The use of intensity-based measures to produce image function metrics (which accommodate Weber's models of perception)

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Primary motivation: perceptual image quality measures

This study represents ongoing work on the development of nonstandard distance functions/metrics between (image) functions. One of the principal motivations for this work is the important problem of objectively measuring **perceptual image quality**.

Example: Traditional best L^2 approximation of a function u in terms of a set of N basis functions.

$$\text{Minimize} \quad \int_a^b \left[u(x) - \sum_{k=1}^{N} c_k \phi_k(x) \right]^2 dx \,. \tag{1}$$

For reasons that may become clearer later in this talk, we may wish to solve the following approximation problem,

$$\text{Minimize} \quad \int_a^b \left[\sqrt{u(x)} - \sqrt{\sum_{k=1}^{N} c_k \phi_k(x)} \right]^2 dx \,. \tag{2}$$

Primary motivation: perceptual image quality measures

It is well known that L^2-based metrics perform poorly in terms of perceptual quality. Much work has been done to develop better image quality measures, most notably, the **Structural Similarity** (SSIM) measure,

> *Z. Wang, A.C. Bovik, H.R. Sheikh and E.P. Simoncelli, Image quality assessment: From error visibility to structural similarity, IEEE Trans. Image Proc.* **13** *(4), 600-612 (2004).*

SSIM generally performs better for two principal reasons:

- One of its three (multiplicative) components is the **correlation** between two image blocks/patches. As such, correlation – extremely important in visual perception – plays a more direct role in SSIM than in the L^2 distance.
- The algebraic forms of the terms composing SSIM – ratios – are designed to accommodate Weber's law/model of perception.

The SSIM index is a **similarity measure**, i.e., the SSIM between two image patches $\mathbf{x}$ and $\mathbf{y}$ behaves as follows,

$$-1 \leq S(\mathbf{x}, \mathbf{y}) \leq 1\,. \tag{3}$$

This suggests that the function

$$T(\mathbf{x}, \mathbf{y}) = 1 - S(\mathbf{x}, \mathbf{y}) \implies 0 \leq T(\mathbf{x}, \mathbf{y}) \leq 2\,, \tag{4}$$

could be related to a distance function. In fact, in the case of zero-mean patches, i.e., $\bar{\mathbf{x}} = \bar{\mathbf{y}} = 0$,

$$T(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x} - \mathbf{y}\|_2^2}{\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2}\,. \tag{5}$$

$T(\mathbf{x}, \mathbf{y})$ is a **normalized** metric. It may also be viewed as an **intensity-weighted** metric.

In this work, however, we are not concerned with SSIM.

Other efforts to incorporate Weber's model of perception in image processing methods

There have been other efforts to incorporate Weber's model of perception in image processing, e.g.,

> *J. Shen, On the foundations of vision modeling I. Weber's law and Weberized TV restoration, Physica D* **175**, *241-251 (2003).*

The basic idea is to "Weberize" the method by dividing by the total variation (TV) $\|\nabla u\|$ by u to produce a modified TV, $\dfrac{\|\nabla u\|}{u}$, which assigns lower/higher weight in regions of higher/lower image intensity u. The TV is now **intensity-dependent**.

The term "Weberize" comes from Weber's law/model of perception:

$$\frac{\Delta I}{I} = C \,, \tag{6}$$

where

- $I > 0$: greyscale background intensity,
- ΔI: minimum change in intensity perceived by human visual system (HVS),
- C: constant, or at least roughly constant, over a significant range of intensities $I > 0$.

In other words, HVS is less/more sensitive to given change in intensity $\Delta I > 0$ in regions of an image at which local intensity $I(x)$ is high/low.

Other efforts to incorporate Weber's model of perception in image processing methods

A "Weberized" method, therefore, should tolerate greater/lesser differences between two functions u and v over regions in which they assume higher/lower intensity values.

This idea motivated our work presented at ICIAR 14:

> *I.A. Kowalik-Urbaniak, D. La Torre, E.R. Vrscay and Z. Wang, Some "Weberized" L^2-based methods of signal/image approximation,* Image Analysis and Recognition, *LNCS* **8814**, *20-29 (2014).*

There, we considered the "Weberization" of L^2 distance between two functions u and v – or any metric involving an integration over some power of $|u(x) - v(x)|$ by introducing an **intensity-based weight function** into the integration.

Other efforts to incorporate Weber's model of perception in image processing methods

One way to "Weberize" the L^2 distance

Consider the usual L^2 distance between two image functions,

$$d_2(u, v) = \left[\int_a^b [u(x) - v(x)]^2 \, dx\right]^{1/2}. \quad (7)$$

If we consider $u(x)$ to be a reference function and $v(x)$ its approximation, introduce $\frac{1}{u(x)^2}$ as intensity-dependent weighting function,

$$\begin{aligned} \Delta(u, v) &= \left[\int_a^b \frac{1}{u(x)^2}[u(x) - v(x)]^2 \, dx\right]^{1/2} \\ &= \left[\int_a^b \left[1 - \frac{v(x)}{u(x)}\right]^2 dx\right]^{1/2}. \end{aligned} \quad (8)$$

If we let

$$v_N(x) = \sum_{k=1}^{N} c_k \phi_k(x) \quad (9)$$

be an approximation to $u(x)$, then minimization of $\Delta(u, v_N)$ yields linear system of equations in c_k, $1 \leq k \leq N$.

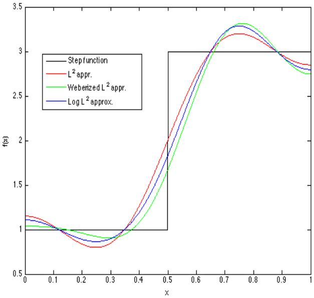
Other efforts to incorporate Weber's model of perception in image processing methods

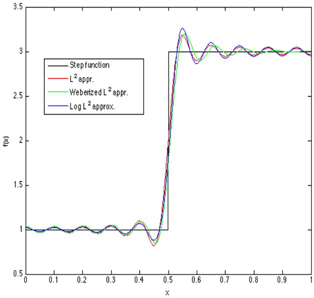
Another way to "Weberize" the L^2 distance: "Logarithmic L^2 distance"

$$
\begin{aligned}
d_{\log}(u, v) &= d_2(\log u, \log v) \\
&= \left[\int_a^b [\log u(x) - \log v(x)]^2 \, dx \right]^{1/2} \\
&= \left[\int_a^b \left[\log \frac{v(x)}{u(x)} \right] dx \right]^{1/2} . \qquad (10)
\end{aligned}
$$

Other efforts to incorporate Weber's model of perception in image processing methods

Example: Step function on [0,1]





Signal/image function metrics generated by greyscale range measures

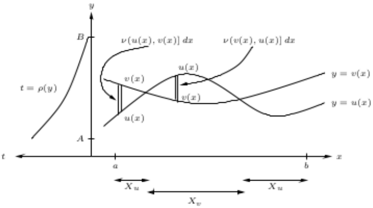
These methods may be viewed as rather *ad hoc*. A more general, and mathematically-based method of defining intensity-dependent metrics between image functions is made possible considering **measures on the greyscale range space** $\mathbb{R}_g$.

Mathematical ingredients:

1. **Base (or pixel) space:** $X \subset \mathbb{R}$. In what follows, we let $X = [a, b]$.
2. **Greyscale range space:** $\mathbb{R}_g = [A, B] \subset (0, \infty)$. Note that this implies that all image functions are **positive-valued**.
3. **Signal/image function space:** $\mathcal{F} = \{u : X \to \mathbb{R}_g \mid u \text{ is measurable}\}$.
4. **Greyscale range measure space:** $\mathcal{M}_g$, set of probability measures on $\mathbb{R}_g$.

Goal: To assign a distance between u and v based on an integration over vertical strips of width dx centered at $x \in [a, b]$.

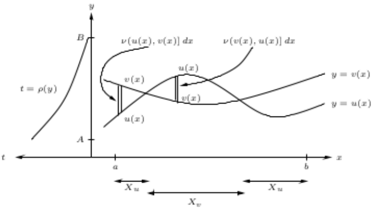
Generic situation:



In most, if not all, traditional integration-based metrics, e.g., the $L^p(X)$ metrics for $p \geq 1$, the contribution of each strip to the integral will be an appropriate power of the height of the strip, $|u(x) - v(x)|$, i.e.,

$$d_p(u, v) = \left[\int_X |u(x) - v(x)|^p \, dx \right]^{1/p} . \qquad (11)$$

This implicitly assumes a **uniform** weighting over intensity axis $\mathbb{R}_g$ since the term $|u(x) - v(x)|$ represents the Lebesgue measure of the strips.



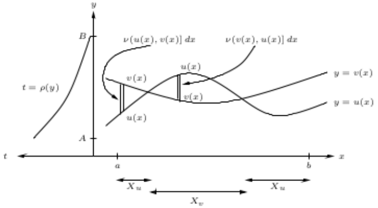
We now wish to use measures which are **nonuniform** over $\mathbb{R}_g$. (If we keep Weber's law/model in mind, then these measures will assign lesser weight at higher intensities.)

Let ν be a measure that is supported on $\mathbb{R}_g = [A, B]$. ν will now be used to define the lengths of the vertical strips as follows:

- strip at left: $\nu(u(x), v(x)]$ since $u(x) < v(x)$.
- strip at right: $\nu(v(x), u(x)]$ since $v(x) < u(x)$.

Now integrate over all strips centered at $x \in X$.

We can express this compactly as follows.



Define the following subsets of $X = [a, b]$,

$$X_u = \{x \in X \mid u(x) \leq v(x)\} \quad X_v = \{x \in X \mid v(x) \leq u(x)\} \tag{12}$$

so that $X = X_u \cup X_v$. The distance between u and v associated with the measure ν is then defined as follows,

$$D(u, v; \nu) = \int_{X_u} \nu(u(x), v(x)]\, dx + \int_{X_v} \nu(v(x), u(x)]\, dx\,. \tag{13}$$

$$D(u, v; \nu) = \int_{X_u} \nu(u(x), v(x)]\, dx + \int_{X_v} \nu(v(x), u(x)]\, dx\,. \tag{14}$$

Special case: $\nu = m_g$, the usual (uniform) Lebesgue measure on $\mathbb{R}_g$, Then for $a < b$,

$$m_g(a, b] = b - a. \tag{15}$$

The sizes of the intervals shown in the above figure become

- strip at left: $\nu(u(x), v(x)] = m_g(u(x), v(x)] = v(x) - u(x)$,
- strip at right: $\nu(v(x), u(x)] = m_g(v(x), u(x)] = u(x) - v(x)$,

so that

$$\begin{aligned} D(u, v; m_g) &= \int_{X_u} [v(x) - u(x)]\, dx + \int_{X_v} [u(x) - v(x)]\, dx \\ &= \int_X |u(x) - v(x)|\, dx\,, \end{aligned} \tag{16}$$

the L^1 distance between u and v.

The natural question is, "What other kind of greyscale measures can/should be considered on $\mathbb{R}_g$?"

For convenience, we consider measures $\nu \in \mathcal{M}_g$ which are defined by continuous, non-negative density functions $\rho(y)$. (This implies that ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_g$.) Given a measure $\nu \in \mathcal{M}_g$ with density function ρ, then for any interval $(y_1, y_2] \subset \mathbb{R}_g$,

$$\nu(y_1, y_2) = \int_{y_1}^{y_2} \rho(y)\, dy = P(y_2) - P(y_1)\,, \tag{17}$$

where $P'(y) = \rho(y)$.

The distance function $D(u, v; \nu)$ becomes

$$\begin{aligned} D(u, v; \nu) &= \int_{X_u} \nu(u(x), v(x)]\, dx + \int_{X_v} \nu(v(x), u(x)]\, dx \\ &= \int_{X_u} [P(u(x)) - P(v(x))]\, dx + \int_{X_v} [P(v(x)) - P(u(x))]\, dx \\ &= \int_X |P(u(x)) - P(u(v))|\, dx\,. \end{aligned} \tag{18}$$

Special case: The density function

$$\rho(y) = \frac{1}{y}, \quad y > 0. \tag{19}$$

In this case, $P(y) = \ln y$ so that distance function becomes

$$D(u, v; \nu) = \int_X |\ln u(x) - \ln v(x)|\, dx = \|\ln u - \ln v\|_1 \quad \left(= \left\| \ln\left(\frac{u}{v}\right) \right\|_1 \right). \tag{20}$$

We now show that the measure associated with this density function accommodates Weber's standard model of perception.

From Kowalik-Urbaniak *et al.* (ICIAR 14):

Let $I_1, I_2 \in \mathbb{R}_g$ be any two greyscale intensities. From Weber's model, minimum changes in perceived intensity, ΔI_1 and ΔI_2, at I_1 and I_2, respectively, are given by

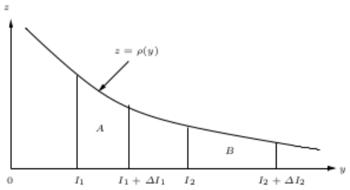
$$\frac{\Delta I_1}{I_1} = \frac{\Delta I_2}{I_2} = C \quad \Longrightarrow \quad \Delta I_1 = CI_1 \, , \Delta_2 = CI_2 \, . \tag{21}$$

A simple calculation shows that

$$\int_{I_1}^{I_1 + \Delta I_1} \frac{1}{y} \, dy = \int_{I_2}^{I_2 + \Delta I_2} \frac{1}{y} \, dy \quad \Longrightarrow \quad \nu(I_1, I_1 + \Delta I_1) = \nu(I_2, I_2 + \Delta I_2) \, . \tag{22}$$

This may be viewed as an invariance result with respect to perception.

Graphical interpretation in terms of equal areas enclosed by density curve $\rho(y) = \dfrac{1}{y}$:



Just to recall: The greyscale range density function $\rho(y) = \dfrac{1}{y}$ decreases with y, i.e., it assigns lesser weight to the distance integral at higher intensity values.

Generalized Weber models of perception

Given a greyscale background intensity $I > 0$, the minimum change in intensity ΔI perceived by the HVS is given by

$$\frac{\Delta I}{I^a} = C\,, \tag{23}$$

where $a > 0$ and C is constant, or at least roughly constant, over a significant range of intensities.

- The case $a = 1$ corresponds to the standard Weber model.
- There are situations in which other values of a, in particular, $a = \frac{1}{2}$ may apply

J.A. Michon, Note on the generalized form of Weber's Law, Perception and Psychophysics **1***, 129-132 (1966).*

Density functions associated with generalized Weber models

Generalized Weber model, $a > 0$,

$$\frac{\Delta I}{I^a} = C \quad \Longrightarrow \quad \Delta I = CI^a \,. \tag{24}$$

ΔI is minimum change in perceived intensity value at I.

In this paper, we show that for $a > 0$, the density function

$$\rho_a(y) = \frac{1}{y^a} \,, \quad y > 0 \,, \tag{25}$$

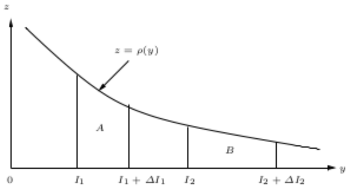
accommodates Weber's generalized model of perception in the following way:

Given two greyscale intensities, $I_1, I_2 \in \mathbb{R}_g$, with associated minimum changes in perceived intensities, $\Delta I_1 = CI_1^a$ and $\Delta I_2 = CI_2^a$, respectively, then to leading order,

$$\int_{I_1}^{I_1+\Delta I_1} \frac{1}{y^a}\, dy \approx \int_{I_2}^{I_2+\Delta I_2} \frac{1}{y^a}\, dy \quad \Longrightarrow \quad \nu(I_1, I_1 + \Delta I_1) \approx \nu(I_2, I_2 + \Delta I_2) \,. \tag{26}$$

Density functions associated with generalized Weber models

Graphical interpretation in terms of equal areas enclosed by density curve $\rho_a(y) = \dfrac{1}{y^a}$:



Once again: For $a > 0$, the greyscale range density function $\rho_a(y) = \dfrac{1}{y^a}$ decreases with y, i.e., it assigns lesser weight to the distance integral at higher intensity values.

Density functions associated with generalized Weber models

The above result may also be extended to include the special case $a = 0$, i.e.,

$$\frac{\Delta I}{I^0} = C \quad \Longrightarrow \quad \Delta I = C\,, \tag{27}$$

essentially an absence of Weber's model. In this case,

$$\rho_0(y) = \frac{1}{y^0} = 1\,, \tag{28}$$

which corresponds to **uniform Lebesgue measure** m_g.

Distance functions associated with generalized Weber model density functions

For $a \geq 0$, $a \neq 1$,

$$\rho_a(y) = \frac{1}{y^a} \quad \Longrightarrow \quad P(y) = \frac{1}{-a+1} y^{-a+1} . \tag{29}$$

Associated distance functions, up to a multiplicative constant, are given by

$$D_a(u, v) = D(u, v; \nu_a) = \int_X \left| u(x)^{-a+1} - v(x)^{-a+1} \right| \, dx . \tag{30}$$

For $a = 1$ (Weber's standard model),

$$\rho_1(y) = \frac{1}{y} \quad \Longrightarrow \quad P(y) = \ln y . \tag{31}$$

Associated distance function, up to a multiplicative constant,

$$D_1(u, v) = D(u, v; \nu_a) = \int_X |\ln u(x) - \ln v(x)| \, dx , \tag{32}$$

discussed earlier.

Function approximation using generalized greyscale measure ν_a

Let $u(x)$ denote a reference function and $v(x)$ an approximation to $u(x)$ having standard form,

$$v(x) = \sum_{k=1}^{N} c_k \phi_k(x) \,, \tag{33}$$

where the set $\{\phi_k\}_{k=1}^{N}$ is assumed to be linearly independent, and perhaps orthogonal, over $X = [a, b]$. Let $Y_N = \text{span}\{\phi_1, \phi_2, \cdots, \phi_N\}$.

Best Y_N-approximation of $u \in \mathcal{F}$ in the metric space $(\mathcal{F}, D_a)$ is found by minimizing the distance $D(u, v; \nu_a)$.

Unfortunately, it is difficult to work with these distance functions, especially because of the appearance of the absolute value in integrand.

It is easier to work with their L^2 analogues:

Function approximation using L^2 analogues of distance functions associated with generalized Weber measures ν_a

For the case $a \geq 0$, $a \neq 1$:

$$D_{2,a}(u, v; \nu_a) = \left[\int_X \left[u(x)^{-a+1} - v(x)^{-a+1} \right]^2 \, dx \right]^{1/2} . \tag{34}$$

And for the case $a = 1$:

$$D_{2,1}(u, v; \nu_a) = \left[\int_X [\ln u(x) - \ln v(x)]^2 \, dx \right]^{1/2} . \tag{35}$$

Special case: $a = \dfrac{1}{2}$

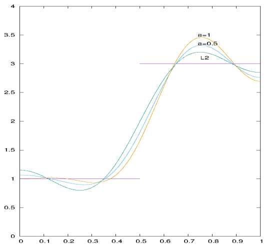
$$D_{2,1/2}(u, v; \nu_a) = \left[\int_X \left[\sqrt{u(x)} - \sqrt{v(x)} \right]^2 \, dx \right]^{1/2} . \tag{36}$$

(See, I told you!)

In all cases, gradients can be computed $\Longrightarrow$ gradient descent methods can be used.

Function approximation using L^2 analogues of distance functions associated with generalized Weber measures ν_a

Example: Return to step function on [0,1]



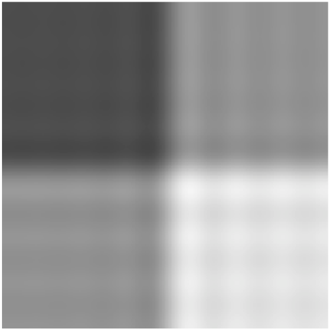
Best approximations to step function $u(x)$ using $N = 5$ orthogonal cosine functions on [0,1] using three greyscale measures ν_a: (i) $a = 0$ (best L^2, green), (ii) $a = 0.5$ (blue) and (iii) $a = 1$ (standard Weber, yellow).

Note that as a increases, the density function $y_a = \dfrac{1}{y^a}$ assigns less and less weight to higher intensity regions. As a result, the approximations are poorer as a increases at higher intensity regions, better as a increases at lower intensity regions.

Function approximation using L^2 analogues of distance functions associated with generalized Weber measures ν_a

Example: 2D step function on $[0,1]^2$

256×256-pixel 8 bpp image composed of four squares with greyscale values 60, 128, 128 and 220.



Left and right, respectively: Best approximations for $a = 0$ (best L^2) and $a = 1$ (standard Weber) obtained with 2D DCT basis set $\Phi_{kl}(n,m) = \phi_k(n)\phi_l(m)$, $0 \leq k, l \leq 14$. The $a = 1$ approximation exhibits greater deviation at higher greyscale levels than the L^2 approximation.