C&O 355: Lecture 17 Notes

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1 Faces of Polyhedra

Definition 1.1. An ***affine space*** $A \subseteq \mathbb{R}^n$ is any set of the form

$$A = \{ \mathbf{x} + \mathbf{z} : \mathbf{x} \in L \},$$

where $L \subseteq \mathbb{R}^n$ is a linear subspace and $\mathbf{z} \in \mathbb{R}^n$ is any vector. The ***dimension*** of A, denoted $\dim A$, is simply the dimension of the corresponding linear subspace L.

Definition 1.2. Let $C \subseteq \mathbb{R}^n$ be any set. The ***dimension*** of C, denoted $\dim C$, is

$$\min \{ \dim A : C \subseteq A \},$$

where the minimum is taken over all affine spaces A.

Strictly speaking, if $C = \emptyset$ then $\dim C$ is not defined. We will adopt the convention that $\dim C = -1$.

Definition 1.3. Let $C \subseteq \mathbb{R}^n$ be any convex set. An inequality $\mathbf{a}^\mathsf{T}\mathbf{x} \leq b$ is called a ***valid inequality*** if $\mathbf{a}^\mathsf{T}\mathbf{x} \leq b$ holds for every point $\mathbf{x} \in C$.

Definition 1.4. Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A ***face*** of P is any set F of the form

$$F = P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\mathsf{T}\mathbf{x} = b \right\}, \tag{1.1}$$

where $\mathbf{a}^\mathsf{T}\mathbf{x} \leq b$ is a valid inequality for P.

Notice that any face of a polyhedron is itself a polyhedron.

As an example, if we take $\mathbf{a} = \mathbf{0}$ and $b = 0$ then this shows that P is itself a face. On the other hand, if we take $\mathbf{a} = \mathbf{0}$ and $b = 1$ then this shows that $\emptyset$ is a face.

Definition 1.5. Let P be a polyhedron and let $F \subseteq P$ be a face. Then F is called a ***d-dimensional face*** or a ***d-face*** if $\dim F = d$.

Note that if F consists of a single point $\mathbf{v}$ (i.e., $F = \{\mathbf{v}\}$) then F is a 0-face. In this case, $\mathbf{v}$ is the unique maximizer of $\mathbf{a}$ over P, where $\mathbf{a}$ is the vector in Eq. (1.1), and thus $\mathbf{v}$ is a ***vertex*** of P. Recalling our earlier results, we see that for polyhedra, vertices, extreme points, basic feasible solutions, and 1-faces are the same concept.

Definition 1.6. If $\dim P = d$ and $F \subset P$ is a $(d-1)$-face then F is called a ***facet***.

Definition 1.7. A 1-face of a polyhedron is called an ***edge***.

Fact 1.8. Let P be a polytope with $\dim P = d$ and let F be a face of P. Then F is itself a polytope and so it too has faces. The faces of F are precisely the faces of P that are contained in F. Now assume that F is a facet. The facets of F are precisely the $(d-2)$-faces of P that are contained in F. Furthermore, each facet of F can be obtained as the intersection of F and another facet of P.

In much the same way that vertices are equivalent to basic feasible solutions, we can give another characterization of edges.

Fact 1.9. Let $P = \left\{ \mathbf{x} : \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \ \forall i \right\}$ be a polyhedron in $\mathbb{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct basic feasible solutions. Recall our notation

$$\mathcal{I}_\mathbf{x} \; = \; \left\{ i : \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \right\}.$$

Suppose that

$$\operatorname{rank}\{\, \mathbf{a}_i : i \in \mathcal{I}_\mathbf{x} \cap \mathcal{I}_\mathbf{y} \,\} \; = \; n-1.$$

Then the line segment

$$L_{\mathbf{x},\mathbf{y}} \; = \; \{\, \lambda\mathbf{x} + (1-\lambda)\mathbf{y} : \lambda \in [0,1] \,\} \tag{1.2}$$

is an edge of P. Moreover, if P is a polytope, then every edge arises in this way.

Definition 1.10. Let $P = \left\{ \mathbf{x} : \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \ \forall i \right\}$ be a polyhedron in $\mathbb{R}^n$. An inequality $\mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i$ is called ***facet-defining*** if the face

$$P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\mathsf{T}\mathbf{x} = b \right\}$$

is a facet.

Fact 1.11. Let $P = \left\{ \mathbf{x} : \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \ \forall i \right\}$ be a polyhedron in $\mathbb{R}^n$. Let

$$\mathcal{I} \; = \; \left\{ i : \text{the inequality ``}\mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i\text{'' is facet-defining} \right\}.$$

Then

$$P \; = \; \left\{ \mathbf{x} : \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i \ \forall i \in \mathcal{I} \right\}.$$

2 Polyhedra and Graphs

Recall from Assignment 2 that every polyhedron has finitely many vertices. Let us now restrict attention to polytopes. By Fact 1.9, every edge of a polytope can be described as the line segment $L_{\mathbf{x},\mathbf{y}}$ connecting two particular vertices $\mathbf{x}$ and $\mathbf{y}$. Thus the vertices and edges of polytopes naturally form a ***graph***.

Definition 2.1. Let P be a polytope and let V be the set of its vertices. Define the graph $G(P) = (V, E)$, where

$$E \; = \; \{\, \{u, v\} : L_{u,v} \text{ is an edge of } P \,\}.$$

This graph is called the ***1-skeleton*** of P.

One may also define $G(P)$ for unbounded polyhedra, but it is slightly messier because not all edges are as in Eq. (1.2); some edges shoot off to infinity. Actually, below we will use $G(P)$ for unbounded polyhedra without rigorously defining it.

For any finite graph $G = (V, E)$, the ***distance*** between two vertices $u, v \in V$, denoted $\text{dist}(u, v)$, is defined to be the minimum number of edges in any path from u to v. The ***diameter*** of G is

$$\operatorname{diam} G \;=\; \max_{u,v \in V} \text{dist}(u, v).$$

Alternatively, $\operatorname{diam} G$ is the smallest number p such that any two vertices can be connected by a path with p edges.

Let P be a polytope with m facets and $\dim P = n$. We are interested in the quantity $\operatorname{diam} G(P)$. In particular, how large can it be? Define

$$\Delta(n, m) \;=\; \max_{P} \operatorname{diam} G(P),$$

where the maximum is taken over all n-dimensional polytopes with m facets.

As an example, it is easy to see that $\Delta(2, m)$ is precisely $\lfloor m/2 \rfloor$.

The following notorious conjecture dates back to 1957.

Conjecture 2.2 (The Hirsch Conjecture). $\Delta(n, m) \leq m - n$.

The following theorem gives (nearly) the best-known progress towards proving the Hirsch conjecture.

Theorem 2.3 (Kalai 1991 & Kalai-Kleitman 1992). $\Delta(n, m) \leq n^{4 \ln m}$.

Before proving this theorem, we must introduce some notation. Consider any n-dimensional polytope P. Let V denote the collection of vertices of P. Let F denote the collection of facets of P.

- For any $v \in V$, let $F(v)$ denote the collection of facets which contain the point v.
- For any two vertices $v, w \in V$, let $\text{dist}(v, w)$ denote the length of the shortest path from v to w in $G(P)$.
- For any vertex v and integer $t \geq 0$, let $B(v, t) = \{\, w \in V \,:\, \text{dist}(v, w) \leq t \,\}$. This can be thought of as the ball of radius t around vertex v in $G(P)$.
- For any vertex v and integer $t \geq 0$, let $F(v, t) = \bigcup_{w \in B(v,t)} F(w)$. This is the set of all facets that can be "touched" by walking from v at most t steps between the vertices of P.

3 Proof of Kalai-Kleitman

Consider any n-dimensional polytope P whose collection of facets is F and $|F| = m$. The distance between any two vertices x and y in $G(P)$ is denote $\text{dist}_P(x, y)$, or simply $\text{dist}(x, y)$. Fix any two vertices u and v of P. Define

$$\begin{aligned} k_u &= \max \{\, t \,:\, |F(u, t)| \leq m/2 \,\} \\ k_v &= \max \{\, t \,:\, |F(v, t)| \leq m/2 \,\} \end{aligned}$$

By the pigeonhole principle, $F(u, k_u + 1) \cap F(v, k_v + 1)$ is non-empty. So there exists a facet f and two vertices $u', v' \in f$ such that

$$\begin{aligned} \text{dist}(u, u') &\leq k_u + 1 \\ \text{dist}(v, v') &\leq k_v + 1. \end{aligned} \tag{3.1}$$

Claim 3.1. $\mathrm{dist}(u', v') \leq \Delta(n-1, m-1)$.

Proof. By definition, f is an $(n-1)$-dimensional polytope. By Fact 1.8, each facet of f is the intersection of f with some other facet of P. So f has at most $m-1$ facets. Since every vertex (or edge) of f is also a vertex (or edge) of P, any path in $G(f)$ is also a path in $G(P)$. Thus $\mathrm{dist}_P(u', v') \leq \mathrm{dist}_f(u', v') \leq \Delta(n-1, m-1)$. ■

Claim 3.2. $k_v \leq \Delta(n, \lfloor m/2 \rfloor)$.

We prove Claim 3.2 below; this is the heart of the theorem. Claim 3.1 and Claim 3.2 lead to the following recursion.

$$\begin{aligned}
\mathrm{dist}_P(u, v) &\leq \mathrm{dist}_P(u, u') + \mathrm{dist}_P(u', v') + \mathrm{dist}_P(v', v) \\
&\leq (k_u + 1) + \Delta(n-1, m-1) + (k_v + 1) \\
&\leq \Delta(n-1, m-1) + 2\Delta(n, \lfloor m/2 \rfloor) + 2
\end{aligned}$$

Since u and v are arbitrary, we have

$$\Delta(n, m) \leq \Delta(n-1, m-1) + 2\Delta(n, \lfloor m/2 \rfloor) + 2. \tag{3.2}$$

The theorem follows by analyzing this recurrence, which we do below.

Proof (of Claim 3.2). Consider any vertex w with $\mathrm{dist}_P(v, w) \leq k_v$. We will obtain a recursive bound on this distance by defining a new polyhedron with fewer facets. Let Q be the polyhedron obtained by deleting all facets in $F \setminus F(v, k_v)$. In other words, let Q be the polyhedron defined by the intersection of all half-spaces induced by the facets in $F(v, k_v)$. By choice of k_v, Q has at most $\lfloor m/2 \rfloor$ facets.

The key step of the proof is to prove that

$$\mathrm{dist}_Q(v, w) \geq \mathrm{dist}_P(v, w). \tag{3.3}$$

Once this is proven, we have $\mathrm{dist}_P(v, w) \leq \mathrm{dist}_Q(v, w) \leq \Delta(n, \lfloor m/2 \rfloor)$, by induction, which is the desired inequality.

So suppose to the contrary that $\mathrm{dist}_Q(v, w) < \mathrm{dist}_P(v, w)$. Consider any shortest path p from v to w in $G(Q)$. Then there must be some edge on path p that is not an edge of P (otherwise path p would be a v-w path in $G(P)$ of length less that $\mathrm{dist}_P(v, w)$). Let $L_{\mathbf{x},\mathbf{y}}$ be the first such edge, i.e., the edge closest to v. Then $\mathbf{x}$ must be a vertex of P (since it is a face of the previous edge). However $\mathbf{y}$ cannot be a vertex of P, otherwise $L_{\mathbf{x},\mathbf{y}}$ would be an edge of P. In fact, the reason that $\mathbf{y}$ is not a vertex of P is that it is not even feasible. To see this, note that the tight constraints of Q at $\mathbf{y}$ have dimension n, and these are a subset of P's constraints. So $\mathbf{y}$ has enough tight constraints to be a vertex of P, so only reason it cannot be a vertex is that it is infeasible.

The line segment $L_{\mathbf{x},\mathbf{y}}$ is feasible for P at $\mathbf{x}$, but infeasible at $\mathbf{y}$, so it must intersect one of the facets of P that is not a facet of Q. Call this facet f and this intersection point z, so we have $f \notin F(v, k_v)$. Then z is a vertex of P and $f \in F(z)$. Furthermore, since the portion of path p from v to x is a path in $G(P)$, we have

$$\mathrm{dist}_P(v, z) \leq \mathrm{dist}_Q(v, y) \leq \mathrm{dist}_Q(v, w) < \mathrm{dist}_P(v, w) \leq k_v.$$

Thus $z \in B(v, k_v)$ and $f \in F(v, k_v)$, which is a contradiction. Thus Eq. (3.3) holds. ■

The final step is to analyze the recurrence in Eq. (3.2).

Claim 3.3. $\Delta(n,m) \le \exp(4\ln(n)\ln(m))$.

Proof. By induction on m, and also using our earlier observation $\Delta(2,m) \le \lfloor m/2 \rfloor$. We have:

$$\begin{aligned}\Delta(n,m) &\le \Delta(n-1,m-1) + 2\Delta(n,\lfloor m/2 \rfloor) + 2 \\ &\le \Delta(n-1,m) + 2\Delta(n,\lfloor m/2 \rfloor) + 2\end{aligned}$$

Let's unroll the recurrence by expanding $\Delta(n-1,m)$.

$$\le \Big(\Delta(n-2,m) + 2\Delta(n-1,\lfloor m/2 \rfloor) + 2\Big) + 2\Delta(n,\lfloor m/2 \rfloor) + 2$$

Now repeatedly unrolling the recurrence until the dimension becomes 2, we have

$$\begin{aligned}&\le \Delta(2,m) + 2\sum_{i=3}^{n} \big(\Delta(i,\lfloor m/2 \rfloor) + 1\big) \\ &\le m + 2\sum_{i=3}^{n} \big(e \cdot \Delta(n,\lfloor m/2 \rfloor)\big) \\ &\le m + e^2(n-2)\Delta(n,\lfloor m/2 \rfloor) \\ &\le m + e^2(n-2)\exp(4\ln(n)\ln(m/2))\end{aligned}$$

One may check that $m \le e^2 \exp(4\ln(n)\ln(m/2))$ holds for all $n \ge 2$ and $m \ge 2$.

$$\begin{aligned}&\le e^2 n \exp(4\ln(n)\ln(m/2)) \\ &\le \exp(4\ln(n)\ln(m/2) + \ln(n) + 2) \\ &\le \exp\Big(4\ln(n)\big(\ln(m) - 1\big) + \ln(n) + 2\Big) \\ &= \exp\big(4\ln(n)\ln(m) - 3\ln(n) + 2\big) \\ &\le \exp\big(4\ln(n)\ln(m)\big)\end{aligned}$$

This completes the inductive proof. ■

Claim 3.3 shows that

$$\Delta(n,m) \;\le\; \exp(4\ln(n)\ln(m)) \;=\; n^{4\ln m}.$$

This proves Theorem 2.3.