Some results on calibrated submanifolds in Euclidean space of cohomogeneity one and two

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Abstract
We study special Lagrangian, associative, coassociative and Cayley submanifolds of Euclidean space with symmetries. We do this by representing the data of the group action through a map, and then solving the differential equations that we get, so that a family of orbits defines a calibrated submanifold. We are able to reproduce some results from [1] using this method. We also prove explicit rigidity results for associatives in $\mathbb{R}^7$ and Cayleys in $\mathbb{R}^8$ under the action of maximal tori. Finally, we also construct cohomogeneity two examples of coassociative submanifolds in $\mathbb{R}^7$ which are invariant under the action of a maximal torus $\mathbb{T}^2 \subset \mathrm{G}_2$.

Calibrated Geometry

Calibrated geometry studies a special class of minimal (vanishing mean curvature) submanifolds within a Riemannian manifold (M, g). Given an oriented k-dimensional vector subspace V of the tangent space T_xM for some $x \in M$, there is a natural volume form vol_V. A closed k-form $\boldsymbol{\alpha}$ is a **calibration** on M if for all oriented tangent k-planes V, we have

$$\boxed{\boldsymbol{\alpha}|_V \leq \mathrm{vol}_V}$$

An oriented k-dimensional submanifold N is **calibrated** if

$$\boxed{\boldsymbol{\alpha}|_{T_xN} = \mathrm{vol}_{T_xN}}$$

for all $x \in N$.

In $\mathbb{R}^7$, the 3-form φ given by

$$\boxed{\boldsymbol{\varphi} = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} - e_{356}}$$

and its Hodge dual $\boldsymbol{\psi} = *\boldsymbol{\varphi}$ are calibrations. Submanifolds calibrated by $\boldsymbol{\varphi}$ are **associative 3-folds**, and those calibrated by $\boldsymbol{\psi}$ are **coassociative 4-folds**. In $\mathbb{R}^8$, the 4-form

$$\boxed{\boldsymbol{\Phi} = e_0 \wedge \boldsymbol{\varphi} + \boldsymbol{\psi}}$$

calibrates **Cayley 4-folds**.

Method

We consider an action of a Lie group G on M which preserves the calibration form. Let the orbits of this action be parametrized by $\theta_1, \dots, \theta_k \in \mathbb{R}$.

We take

$$\alpha : (-\epsilon_1, \epsilon_1) \times \cdots \times (-\epsilon_l, \epsilon_l) \to M$$

to be a smooth map where $\epsilon_i > 0$ and package the information about the group acting on α through the map

$$F(t_1, \dots, t_l, \theta_1, \dots, \theta_k) = A_{\theta_1, \dots, \theta_k} \cdot \alpha(t_1, \dots, t_l),$$

where $A_{\theta_1, \dots, \theta_k} \cdot \alpha(t_1, \dots, t_l)$ is just the point on the orbit through $\alpha(t_1, \dots, t_l)$. Then, we define the following vector fields along F: $V_i = \frac{\partial F}{\partial \theta_i}$, and $V_j = \frac{\partial F}{\partial t_j}$.

We want the V_i, V_j to span a G-invariant calibrated submanifold. To achieve this, we want to ensure that F is an immersion, and through that process, we obtain certain necessary conditions that α has to satisfy. Once we impose those conditions on α, we demand that the tangent space to the submanifold determined by F be calibrated, to obtain differential equations for α.

Rigidity of Associatives, Cayleys and Coassociatives

Motivated by the $\mathbb{T}^{n-1}$-invariant SL example in [1], we look at calibrated submanifolds invariant under the action of a maximal torus.

Theorem 6.1

An associative submanifold in $\mathbb{R}^7$ invariant under the action of a maximal torus $\mathbb{T}^2 \subset G_2$ has to be a special Lagrangian submanifold in $\mathbb{C}^3$.

Similarly, extending this framework into $\mathbb{R}^8$ yields a parallel rigidity constraint for Cayley geometries.

Theorem 7.1

A Cayley submanifold in $\mathbb{R}^8$ invariant under the action of a maximal torus $\mathbb{T}^3 \subset \mathrm{Spin}(7)$ has to be a special Lagrangian submanifold in $\mathbb{C}^4$.

We have the following rigidity result for the coassociatives invariant under the unit quaternions constructed in [1], whose solution curve lies in a 2-plane $\mathbb{R}\vec{e} \oplus e\mathbb{R}^+$:

Theorem 8.1

A coassociative submanifold in $\mathbb{R}^7$ invariant under the action of $\mathrm{Sp}(1) \subset \mathbb{H}$, with the solution curve lying in a 4-plane $\mathrm{Im}\mathbb{H} \oplus e\mathbb{R}^+ \subset \mathrm{Im}\mathbb{O}$, has to be exactly the $\mathrm{Sp}(1)$-invariant coassociative in [1].

Cohomogeneity two coassociatives

We construct examples of coassociative submanifolds in $\mathbb{R}^7$ which are invariant under the action of a maximal torus $\mathbb{T}^2 \subset \mathrm{G}_2$, which is new. These examples have cohomogeneity two.

Theorem 9.1

Let V_1, V_2, V_3, V_4 be linearly independent vector fields generated by the $\mathbb{T}^2$-action, $\alpha(s, t)$ be the surface in $\mathbb{R}^7$, and w_1, w_2, w_3 be the corresponding complex coordinates given by $w_i = \alpha_{2i} + \alpha_{2i+1}$. The vector fields V_i span a coassociative subspace if and only if

$$\boxed{\mathbf{Re}(w_1 w_2 w_3) = c,}$$

for some constant $c \in \mathbb{R}$ and

$$\boxed{\begin{aligned} &\frac{\partial \alpha_1}{\partial s}\left(\frac{\partial |w_i|^2}{\partial t} - \frac{\partial |w_3|^2}{\partial t}\right) - \frac{\partial \alpha_1}{\partial t}\left(\frac{\partial |w_i|^2}{\partial s} - \frac{\partial |w_3|^2}{\partial s}\right) \\ &- \mathbf{Im}\left(w_i\left(\frac{\partial w_i}{\partial s}\frac{\partial w_3}{\partial t} - \frac{\partial w_i}{\partial t}\frac{\partial w_3}{\partial s}\right) + w_3\left(\frac{\partial w_1}{\partial t}\frac{\partial w_2}{\partial s} - \frac{\partial w_1}{\partial s}\frac{\partial w_2}{\partial t}\right)\right) = 0 \end{aligned}}$$

for $i \neq j \in \{1, 2\}$.

We can choose arbitrary real-valued functions $\alpha_2, \alpha_4, \alpha_7$ and then α_6 is determined from α_2 and α_4, from which we can solve the above two linear PDEs for $\frac{\partial \alpha_1}{\partial s}$ and $\frac{\partial \alpha_1}{\partial t}$ (which may not always be solvable) to obtain examples of $\mathbb{T}^2$-invariant coassociatives.

References

[1] R. Harvey and H. B. Lawson Jr. Calibrated geometries. *Acta Mathematica*, 148:47–157, 1982.