Iterated functions with place-dependent probabilities and the inverse problem of measure approximation using moments

Erik Maki[1], Davide La Torre[2], Franklin Mendivil[3] and **Edward R. Vrscay**[1]

[1]Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario Canada.
[2]Department of Economics, University of Milan and Department of Mathematics, Nazarbaev University, Astana, Kazakhstan.
[3]Department of Mathematics and Statistics, Acadia University, Wolfville, Nova Scotia, Canada.

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“Waterloo Fractal Analysis and Coding Project”

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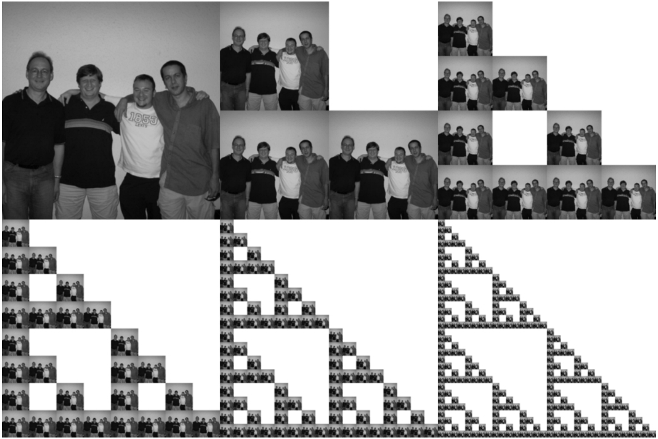

ERV, Herb Kunze (Guelph), Davide La Torre (Milan & Abu Dhabi), Franklin Mendivil (Acadia), ERV, Herb Kunze (Guelph), Davide La Torre (Milan & Abu Dhabi), Franklin Mendivil (Acadia), ERV, Herb Kunze (Guelph), Davide La Torre (Milan & Abu Dhabi), Franklin Mendivil (Acadia), ...

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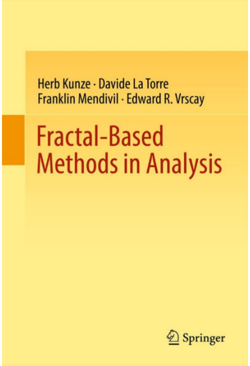


In memoriam:

Bruno Forte
Department of Applied Mathematics, UW (to 1993)
Universita Degli Studi de Verona, Verona, Italia (1993-2002)

Iterated Function Systems, or "Map Bags" (B. Mandelbrot)

A collection of contraction maps that operate in a parallel fashion.

This idea - in some way, shape or form - was around for quite some time, e.g.

- R.F. Williams, Composition of contractions, Bol. Soc. Brasil Mat. **2**, 55-59 (1971). Fixed points of finite compositions of contraction maps.
- S. Nadler, Multi-valued contraction mappings, Pacific J. Math. **30**, 475-488 (1969). Systems of contraction maps considered as defining "multifunctions."
- S. Karlin, Some random walks arising in learning models, I, Pacific J. Math. **3**, 725-756 (1953). Random walks over Cantor-like sets on $[0, 1]$ and associated measures - essentially the "Chaos Game" of Barnsley-Demko.

Two seminal works:

- J. Hutchinson, Fractals and self-similarity, Indiana Univ. J. Math. **30**, 713-747 (1981). Geometric and measure theoretic aspects of systems of contractive maps with associated probabilities, incl. invariant sets and probability measures supported on these sets.
- M.F. Barnsley and S.G. Demko, Iterated function systems and the global construction of fractals, Proc. Roy. Soc. London A **399**, 243-275 (1988). An independent discovery of such systems of mappings and associated attractors and invariant measures, but in a more probabilistic setting, i.e., random process. Perhaps the first solution of an “inverse problem” of fractal construction – and they tried to “match moments”.

And then:

- M.F. Barnsley, V. Ervin, D. Hardin and J. Lancaster, Solution of an inverse problem for fractals and other sets, Proc. Nat. Acad. Sci. USA **83**, 1975-1977 (1985). The use of the “Collage Theorem” to address the inverse problem.
- A. Jacquin, Image coding based on a fractal theory of iterated contractive image transformations, IEEE Trans. Image Proc. **1**, 18-30 (1992). Perhaps the first paper on fractal image coding.

Back to Iterated Function Systems (IFS):

Ingredients:

- (X, d): A complete metric space (e.g., $[0, 1]^n$ with Euclidean metric)
- $(\mathcal{H}(X), h)$: Complete metric space of non-empty compact subsets of X with Hausdorff metric h
- $w_i : X \to X$, $1 \leq i \leq N$: Set of contraction maps on X with contraction factors $c_i \in [0, 1)$.

Associated with each w_i is a set-valued mapping $\hat{w}_i : \mathcal{H}(X) \to \mathcal{H}(X)$, where

$$\hat{w}_i(S) = \{w_i(x),\, x \in S\} \ \ \forall\, S \in \mathcal{H}(X)\,. \tag{1}$$

IFS operator $\hat{\mathbf{w}}$ associated with N-map IFS $\mathbf{w}$ defined as follows:

$$\hat{\mathbf{w}}(S) = \bigcup_{i=1} \hat{w}_i(S)\,, \quad S \in \mathcal{H}(X)\,. \tag{2}$$

Theorem (Hutchinson): $\hat{\mathbf{w}} : \mathcal{H}(X) \to \mathcal{H}(X)$ is contractive,

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \leq c\, h(A, B) \ \ \forall\, A, B \in \mathcal{H}(X)\,, \tag{3}$$

where

$$c = \max_{1 \leq i \leq N} c_i < 1\,. \tag{4}$$

Important consequence:

From Banach's Fixed Point Theorem, there exists a unique compact set $A \in \mathcal{H}(X)$ which is the fixed point of $\hat{\mathbf{w}}$, i.e., $\hat{\mathbf{w}}(A) = A$, i.e.,

$$A = \bigcup_{i=1} A_i \quad \text{where} \quad \hat{w}_i(A)\,, 1 \leq i \leq N. \tag{5}$$

A is "self-similar," i.e., a union of contracted copies of itself.

Furthermore: For any $S_0 \in \mathcal{H}(X)$, define the iteration sequence

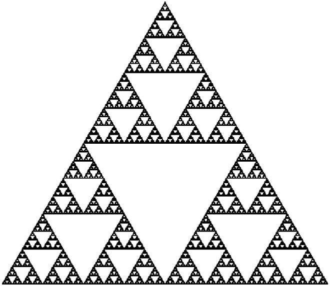
$$S_{n+1} = \hat{w}(S_n) = \bigcup_{i=1}^{N} \hat{w}_i(S_n)\,. \tag{6}$$

Then

$$\lim_{n \to \infty} h(S_n, A) = 0\,. \tag{7}$$

A is the unique (global) *attractor* of the IFS $\hat{\mathbf{w}}$.

Celebrated example: “Sierpinski triangle (gasket)”



Attractor of a 3-map affine IFS in $\mathbb{R}^2$: $w_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$,
$w_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$, $w_3(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)$.

Another celebrated example: Barnsley's spleenwort fern

Attractor of a 4-map affine IFS in $\mathbb{R}^2$.

Of course, this leads to the question, "Can we use IFS to generate other interesting sets? Plants? Trees? Faces? ... Anything?"

This is an inverse problem

First thoughts on how to solve such an inverse problem

Suppose we have a (bounded) set $S \subset \mathbb{R}^2$, for example, another leaf-like set: Do we just start playing around with (affine) contraction maps in the plane, perturbing them, generating attractors, etc.?

Perhaps a more clever approach: We're trying to approximate S by the attractor A of an N-map IFS, i.e.,

$$S \approx A = \bigcup_{i=1}^{N} \hat{w}_i(A) = \hat{\mathbf{w}}(A) \,. \tag{8}$$

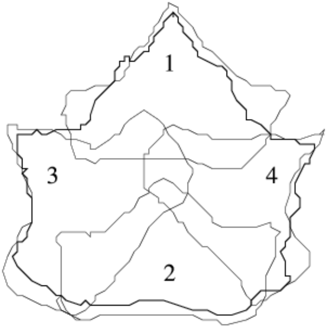
If $S \approx A$, then $\hat{w}_i(A) \approx \hat{w}_i(S)$, which implies that

$$S \approx \bigcup_{i=1}^{N} \hat{w}_i(S) = \hat{\mathbf{w}}(S) \,. \tag{9}$$

In other words: **We try to approximate S as a union of contracted copies of itself.**

Presumably, the closer S is to $\hat{\mathbf{w}}(S)$, a union , or "collage," of contracted copies of itself, the closer it is to the attractor A of the IFS $\hat{\mathbf{w}}$, i.e., the better it is approximated by A. Maybe we can express this more mathematically ... a little later.

Inverse problem by “collaging”



Left: Approximating a leaf S (darker boundary) with four contracted copies $\hat{w}_i(S)$, $1 \leq i \leq 1$ of itself. **Right:** The attractor of the resulting IFS $\hat{\mathbf{w}}$.

Banach's Fixed Point Theorem*

(Contraction Mapping Theorem ...)

is central to most (almost all?) fractal-based methods:

Let $Y, d_Y)$ be a complete metric space and $T : Y \to Y$ a contraction mapping, i.e., $d_Y(Ty_1, Ty_2) \leq c d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$ where $c \in [0, 1)$. Then there exists a unique $\bar{y} \in Y$ such that

- $T\bar{y} = \bar{y}$ (fixed point of T)
- $d_Y(T^n y_0, \bar{y}) \to 0$ as $n \to \infty$ (attractive fixed point)

*S. Banach, Sur les opérations dans les ensembles abstraites et leurs applications aux équations integrales, Fund. Math. **3** 133-181 (1922). The CMT is in an appendix to this paper, which is based on Banach's Ph.D. thesis.

Inverse problem of approximation by fixed points of contraction mappings

Let (Y, d_Y) be a complete metric space and $Con(Y)$ the set of all contraction maps $T : X \to X$. Now let $Con'(Y) \subset Con(Y)$ be a particular class of contraction maps that we wish to consider. (For example, in $\mathbb{R}^2$, the set of all N-map affine IFS, $N = 1, 2, \cdots$).

Then given a $y \in Y$ (our "target") and an an $\epsilon > 0$, can we find a $T \in Con'(Y)$ with fixed point $\bar{y}$ such that

$$d_Y(y, \bar{y}) < \epsilon \, ? \tag{10}$$

In other words, can we approximate y with the fixed point $\bar{y}$ to ϵ-accuracy?

In general, especially for fractal transforms, this problem is intractable. The following "collaging" result simplifies the problem.

The "Collage Theorem"

(That's what it's called in the fractal coding literature.)

Theorem: Let (Y, d_Y) be a complete metric space and T a contraction map on Y with contraction factor $c_T \in [0, 1)$ and fixed point $\bar{y}$. Then for any $y \in Y$,

$$d_Y(y, \bar{y}) \le \frac{1}{1 - c_T} d_Y(Ty, y) \,. \tag{11}$$

$$\left[\begin{array}{c} \text{Error in approximating} \\ y \text{ with } \bar{y} \end{array} \right] \le K(T) \left[\begin{array}{c} \text{“Collage error” in} \\ \text{approximating } Ty \text{ with } y \end{array} \right] \tag{12}$$

"Collage coding:" Try to make $d_Y(y, \bar{y})$ by finding a T that makes the collage error $d_y(y, Ty)$ as small as possible. Or rephrase as: Given a $y \in Y$ and a $\delta > 0$, find T so that

$$d_Y(Ty, y) < \delta \,. \tag{13}$$

Note: T does NOT have to be a fractal-type operator. More on this later.

The "Collage Theorem" was proved in an IFS setting in

- M.F. Barnsley, V. Ervin, D. Hardin and J. Lancaster, Solution of an inverse problem for fractals and other sets, Proc. Nat. Acad. Sci. USA **83**, 1975-1977 (1985).

It is presented as a Remark to Banach's Theorem in

- D. Smart, *Fixed Point Theorems*, Cambridge University Press, London (1974).

Simple proof: Just play around – in the right way – with y, $\bar{y}$ and Ty, using – what else? – the triangle inequality:

$$\begin{aligned} d_Y(y,\bar{y}) &\leq d_Y(y,Ty) + d_Y(Ty,\bar{y}) \\ &= d_Y(y,Ty) + d_Y(Ty,T\bar{y}) \\ &\leq d_Y(y,Ty) + c_T d_Y(y,\bar{y}), \end{aligned} \tag{14}$$

and the desired result follows, i.e.,

$$d_Y(y,\bar{y}) \leq \frac{1}{1-c_T}\, d_Y(Ty,y). \tag{15}$$

But, like, what if you play with y, Ty and $\bar{y}$ in the "wrong way", i.e., start with y and Ty:

$$\begin{aligned} d_Y(y, Ty) &\leq d_Y(y, \bar{y}) + d_Y(\bar{y}, Ty) \\ &= d_Y(y, \bar{y}) + d_Y(T\bar{y}, Ty) \\ &\leq d_Y(y, \bar{y}) + c_T d_Y(\bar{y}, y)\,, \end{aligned} \tag{16}$$

which yields

The "Anti-Collage Theorem"

$$d_Y(y, \bar{y}) \geq \frac{1}{1 + c_T} d_Y(Ty, y)\,. \tag{17}$$

Net result:

$$\frac{1}{1 + c_T} d_Y(Ty, y) \leq d_Y(y, \bar{y}) \leq \frac{1}{1 - c_T} d_Y(Ty, y)\,. \tag{18}$$