Lecture 10: Ellipsoids

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Recall $B = \{x : x^T x \leq 1\}$ is the unit ball.

<u>Definition:</u> For any non-singular matrix A and vector b, the set $T(B)$ is an <u>ellipsoid</u>.

$$
\begin{aligned}
T(B) &= \{ x : x \in B \} \\
&= \{ x : x^T x \leq 1 \} \\
&= \{ y : (y-b)^T \underbrace{(A^{-1})^T A^{-1}} (y-b) \leq 1 \}
\end{aligned}
$$

Its center is $T(0) = b$.
Its volume is $|\det A| \cdot \mathrm{vol}(B)$.

What is this matrix?

<u>Definition:</u> Any matrix of the form $M = V^T V$ is called a <u>symmetric, positive semi-definite matrix</u> (PSD)

If M is also non-singular, it is called a <u>symmetric, positive definite matrix</u> (PD)

Such matrices are very important and useful.
There are many equivalent ways to define them.
None of these are terribly intuitive.
For convenience we will omit the word "symmetric"

Here is are two useful facts.

<u>Fact 1:</u> Let M be a positive definite matrix.
Then M^{-1} is also positive definite.

Fact 2: Let M be PD. Then there is a unique

<u>Fact 2:</u> Let M be PD. Then there is a unique PD matrix V s.t. $M = V^T V = V^2$. (V is symmetric)

The matrix V is called the square root of M, and is denoted $V = M^{1/2}$.

<u>Note:</u> $\det(M) = \det(V \cdot V) = \det(V) \cdot \det(V)$
$\Rightarrow \det(V) = \sqrt{\det M}$

<u>Note:</u>
$$\begin{aligned} &(V^{-1})^T (V^{-1}) \cdot M \\ &= V^{-1} \cdot V^{-1} \cdot V \cdot V \qquad \text{(since } V^{-1} \text{ symmetric)} \\ &= I \end{aligned}$$
$\Rightarrow M^{-1} = V^{-1} \cdot V^{-1}$
$\Rightarrow V^{-1}$ is the square root of M^{-1}
$\Rightarrow (M^{-1})^{1/2} = (M^{1/2})^{-1}$

So we can also define ellipsoids as follows.

<u>Definition:</u>
Given a PD matrix M and $b \in \mathbb{R}^n$, let $V = M^{1/2}$.
Then

$$E(M, b) = \{ x : (x-b)^T \underbrace{M^{-1}}_{=(V^{-1})^T V^{-1}} (x-b) \leq 1 \} =$$

is an ellipsoid. Here we used this

Note: $E(M,b) = T(B)$, where $T(x) = Vx + b$.

As above, $\dfrac{\text{vol } E(M,b)}{\text{vol } B} = |\det V| = \sqrt{|\det M|}$

Rank-1 Updates

Fact 3: Let $M = I + \alpha z z^T$. Suppose $\alpha \neq \frac{-1}{z^T z}$.

Then $M^{-1} = I + \beta z z^T$, where $\beta = \frac{-\alpha}{1+\alpha z^T z}$.

Proof: Just verify it:

$$(I + \alpha z z^T) \cdot (I + \beta z z^T)$$

$$= I + \alpha z z^T + \beta z z^T + \alpha \beta z \underbrace{z^T z}_{} z^T$$

This is a scalar. It commutes

$$= I + \underbrace{(\alpha + \beta + \alpha \beta z^T z)}_{} z z^T$$

$$\alpha + \beta(1 + \alpha z^T z) = \alpha + \left(\frac{-\alpha}{1 + \alpha z^T z}\right)(1 + \alpha z^T z) = 0$$

$$= I$$

$\square$

Fact 4: Let $M = I + \alpha z z^T$.

If $\alpha \geq -1/z^T z$ then M is PSD.

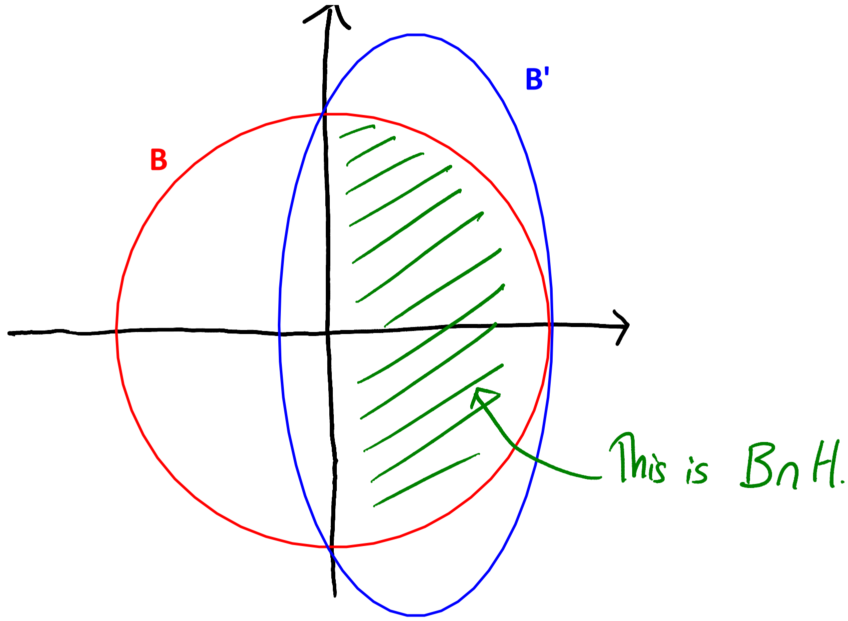
If $\alpha > -1/z^T z$ then M is PD.

Fact 5: Let $M = I + \alpha z z^T$. Then $\det M = 1 + \alpha z^T z$

Fact 6: If M has size $n \times n$ then $\det(cM) = c^n \cdot \det M$

Fact 7: $1 + x \leq e^x \quad \forall x \in \mathbb{R}$.

The next topic is "covering hemispheres by balls".



Let H be the halfspace $H = \{ x : x_1 \geq 0 \}$
$= \{ x : e_1^T x \geq 0 \}$

<u>Lemma:</u> Let $M = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} e_1 e_1^T \right)$

Let $b = \frac{e_1}{n+1}$.

Let $B' = E(M, b)$.

Then ① $B \cap H \subseteq B'$

② $\frac{\mathrm{vol}(B')}{\mathrm{vol}(B)} < e^{-1/2(n+1)} \leq 1 - \frac{1}{4(n+1)}$.

The notation $E(M, b)$ is only defined if M is PD.
That follows from Fact 4.

<u>Proof:</u>

From Fact 3 and Fact 6,

$$M^{-1} = \frac{n^2-1}{n^2}\left(I + \frac{2}{n-1} e_1 e_1^T \right).$$

<u>Claim:</u> $B' = \left\{ x : \left(\frac{n^2-1}{n^2} \right) x^T x + \frac{1}{n^2} + \frac{2(n+1)}{n^2} x_1 (x_1 - 1) \leq 1 \right\}$

$$M^{-1} = \frac{n^2-1}{n^2}\left(I + \frac{2}{n-1} e_1 e_1^T\right).$$

<u>Claim:</u> $B' = \left\{ x : \left(\frac{n^2-1}{n^2}\right) x^T x + \frac{1}{n^2} + \frac{2(n+1)}{n^2} x_1 (x_1 - 1) \leq 1 \right\}$

Proof:

$$B' = \left\{ x : \left(x - \frac{e_1}{n+1}\right)^T \left(\frac{n^2-1}{n^2}\right)\left(I + \frac{2}{n-1} e_1 e_1^T\right)\left(x - \frac{e_1}{n+1}\right) \leq 1 \right\}$$

$$= \left\{ x : \left(\frac{n^2-1}{n^2}\right)\left[\left(x - \frac{e_1}{n+1}\right)^T\left(x - \frac{e_1}{n+1}\right) + \left(\frac{2}{n-1}\right)\left(e_1^T\left(x - \frac{e_1}{n+1}\right)\right)^2\right] \leq 1 \right\}$$

$$= \left\{ x : \left(\frac{n^2-1}{n^2}\right)\left[x^T x \underbrace{- \frac{2x_1}{n+1} + \frac{1}{(n+1)^2} + \left(\frac{2}{n-1}\right)\left(x_1 - \frac{1}{n+1}\right)^2} \right] \leq 1 \right\}$$

Coefficient of $x_1^2 = \frac{(n^2-1)\cdot 2}{n^2(n-1)} = \frac{2(n+1)}{n^2}$

Coefficient of $x_1 = \frac{(n^2-1)}{n^2}\left(\frac{-2}{n+1} + 2\left(\frac{2}{n-1}\right)\left(\frac{-1}{n+1}\right)\right) = -2\frac{(n-1)}{n^2}\left(1 + \frac{2}{n-1}\right) = -2\frac{(n+1)}{n^2}$

Constant coefficient $= \frac{(n^2-1)}{n^2}\left(\frac{1}{(n+1)^2} + \left(\frac{2}{n-1}\right)\frac{1}{(n+1)^2}\right) = \frac{(n-1)}{n^2(n+1)}\left(1 + \frac{2}{n-1}\right) = \frac{1}{n^2}$ $\square$

<u>Claim:</u> $B \cap H \subseteq B'$

Proof: Let $x \in B \cap H$.

Then $x^T x \leq 1$ and $0 \leq x_1 \leq 1 \Rightarrow x_1(x_1 - 1) \leq 0$.

So $\left(\frac{n^2-1}{n^2}\right)\underbrace{x^T x}_{\leq 1} + \frac{1}{n^2} + \frac{2(n+1)}{n^2}\underbrace{x_1(x_1 - 1)}_{\leq 0}$

$\leq \frac{n^2-1}{n^2} + \frac{1}{n^2} = 1.$ So $x \in B'$. $\square$

Let's assume $n \geq 3$.

<u>Claim:</u> $\frac{\text{vol } B'}{\text{vol } B} < \exp\left(-\frac{1}{4(n+1)}\right)$.

Proof: $(\text{vol } B')^2$ $|\det M|$

Proof: $$\left(\frac{\text{vol } B'}{\text{vol } B}\right)^2 = |\det M|$$

$$= \left(\frac{n^2}{n^2-1}\right)^n \left(1 - \frac{2}{n+1}\right) \qquad \text{(by Fact 5)}$$

$$= \left(1 + \frac{1}{n^2-1}\right)^n \left(1 - \frac{2}{n+1}\right)$$

$$\leq \left(\exp\left(\frac{1}{n^2-1}\right)\right)^n \left(\exp\left(-\frac{2}{n+1}\right)\right) \qquad \text{(by Fact 7)}$$

$$= \exp\left(\frac{n}{(n+1)(n-1)} - \frac{2}{n+1}\right)$$

$$= \exp\left(\frac{1}{n+1}\left(\frac{n}{n-1} - 2\right)\right)$$

$$\leq \exp\left(\frac{-1}{2(n+1)}\right).$$

Because $n \geq 3 \Rightarrow n/(n-1) \leq 3/2 \Rightarrow \frac{n}{n-1} - 2 \leq -\frac{1}{2}$ $\square$

This completes proof of the lemma.