Q1. For $a, b \in \mathbb{R}^\times$, define the generalized quaternion algebra $\mathbb{H}_{a,b}$ to be the 4-dimensional $\mathbb{R}$-vector space with basis $1, i, j, k$ and multiplication satisfying

$$i^2 = a, \quad j^2 = b \quad \text{and} \quad ij = -ji = k.$$

Note that the Hamilton quaternion algebra is $\mathbb{H} = \mathbb{H}_{-1,-1}$.

(a) *Do not submit.* Convince yourself that the above really defines an $\mathbb{R}$-algebra. In particular, what are the products ik, ki, jk, kj? Show that $k^2 = -ab$.

(b) Show that there are isomorphisms of $\mathbb{R}$-algebras $\mathbb{H}_{a,b} \cong \mathbb{H}_{b,a}$ and $\mathbb{H}_{u^2a,v^2b} \cong \mathbb{H}_{a,b}$ for all $u, v \in \mathbb{R}^\times$. Hence deduce that $\mathbb{H}_{a,b}$ is isomorphic to one of $\mathbb{H}_{1,1}$, $\mathbb{H}_{1,-1}$ and $\mathbb{H}_{-1,-1}$.

(c) Show that $\mathbb{H}_{1,1} \cong \mathbb{H}_{1,-1} \cong M_2(\mathbb{R})$ and that $\mathbb{H}_{-1,-1} \not\cong M_2(\mathbb{R})$.

[So this "general" construction doesn't give us anything new—we either get $M_2(\mathbb{R})$ or $\mathbb{H}$! However, note that the recipe for $\mathbb{H}_{a,b}$ works over other fields, in which case it can produce interesting algebras.]

Q2. Let $R = F[x]/(f(x))$, where F is a field and $\deg f \geq 1$. Suppose $f(x) = p_1(x)^{a_1} \cdots p_k(x)^{a_k}$ is the factorization of f into distinct irreducibles $p_i(x) \in F[x]$. Set $S_i := F[x]/(p_i(x))$.

(a) Show that S_i is a simple R-module.

(b) Show, conversely, that every simple R-module is isomorphic to some S_i.

(c) Conclude that there are k distinct simple R-modules up to isomorphism, and representatives for the isomorphism classes are given by S_i for $1 \leq i \leq k$.

Q3. (a) Prove that $\mathbb{R}C_n \cong \mathbb{R}[x]/(x^n - 1)$ as rings.

(b) Hence deduce:

i. If n is odd, $\mathrm{Irr}_{\mathbb{R}}(C_n)$ consists of the trivial representation and $\frac{n-1}{2}$ two-dimensional representations.

ii. If n is even, $\mathrm{Irr}_{\mathbb{R}}(C_n)$ consists of two one-dimensional representations and $\frac{n-2}{2}$ two-dimensional representations.

Q4. Let M be an R-module and let A, B, and C be submodules of M such that $C \subseteq A$. Prove:

(a) $A \cap (B + C) = (A \cap B) + C$.

(b) If there is a submodule C' such that $M = C \oplus C'$ then there is a submodule C'' such that $A = C \oplus C''$.