Q1. (a) Determine the character table of A_5.

(b) Deduce that A_5 is a simple group.

Here are some suggestions for part (a). You will want to use the character table of S_5.

- Given $\pi \in A_n$, we know how to determine its conjugacy class C in S_n: it consists of all permutations with the same cycle type as π. There are now two scenarios:
  - The centralizers of π in S_n and A_n coincide. (Equivalently, π does not commute with any odd permutation.) In this case, C splits into the disjoint union of two conjugacy classes in A_n with representatives π and $(1\ 2)\pi(1\ 2)$.
  - The centralizers of π in S_n and A_n differ. (Equivalently, π commutes with some odd permutation.) In this case, the conjugacy class of π in S_n coincides with its conjugacy class in A_n.

  Note that the first scenario occurs if and only if the cycle decomposition of π consists of odd cycles of different lengths. This should allow you to determine the conjugacy classes of A_5. You should find that there are five. [You may use the above facts without proof on the assignment. But you should think about why they are true. The key is that A_n has index 2 in S_n.]

- Restrict the irreducible characters of S_5 to A_5. Determine which of them remain irreducible. Use the dimension formula to find the degrees of the characters that you're missing. Finally, the orthogonality relations should take you home.

Q2. [*Context:* Although we have been working only over $\mathbb{C}$ lately, a natural question you might ask is whether a given representation $\rho\colon G \to GL(V)$ can be **defined over** $\mathbb{R}$. That is, can we find a basis $\mathcal{B}$ for V with respect to which all the matrices of $[\rho(g)]_{\mathcal{B}}$ have real entries? A necessary (but not sufficient!) condition is that χ_V should be real-valued. The exercises below (and A3Q3) will help answer this question. This story will be concluded later.]

Let χ be a character of G. The **Frobenius–Schur indicator** of χ is the number

$$\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Assume V is an irreducible $\mathbb{C}G$-module and let $\chi = \chi_V$.

(a) Prove $\text{mult}(V_{\text{triv}}, V \otimes V)$ is either 1 or 0 depending on whether $V \cong V^*$ or not.

(b) Express $\varepsilon(\chi)$ in terms of the characters of $\text{Sym}^2(V)$ and $V \otimes V$.

(c) Hence show that

$$\varepsilon(\chi) = \begin{cases} \pm 1 & \text{if and only if } \chi_V \text{ is real valued} \\ 0 & \text{otherwise.} \end{cases}$$

(d) Explain briefly how the space of bilinear forms on V can be identified with $(V \otimes V)^*$. Deduce that the space of G-invariant bilinear forms on V (as defined in A3Q3) can be identified with $(V^* \otimes V^*)^G \cong \mathrm{Sym}^2(V^*)^G \oplus \mathrm{Alt}^2(V^*)^G$.

(e) Show that if there exists a non-zero G-invariant bilinear form B on V then it must belong to either $\mathrm{Sym}^2(V^*)^G$ (in which case it is a symmetric form, i.e. $B(x, y) = B(y, x)$) or else it must belong to $\mathrm{Alt}^2(V^*)^G$ (in which case it is a skew-symmetric form, i.e. $B(x, y) = -B(y, x)$).

[Hint: What is $\dim (V^* \otimes V^*)^G$?]

(f) Show that c_V from A3Q3 is the Frobenius–Schur indicator of χ_V. Hence conclude that

$$\varepsilon(\chi_V) = \begin{cases} 1 & \text{if there exists a nonzero symmetric } G\text{-invariant bilinear form on } V; \\ -1 & \text{if there exists a nonzero skew-symmetric } G\text{-invariant bilinear form on } V; \\ 0 & \text{if } V \text{ has no nonzero } G\text{-invariant bilinear forms.} \end{cases}$$

Q3. (a) For each partition $\lambda \vdash 4$, match up the Specth module S_λ with the appropriate irreducible representation $V \in \mathrm{Irr}_{\mathbb{C}}(S_4)$ that you determined in A4Q3.

(b) Let V be the standard representation of S_4. Determine the isotypic decomposition of $\mathrm{Sym}^2(V)$. Express your answer in terms of the Specht modules S_λ with $\lambda \vdash 4$.

(c) Determine the dimension of the Specht module $S_{(n-1,1)} \in \mathrm{Irr}_{\mathbb{C}}(S_n)$ by using:

i. The Hook Length formula.

ii. The Frobenius character formula.

(d) Let $V \in \mathrm{Irr}_{\mathbb{C}}(S_n)$. Prove that χ_V is integer-valued by using:

i. The Specht module construction.

ii. The Frobenius character formula.

Q4. (Bonus.) Let V be the standard representation of S_n.

(a) Show that $\mathrm{Alt}^{n-1}(V)$ is isomorphic to the alternating representation.

(b) More generally, show that for $1 \leq k \leq n-1$, $\mathrm{Alt}^k(V)$ is isomorphic to the Specht module $S_{(n-k,1,\ldots,1)}$. [This is challenging. For partial credit, you may submit only part (a).]