geq 0 & \forall e \in E \end{array}$$

- **Theorem**: Every BFS of (LP) is actually an (IP) solution.
- What is the dual of (LP)?

(LP-Dual)
$$\begin{array}{llll} \min & \sum_{v\in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array}$$

# Dual of Bipartite Matching LP

- What is the dual LP?

(LP-Dual)
$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array}$$

- Note that any optimal solution must satisfy $y_v \leq 1 \ \forall v \in V$
- Suppose we impose integrality constraints:

(IP-Dual)
$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \in \{0,1\} & \forall v \in V \end{array}$$

- **Claim:** If y is feasible for IP-dual then $C = \{ v : y_v = 1 \}$ is a vertex cover. Furthermore, the objective value is $|C|$.
- So IP-Dual is precisely the minimum vertex cover problem.
- **Theorem**: Every optimal BFS of (LP-Dual) is an (IP-Dual) solution.

- Let G=(U∪V, E) be a bipartite graph. Define A by

$$A_{v,e} = \begin{cases} 1 & \text{if vertex v is an endpoint of edge e} \\ 0 & \text{otherwise} \end{cases}$$

- **Lemma:** A is TUM.
- **Claim:** If A is TUM then $A^T$ is TUM.
- **Proof:** Exercise on Assignment 5.
- **Corollary:** Every **BFS** of P = { x : $A^T y \geq \mathbf{1}$, $y \geq 0$ } is integral.
- But LP-Dual is

$$\begin{array}{llll} \min & \sum_{v \in V} y_v & & \\ \text{s.t.} & y_u + y_v & \geq 1 & \forall \{u,v\} \in E \\ & y_v & \geq 0 & \forall v \in V \end{array} \quad = \quad \begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \text{s.t.} & A^T y & \geq \mathbf{1} \\ & y & \geq 0 \end{array}$$

- So our Corollary implies every BFS of LP-dual is integral
- Every optimal solution must have $y_v \leq 1$ $\forall v \in V$

  $\Rightarrow$ every optimal BFS has $y_v \in \{0,1\}$ $\forall v \in V$, and hence it is a feasible solution for IP-Dual. ■

# Proof of Konig's Theorem

- **Theorem** (Konig's Theorem): If G is bipartite then there exists a matching M and a vertex cover C s.t. |M|=|C|.
- **Proof:**

  Let x be an optimal BFS for (LP).

  Let y be an optimal BFS for (LP-Dual).

  Let M = { e : $x_e$=1 }.

  M is a matching with |M| = objective value of x. (By earlier theorem)

  Let C = { v : $y_v$ = 1 }.

  C is a vertex cover with |C| = objective value of y. (By earlier theorem)

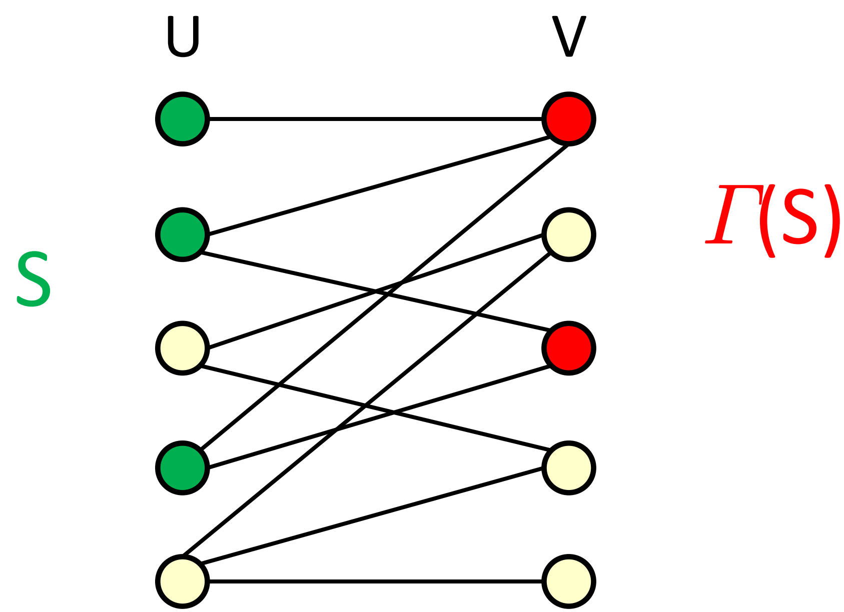
  By Strong LP Duality:

  |M| = LP optimal value = LP-Dual optimal value = |C|. ■

# Hall's Theorem

- Let G=(U∪V, E) be a bipartite graph.
- **Notation:** For S⊆U, $\Gamma(S) = \{\ v\ :\ \exists u \in S \text{ s.t. } (u,v) \in E\ \}$
- **Theorem**: There exists a matching covering all vertices in U $\Leftrightarrow$ $|\Gamma(S)| \geq |S|$ $\forall S \subseteq U$.
- **Proof:** $\Rightarrow$: This is the easy direction.

If $|\Gamma(S)| < |S|$ then there can be no matching covering S.



- **Theorem**: There exists a matching covering all vertices in U $\Leftrightarrow$ $|\Gamma(S)| \geq |S|$ $\forall S \subseteq U$.
- **Proof:** $\Leftarrow$: Suppose $|\Gamma(S)| \geq |S|$ $\forall S \subseteq U$.
- **Claim:** Every vertex cover C has $|C| \geq |U|$.
- Then Konig's Theorem implies there is a matching of size $\geq |U|$; this matching obviously covers all of U.
- **Proof of Claim:**

  Suppose C is a vertex cover with $|C \cap U| = k$ and $|C \cap V| < |U| - k$.

  Consider the set $S = U \setminus C$.

  Then $|\Gamma(S)| \geq |S| = |U| - k > |C \cap V|$.

  So there must be a vertex v in $\Gamma(S) \setminus (C \cap V)$.

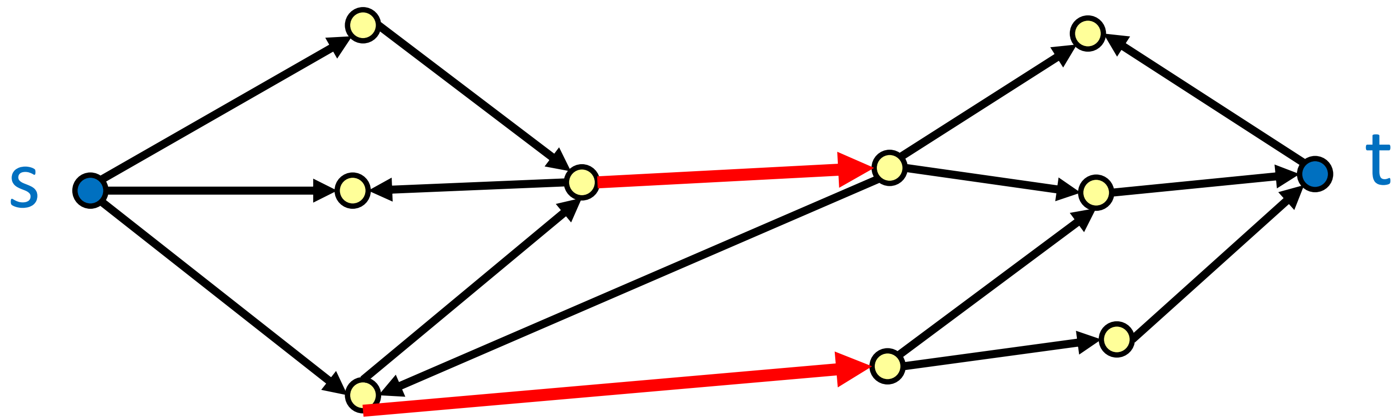
  There is an edge {s,v} with $s \in S$. (since $v \in \Gamma(S)$)

  But $s \notin C$ and $v \notin C$, so {s,v} is not covered by C.

  This contradicts C being a vertex cover. ■

# Minimum s-t Cuts

- Let $G=(V,A)$ be a digraph. Fix two vertices $s,t \in V$.
- An **s-t cut** is a set $F \subseteq A$ s.t. no s-t dipath in $G \setminus F = (V, A \setminus F)$




These edges are a **minimum** s-t cut

# Minimum s-t Cuts

- Let G=(V,A) be a digraph. Fix two vertices s,t∈V.
- An **s-t cut** is a set F⊆A s.t. no s-t dipath in G\F = (V,A\F)
- Make variable $y_a$ ∀a∈A. Let $\mathcal{P}$ be set of all s-t dipaths.

(IP)
$$\begin{aligned} \min \quad & \sum_{a \in A} y_a & & \\ \text{s.t.} \quad & \sum_{a \in p} y_a \geq 1 & & \forall p \in \mathcal{P} \\ & y_a \in \{0,1\} & & \forall a \in A \end{aligned}$$

(LP)
$$\begin{aligned} \min \quad & \sum_{a \in A} y_a & & \\ \text{s.t.} \quad & \sum_{a \in p} y_a \geq 1 & & \forall p \in \mathcal{P} \\ & y_a \geq 0 & & \forall a \in A \end{aligned}$$

Delbert Ray Fulkerson

This proves half of the famous **max-flow min-cut theorem**, due to [Ford & Fulkerson, 1956].

**Theorem:** (Fulkerson 1970)
There is an optimal solution to (LP) that is feasible for (IP)

**Theorem:** There is an optimal solution to (LP) that is feasible for (IP)

(Fulkerson's Proof is much more general and sophisticated than ours.)

(LP)

$$\begin{array}{llll} \min & \sum_{a \in A} y_a & & \\ \text{s.t.} & \sum_{a \in p} y_a & \geq 1 & \forall p \in \mathcal{P} \\ & y_a & \geq 0 & \forall a \in A \end{array}$$

$= \text{length}_y(p)$

(LP-Dual)

$$\begin{array}{llll} \max & \sum_{p \in \mathcal{P}} x_p & & \\ \text{s.t.} & \sum_{p \,:\, a \in p} x_p & \leq 1 & \forall a \in A \\ & x_p & \geq 0 & \forall p \in \mathcal{P} \end{array}$$

- We can think of $y_a$ as the "length" of arc a

- **Notation:** $\text{length}_y(p)$ = total length of path p

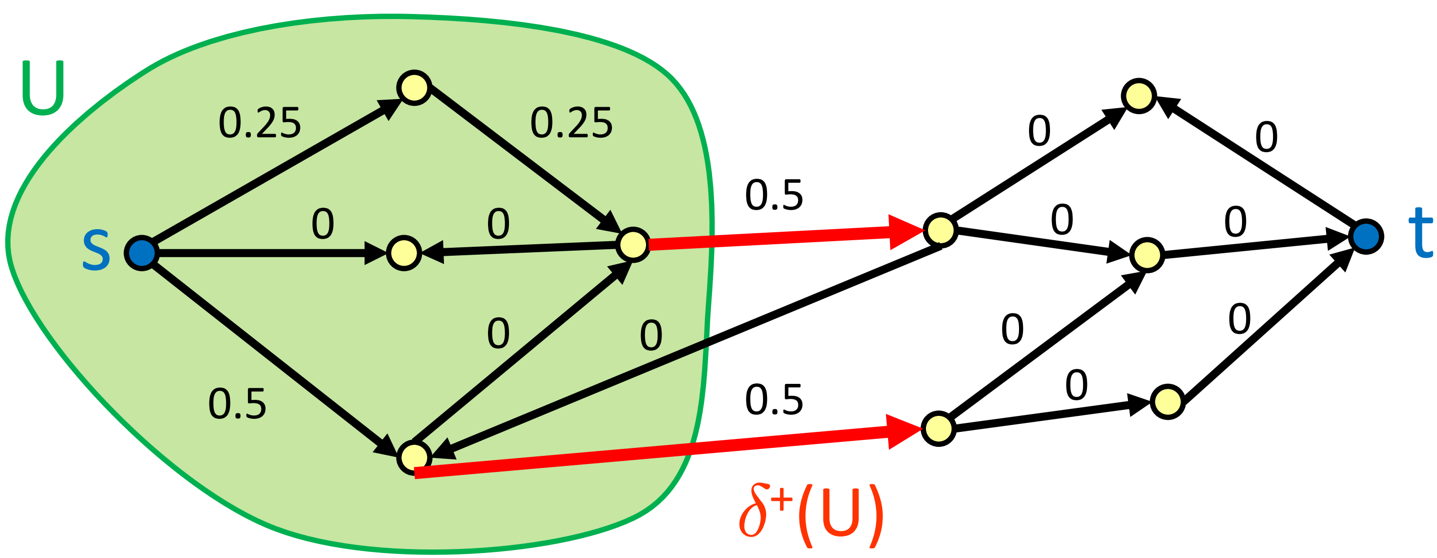
  $\text{dist}_y(u,v)$ = shortest-path distance from u to v

For any U⊆V:

$$\delta^+(U) = \{ (u,v) \in A : u \in U, v \notin U \}$$
$$\delta^-(U) = \{ (v,u) \in A : u \in U, v \notin U \}$$

- **Theorem:** Let y be optimal for (LP).
  Let U = { u : $\mathrm{dist}_y(s,u)<1$ }. Then $\delta^+(U)$ is also optimal for (LP).

- **Note:**
  - $s \in U$, since $\mathrm{dist}_y(s,s) = 0$.
  - $t \notin U$, since $\mathrm{length}_y(p) \geq 1$ for every s-t path p $\Rightarrow$ $\mathrm{dist}_y(s,t) \geq 1$
- **Claim 1:** For every path $p \in \mathcal{P}$, $|p \cap \delta^+(U)| \geq 1$.
- **Proof:** Every path $p \in \mathcal{P}$ starts at $s \in U$ and ends at $t \notin U$.
  So some arc of p must be in $\delta^+(U)$. □



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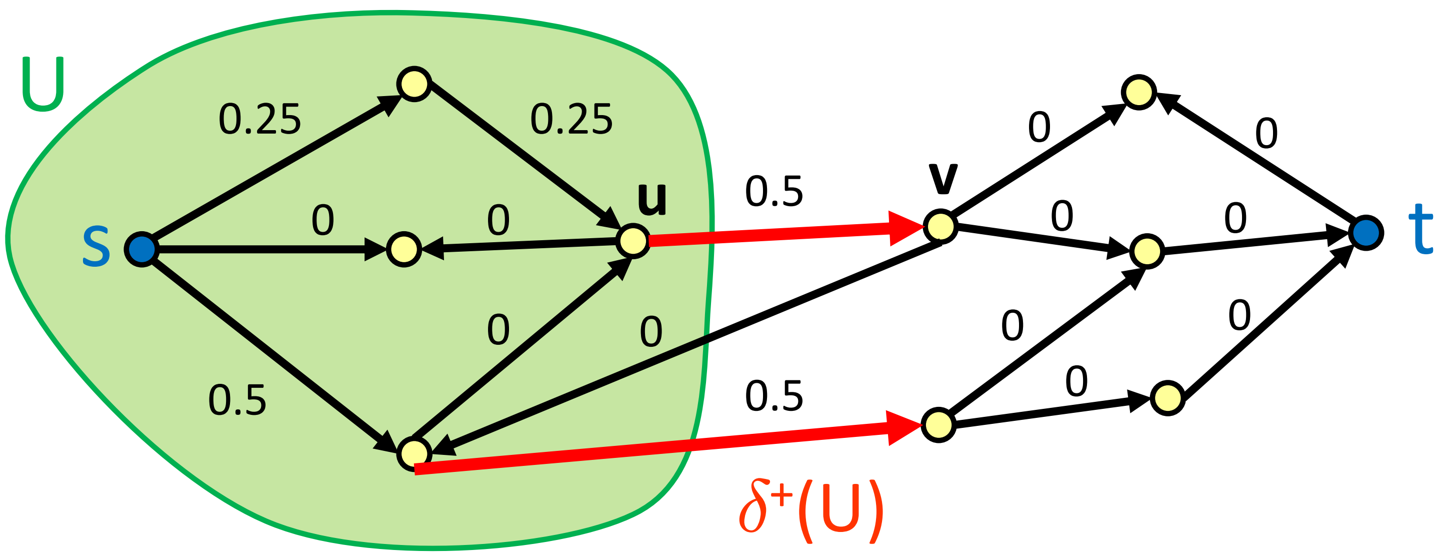
- **Claim 1:** For every path $p\in\mathcal{P}$, $|p \cap \delta^+(U)| \geq 1$.
- Let x be optimal for (LP-Dual).
- **Claim 2:** For every $(u,v) \in \delta^+(U)$, we have $y_{(u,v)}>0$ and $\sum_{p\,:\,(u,v)\in p} x_p = 1.$
- **Proof:** $1 \leq \text{dist}_y(s,v) \leq \underbrace{\text{dist}_y(s,u)}_{<1} + y_{(u,v)}$.

since v∉U   triangle inequality

This implies $y_{(u,v)} > 0$



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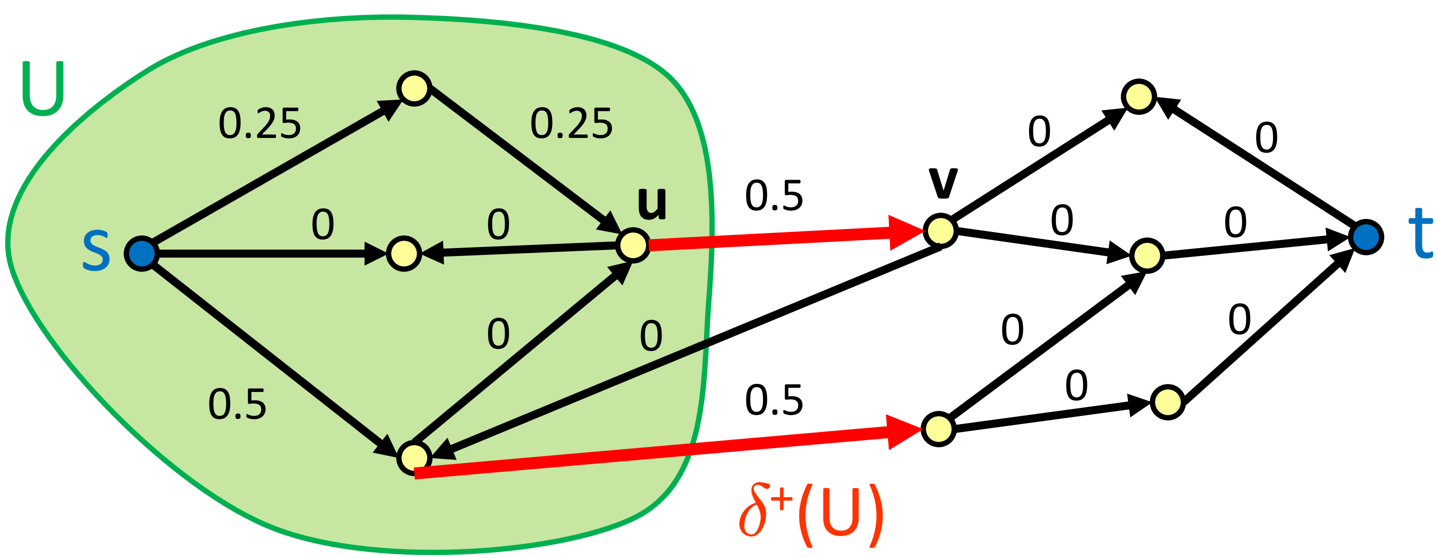
- **Proof:** $1 \leq \mathrm{dist}_y(s,v) \leq \mathrm{dist}_y(s,u) + y_{(u,v)}$.

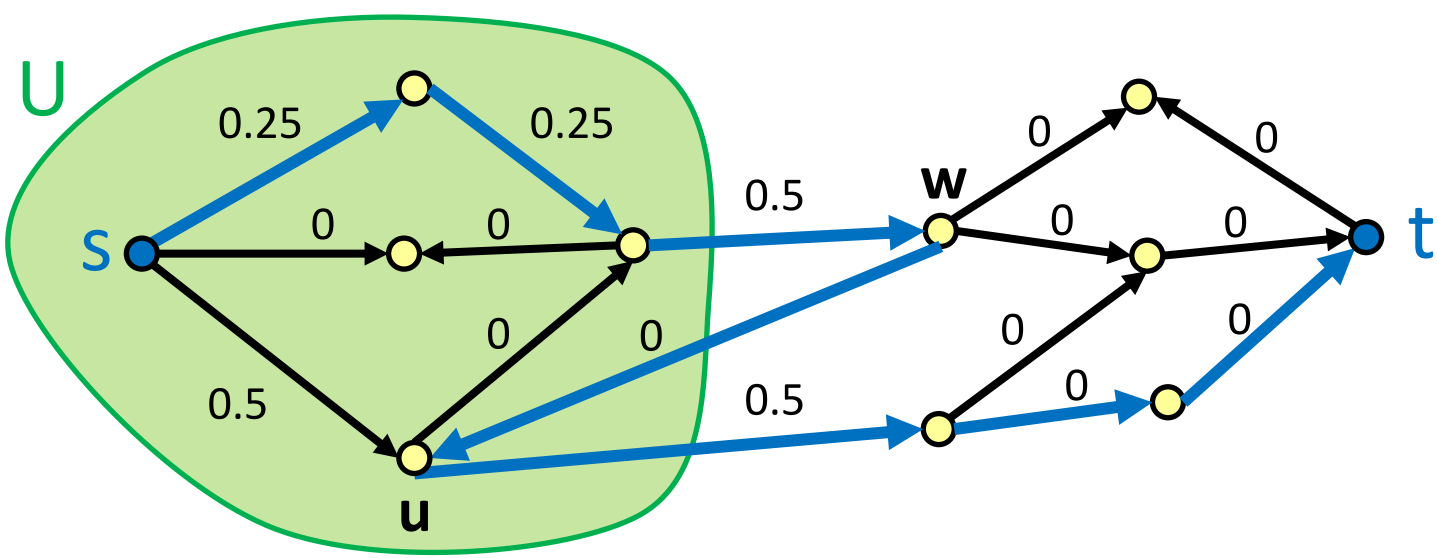
  Since $y_{(u,v)}>0$, complementary slackness implies $\sum_{p\,:\,(u,v)\in p} x_p = 1$. □



- **Claim 1:** For every path $p \in \mathcal{P}$, $|p \cap \delta^+(U)| \geq 1$.
- **Claim 2:** For every $(u,v) \in \delta^+(U)$, we have $y_{(u,v)} > 0$ and $\sum_{p\,:\,(u,v) \in p} x_p = 1.$
- **Claim 3:** Every path $p \in \mathcal{P}$ with $x_p > 0$ has $|p \cap \delta^+(U)| = 1$.
- **Proof:** Consider a path p s.t. $|p \cap \delta^+(U)| \geq 2$. (i.e., p leaves U at least twice)

  Let (w,u) be any arc in p that re-enters U, i.e., $(w,u) \in p \cap \delta^-(U)$.

  $$\text{length}_y(p) \geq \underbrace{\text{dist}_y(s,w)}_{\geq 1} + y_{(w,u)} + \underbrace{\text{dist}_y(u,t)}_{> 0} > 1$$



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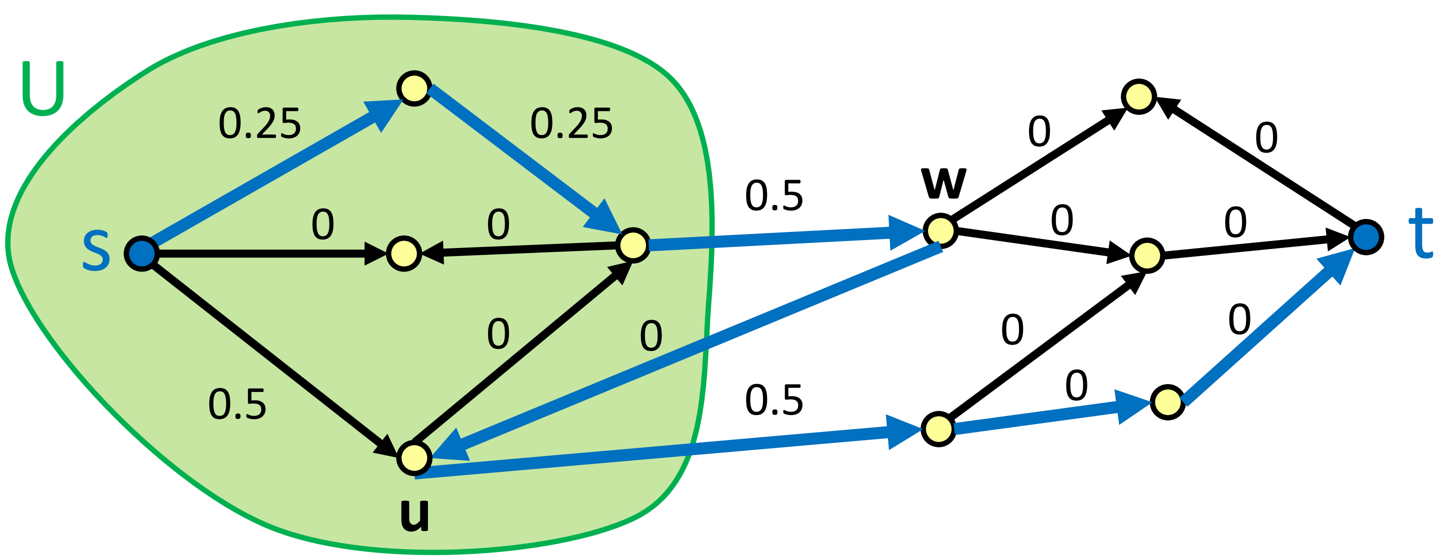
  $\text{length}_y(p) \geq \text{dist}_y(s,w) + y_{(w,u)} + \text{dist}_y(u,t) > 1$

  So $p^{th}$ constraint of (LP) is **not tight**.

  So complementary slackness implies that $x_p = 0$. □



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- **Claim 2:** For every $(u,v) \in \delta^+(U)$, we have $y_{(u,v)} > 0$ and $\sum_{p:(u,v)\in p} x_p = 1$.
- **Claim 3:** Every path $p \in \mathcal{P}$ with $x_p > 0$ has $|p \cap \delta^+(U)| = 1$.

Define the vector z by $z_{(u,v)}=1$ if $(u,v) \in \delta^+(U)$ and $z_{(u,v)}=0$ otherwise.

Note that z is feasible for (LP) and (IP). (by Claim 1)

The LP objective value at z is:

by Claim 2

$$\sum_{(u,v)\in A} z_{(u,v)} = \sum_{(u,v)\in\delta^+(U)} 1 = \sum_{(u,v)\in\delta^+(U)} \sum_{p:(u,v)\in p} x_p$$

$$= \sum_p \sum_{(u,v)\in p\cap\delta^+(U)} x_p = \sum_p x_p \cdot |p \cap \delta^+(U)|$$

by Claim 3

$$= \sum_p x_p$$ = Optimal value of (LP-Dual)

So z is optimal for (LP). ■