Q1. Let $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ and set $n = \dim_{\mathbb{C}}(V)$. The exercises below give a proof of Proposition 26.22.

(a) Show that $\dim_{\mathbb{R}} \mathrm{End}_{\mathbb{R}G}(V_{\mathbb{R}}) = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}G}(V \oplus \overline{V})$.

(b) Assume V cannot be defined over $\mathbb{R}$. Show that $V_{\mathbb{R}}$ must be irreducible. [Hint: Go for a contradiction and consider $(V_{\mathbb{R}})_{\mathbb{C}}$.]

(c) Assume χ_V is not real-valued.

(i) Show that $\dim_{\mathbb{R}} \mathrm{End}_{\mathbb{R}G}(V_{\mathbb{R}}) = 2$ hence deduce that $\mathrm{End}_{\mathbb{R}G}(V_{\mathbb{R}}) \cong \mathbb{C}$. [Hint: Lemma 23.8. For the last bit, note that $\mathrm{End}_{\mathbb{R}G}(V_{\mathbb{R}})$ contains a copy of $\mathbb{C}$ since V is a $\mathbb{C}$-vector space.]

(ii) Explain why $V_{\mathbb{R}}$ is irreducible and show that the corresponding component in the Wedderburn decomposition of $\mathbb{R}G$ is $M_n(\mathbb{C})$.

(d) Assume χ_V is real-valued.

(i) Show that $\dim_{\mathbb{R}} \mathrm{End}_{\mathbb{R}G}(V_{\mathbb{R}}) = 4$. [Hint: Lemma 23.6.]

(ii) Assume that V can be defined over $\mathbb{R}$; say $V = W_{\mathbb{C}}$ with $W \in \mathrm{Irr}_{\mathbb{R}}(G)$. Show that $\mathrm{End}_{\mathbb{R}G}(W) = \mathbb{R}$ and that the corresponding Wedderburn component in $\mathbb{R}G$ is $M_n(\mathbb{R})$. [Hint: Consider $\mathbb{C} \otimes \mathrm{End}_{\mathbb{R}G}(W)$.]

(iii) Assume that V cannot be defined over $\mathbb{R}$. By part (b), $V_{\mathbb{R}}$ is irreducible. Show that $\mathrm{End}_{\mathbb{R}G}(V_{\mathbb{R}}) = \mathbb{H}$ and that the corresponding Wedderburn component is $M_{\frac{n}{2}}(\mathbb{H})$. [Hint: Frobenius.]

Q2. (a) Determine the Wedderburn decompositions of $\mathbb{C}A_4$, $\mathbb{C}A_5$, $\mathbb{R}A_4$ and $\mathbb{R}A_5$. [You are free to look up the character table of A_4. It appears in Q2a of Sample Test 2.]

(b) Give a list of all semisimple $\mathbb{R}$-algebras of dimension 4 (up to isomorphism). For each algebra A in your list, either find a finite group G such that $\mathbb{R}G \cong A$ or else prove that no such group exists.

(c) Same problem as (b) but over $\mathbb{C}$ and of dimension 10.

[Note: For (b) and (c) give an individual reason for why A cannot be a group ring. Do not argue along the lines of "The only groups of order N are (...), and their groups rings are (...)."]

Q3. Let $H \leq G$ and suppose that $\chi_1, \ldots, \chi_k$ are the irreducible characters of G. Let ψ be an irreducible character of H. Show that if $\mathrm{Ind}_H^G \psi = \sum_{i=1}^k d_i \chi_i$ then $\sum_{i=1}^k d_i^2 \leq (G : H)$.

Q4. Let $G = S_4$ and $H = \langle (1\ 2), (3\ 4) \rangle \cong S_2 \times S_2$. Let ψ be the trivial character of H.

(a) Calculate $\left\langle \mathrm{Ind}_H^G \psi, \chi \right\rangle$ for $\chi \in \{\chi_{\mathrm{triv}}, \chi_{\mathrm{sgn}}, \chi_{\mathrm{std}}\}$.

(b) Determine the isotypic decomposition of $\mathrm{Ind}_H^G(\mathrm{triv})$.

Q5. (Bonus) Give a "direct" (i.e. no FS indicators, etc.) argument that proves that the two-dimensional complex irrep of Q_8 cannot be defined over $\mathbb{R}$.