# Lecture 11: Ellipsoids

Monday, October 12, 2009
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Let's continue our study of applying linear transformations to the unit ball.
Consider $T(x) = Ax$ where $A^TA = I$.

Such a matrix $A$ is called an orthogonal matrix.

Geometrically, it corresponds to a rotation/reflection.

Fact 8: If $T$ is orthogonal then $T(B) = B$.

Let's try to find an orthogonal matrix $A$ that aligns an arbitrary vector $u$ with the vector $e_1 = (1,0,0,\dots,0)$.

One beautiful (and very useful) way to do this is called a "Householder Reflection."

Lemma: Given $u \in \mathbb{R}^n$, let $v = \|u\|e_1 - u$.
Define $R := I - \left(\frac{2}{v^Tv}\right) vv^T$.

Then: ① $R$ is orthogonal
② $R \cdot u = \|u\| \cdot e_1$

Proof:

① $R^TR = \left(I - \left(\frac{2}{v^Tv}\right)vv^T\right)\left(I - \left(\frac{2}{v^Tv}\right)vv^T\right)$

$= I - 2\left(\frac{2}{v^Tv}\right)vv^T + \left(\frac{2}{v^Tv}\right)^2 v v^Tv v^T$

$= I + \left(-2\left(\frac{2}{v^Tv}\right) + \frac{4}{v^Tv}\right) vv^T$

$= I$

② $$\frac{v^Tu}{v^Tv} = \frac{(\|u\|e_1 - u)^T u}{(\|u\|e_1 - u)^T(\|u\|e_1 - u)}$$
$$= \frac{\|u\| e_1^T u - \|u\|^2}{\|u\|^2 - \|u\| e_1^T u - \|u\| u^T e_1 + \|u\|^2}$$
$$= \frac{\|u\| e_1^T u - \|u\|^2}{2\|u\|^2 - 2\|u\| e_1^T u}$$
$$= -\frac{1}{2}$$

So $R \cdot u = I \cdot u - \frac{2}{v^Tv} v v^T u = u - 2\left(\frac{v^Tu}{v^Tv}\right) v$
$= u + v = \|u\| e_1.$ $\square$

Note: $R$ is actually symmetric, so $R^2 = R^T R = I$
$\Rightarrow R^{-1} = R.$

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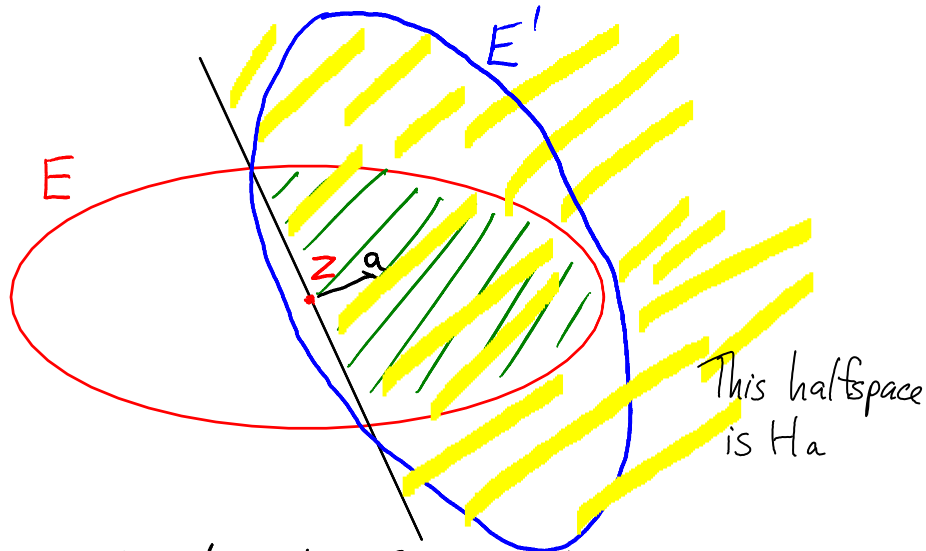
Now we generalize our earlier result to

Covering Half-Ellipsoids by Ellipsoids




Want ellipsoid $E'$ s.t. $E \cap H_a \subseteq E'$.

Suppose $E = E(N,z) = T(B)$, where $T(x) = N^{1/2}x + z$.

The only trouble is: $T(H) \neq H_a$.

So, we first want to rotate/reflect $e_1$ to align with $u := \underbrace{N^{1/2}a}$. This is a bit unmotivated, but you'll see.

We know how to rotate $u$ to align with $e_1$, using Householder Reflections. But they are self-inverse, so they also align $e_1$ with $u$.

Set $v := \|u\|e_1 - u$
Set $R := I - \left(\frac{2}{v^Tv}\right)vv^T$.

So $Ru = \|u\|e_1 \Rightarrow Re_1 = R^{-1}e_1 = u/\|u\|$

So let $S(x) = T(R(x))$
Note: $S^{-1}(y) = R^{-1}(T^{-1}(y)) = RN^{-1/2}(y-z)$

<u>Claim:</u> $S(B) = E$
Proof: $R(B) = B$ by Fact 8

$T(\bar{B}) = E$ by definition. $\square$

<u>Claim:</u> $S(H) = H_a$

<u>Proof:</u>

$$\begin{aligned}
S(H) &= \{ S(x) : x \in H \} \\
&= \{ S(x) : x^T e_1 \geq 0 \} \\
&= \{ y : S^{-1}(y)^T e_1 \geq 0 \} \\
&= \{ y : (y-z)^T N^{-\frac{1}{2}} R e_1 \geq 0 \} \\
&= \{ y : (y-z)^T N^{-\frac{1}{2}} u / \|u\| \geq 0 \} \\
&= \{ y : (y-z)^T a \geq 0 \} \\
&= H_a
\end{aligned}$$

See? $\square$

Now $B \cap H \subseteq B' \Rightarrow S(B) \cap S(H) \subseteq S(B')$.

$\Rightarrow E \cap H_a \subseteq S(B')$.

<u>Claim:</u> $S(B')$ is an ellipsoid.

<u>Proof:</u> It is a linear transformation of an ellipsoid, so $S(B')$ is also an ellipsoid! $\square$

<u>More Detailed Proof:</u>

Recall: $B' = E(M, b)$

where $$M = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} e_1 e_1^T\right) \qquad (*)$$

and $$b = \frac{e_1}{n+1}.$$

So $B' = L(B)$, where $L(x) = M^{1/2} x + b$.

So $S(B') = S(L(B))$.

Since $x \mapsto S(L(x))$ is a linear map, $S(B')$ is an ellipsoid. $\square$

So we'll <u>define</u> the ellipsoid $E'$ to be $S(B')$.

<u>Claim:</u> $$\frac{\text{vol } E'}{\text{vol } E} < \exp\left(-\frac{1}{4(n+1)}\right).$$

**Claim:** $\dfrac{\text{vol } E'}{\text{vol } E} < \exp\left(-\dfrac{1}{4(n+1)}\right)$.

**Proof:** $\dfrac{\text{vol } S(B')}{\text{vol } E} = \dfrac{\text{vol } S(B')}{\text{vol } S(B)} = \dfrac{\text{vol } B'}{\text{vol } B} < \exp\left(-\dfrac{1}{4(n+1)}\right)$.

Since $S$ scales volume of $B'$ and $B$ by same factor.

From Lecture 10.

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Let $E_i$ be the ellipsoid in step $i$ of algorithm.

Initially $E_0 = B(0,R)$

By above claim, $\dfrac{\text{vol } E_{i+1}}{\text{vol } E_i} < \exp\left(-\dfrac{1}{4(n+1)}\right)$.

$$\Rightarrow \frac{\text{vol } E_q}{\text{vol } E_0} < \prod_{i=1}^{q} \exp\left(-\frac{1}{4(n+1)}\right) = \exp\left(-\frac{q}{4(n+1)}\right).$$

$$\Rightarrow \text{vol } E_q < \text{vol } B(0,R) \cdot \exp\left(-\frac{q}{4(n+1)}\right)$$

Note that the algorithm will halt if

$$\text{vol } E_q < \text{vol } B(0,r).$$

This certainly holds if:

$$\text{vol } B(0,R) \cdot \exp\left(-\frac{q}{4(n+1)}\right) < \text{vol } B(0,r).$$

$$\Leftrightarrow \frac{\text{vol } B(0,R)}{\text{vol } B(0,r)} < \exp\left(\frac{q}{4(n+1)}\right)$$

$$\Leftrightarrow \left(\frac{R}{r}\right)^n < \exp\left(\frac{q}{4(n+1)}\right)$$

$$\Leftrightarrow n \log\left(\frac{R}{r}\right) < \frac{q}{4(n+1)}$$

Conclusion: The algorithm cannot run for more than

$4n(n+1)\log\left(\frac{R}{r}\right)$ iterations