Some "Weberized" L^2-based methods of signal/image approximation

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ICIAR 2014, Villamoura, Portugal, October 22-24, 2014

Primary motivation for this study:

To "Weberize" L^2-based methods of approximation so that they conform as much as possible to **Weber's model of perception:**

Given a greyscale background intensity $I > 0$, the minimum change in intensity ΔI perceived by the human visual system (HVS) is

$$\frac{\Delta I}{I} = C\,, \tag{1}$$

where C is constant, or at least roughly constant over a significant range of intensities. Therefore,

A "Weberized" L^2 distance between two image functions u and v should tolerate greater/lesser differences over regions in which they assume higher/lower intensity values.

In what follows:

$X \subset \mathbb{R}^2$ or $\mathbb{Z}^2$ denotes the **base** or **pixel space** of the image functions.
$\mathbb{R}_g = [A, B] =\subset (0, \infty)$ denotes the **greyscale range space** ($A > 0$).

The L^2 distance between u and v is then given by

$$d_2(u,v) = \left[\int_X [u(x) - g(x)]^2 \, dx \right]^{1/2} . \tag{2}$$

One way to "Weberize" this metric is to insert an intensity-dependent weighting functions into the integrand, i.e.,

$$d_{2W}(u,v) = \left[\int_X g(u(x), v(x)) \, [u(x) - g(x)]^2 \, dx \right]^{1/2} , \tag{3}$$

where $g : \mathbb{R}_g \times \mathbb{R}_g \to \mathbb{R}_+ = [0, \infty)$.

In order that the d_{2W} conform to Weber's model:

$g(u,v)$ should be **decreasing** in u and v.

Possible family of weighting functions:

$$g(u,v) = [uv]^{-q} , \quad q > 0 . \tag{4}$$

Unfortunately, such functions are difficult to work with, especially when we consider one of the functions, say $v(x)$, to be an approximation of the other, e.g. a linear combination of basis elements.

A significant simplification is achieved if we consider g to be a function of only one intensity function. In particular, if we let

$$g(u,v) = g(u) = \frac{1}{u^2}\,, \tag{5}$$

then the weighted L^2 distance becomes

$$d_{2W}(u,v) = \left[\int_X \left[1 - \frac{v(x)}{u(x)}\right]^2 dx\right]^{1/2} =: \Delta\,(u,v)\,. \tag{6}$$

Of course, $\Delta(u,v)$ is not symmetric in u and v. Here, we consider u, which defines the weighting function g, to be the *reference function* so that

$\Delta(u,v)$ is the weighted L^2 error in approximating u by v.

Similarly, if we let

$$g(u, v) = g(v) = \frac{1}{v^2}\,, \tag{7}$$

we obtain

$$d_{2W}(u, v) = \left[\int_X \left[1 - \frac{u(x)}{v(x)}\right]^2 dx\right]^{1/2} =: \Delta\,(v, u)\,. \tag{8}$$

$\Delta(v, u)$ is the weighted L^2 error in approximating v by u.

Clearly, $\Delta(u, v) \neq \Delta(v, u)$ but this is not a problem:

Theorem: Let $\mathbb{R}_g = [A, B]$, with $A > 0$. Then

$$\frac{1}{B}\, d_2(u, v) \leq \left\{ \begin{array}{c} \Delta(u, v) \\ \Delta(v, u) \end{array} \right\} \leq \frac{1}{A}\, d_2(u, v)\,, \tag{9}$$

from which it follows that

$$\left[2 - \frac{B}{A}\right] \Delta(u, v) \leq \Delta(v, u) \leq \frac{B}{A}\, \Delta(u, v)\,. \tag{10}$$

Since the distance function

$$d_{2W}(u,v) = \left[\int_X \left[1 - \frac{v(x)}{u(x)}\right]^2 dx\right]^{1/2} =: \Delta\,(u,v) \tag{11}$$

involves only a ratio of greyscale intensity functions, we might expect that Weber's model of perception is accommodated. The following simple example illustrates this.

Example: Consider the constant reference image $u(x) = I$, where $I \in \mathbb{R}_g$. Now let $v(x) = I + \Delta I$, with $\Delta I > 0$, to be the constant approximation to $u(x)$, where $\Delta I = CI$ is the minimum perceived change in intensity corresponding to I, according to Weber's model in Eq. (1). The L^2 distance between u and v is

$$d_2(u,v) = K\Delta I = KCI \quad \text{where} \quad K = \left[\int_X dx\right]^{1/2} \tag{12}$$

which is DEPENDENT ON I. A simple computation shows that

$$\Delta\,(u,v) = K\frac{\Delta I}{I} = KC\,, \tag{13}$$

which is INDEPENDENT OF I. Hence Weber's model is better accommodated by $\Delta\,(u,v)$.

Best approximation in terms of $\Delta(u, v)$

Let $\{\phi_k\}_{k=1}^{\infty}$ denote a complete orthonormal basis set of $L^2(X)$. Given an $N > 0$, let

$$A_N = \text{span}\,\{\phi_1, \phi_2, \cdots \phi_N\} \subset L^2(X)\,. \tag{14}$$

Then $v \in A_N$ implies that

$$v = \sum_{k=1}^{N} c_k \phi_k\,. \tag{15}$$

Given a $u \in L^2(X)$, its best Weberized L^2 approximation in A_N is

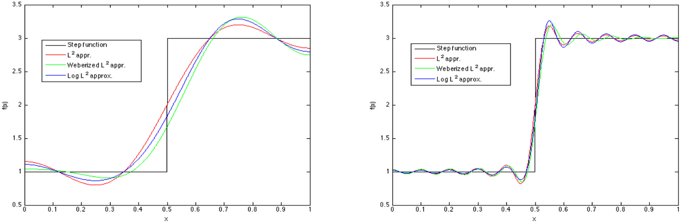
$$u_N = \arg\min_{v \in A_N} \Delta^2(u, v)\,. \tag{16}$$

This yields a linear system of equations in the unknowns c_k of the form,

$$\mathbf{Ac} = \mathbf{b}\,. \tag{17}$$

Details are presented in paper.

SOME RESULTS



Best L^2 (u_N, dotted), Weighted L^2 (u_N^W) and Logarithmic L^2 (u_N^L) approximations to step function using cosine basis functions. **Left:** $N = 5$. **Right:** $N = 20$.

Approximation errors:

N	$\|u - u_N\|_2$	$\|u - u_N^W\|_2$	$\|u - u_N^L\|_2$
5	0.315	0.399	0.345
20	0.142	0.194	0.156

Weberized methods tolerate greater error at higher intensity values.

Best L^2 (left), Weberized L^2 (center) and Logarithmic L^2 (right) approximations to *Lena* image using $N = 66$ 2D DCT basis functions over 32×32-pixel blocks comprising the shoulder region of *Lena* image.

Logarithmic L^2 metric

Since distance functions involving ratios of intensity functions appear to accommodate Weber's model of perception, what about **logarithmic L^2 distance**, i.e.,

$$\begin{aligned} d_{log}(u,v) &= \left[\int_X [\log u(x) - \log v(x)]^2 \, dx\right]^{1/2} \\ &= \left[\int_X \left[\log \frac{u(x)}{v(x)}\right]^2 dx\right]^{1/2} = \left[\int_X \left[\log \frac{v(x)}{u(x)}\right]^2 dx\right]^{1/2} ? \end{aligned} \tag{18}$$

The choice of the logarithmic L^2 distance as a Weberized L^2 distance can be justified mathematically, following some earlier work by Forte and ERV (1995):

Distance functions involving measures over greyscale space

Consider a measure ν defined over the greyscale space $\mathbb{R}_g$. Then define the following intensity-weighted distance between two image functions u and v:

$$D(u, v : \nu) = \int_{X_u} \nu(u(x), v(x)]\, dx + \int_{X_v} \nu(v(x), u(x)]\, dx\,, \tag{19}$$

where

$$X_u = \{x \in X \mid u(x) < v(x)\} \subset X \quad X_v = \{x \in X \mid u(x) \geq v(x)\} \subset X\,. \tag{20}$$

This distance involves an integration of measures of the greyscale intervals $(u(x), v(x)]$ or $(v(x), u(x)]$ over X.

In the special case $\nu = m_g$, the uniform Lebesgue measure on $\mathbb{R}_g$, $D(u, v; m_g)$ is the L^1 distance between u and v (Forte and ERV 1995).

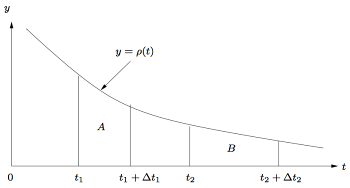
Theorem: The unique measure ν on $\mathbb{R}_g$ which accommodates Weber's model of perception over the greyscale space $\mathbb{R}_g \subset \mathbb{R}$ is (up to a normalization constant) defined by the continuous density function $\rho(t) = \frac{1}{t}$.

Idea behind proof: For any two greyscale intensities $I_1, I_2 \in \mathbb{R}_g$,

$$\int_{t_1}^{t_1+\Delta t_1} \frac{1}{t}\, dt = \int_{t_2}^{t_2+\Delta t_2} \frac{1}{t}\, dt \quad \Longrightarrow \quad \nu[I_1, I_1 + \Delta I_1] = \nu[I_2, I_2 + \Delta I_2]\,.$$

Area A Area B

where $\Delta I_1 = CI_1$ and $\Delta I_2 = CI_2$ are the minimum changes in perceived intensity at I_1 and I_2, respectively, according to Weber's model.



This is a kind of invariance result with respect to perception.

Using the measure ν with density $\rho(t) = \dfrac{1}{t}$, the distance between u and v becomes

$$\begin{aligned} D(u, v; \nu) &= \int_{X_u} \left[\int_{u(x)}^{v(x)} \frac{1}{t}\, dt \right] dx + \int_{X_v} \left[\int_{v(x)}^{u(x)} \frac{1}{t}\, dt \right] dx \\ &= \int_X |\ \ln u(x) - \ln v(x)\, |\, dx\,, \end{aligned} \tag{21}$$

the logarithmic L^1 distance between u and v. All other logarithmic L^p distances, $p > 1$, including the logarithmic L^2 distance in Eq. (18) may be viewed as extensions of this result.

Best approximation in terms of logarithmic L^2 metric

Given a $u(x)$, we seek to approximate it as a linear combination of orthonormal basis functions $\{\phi_1, \phi_2, \cdots, \phi_N\}$.

Minimization of $d_{\log}(u, u_N)$ in Eq. (18) is a very complicated nonlinear problem.

A huge simplification is accomplished if we consider L^2 approximations of the logarithmic function, i.e.,

$$U(x) = \log u(x)\,. \tag{22}$$

Then

$$U \cong U_N = \sum_{k=1}^{N} a_k \phi_k\,, \tag{23}$$

where

$$a_k = \langle U, \phi_k \rangle = \int_X U(x)\phi_k(x)\,dx\,. \tag{24}$$

The resulting logarithmic L^2-based approximations to u are then given by

$$u_N^L(x) = \exp(U_N(x)) = exp\left(\sum_{k=1}^{N} a_k \phi_k(x))\right)\,. \tag{25}$$