Continuous evolution of fractal transforms and nonlocal PDE imaging

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ICIAR 2006 – September 18-20, 2006

Research supported in part by Natural Sciences and Engineering Research Council of Canada (NSERC)

1. The mathematical setting:

In what follows, X will denote a closed and bounded subset of $\mathbf{R}^n$, $n = 1, 2, \cdots$, with d the Euclidean metric on X. Let $B(X)$ denote a Banach space of functions defined on X.

Eventually, X will represent the "canvas" or "computer screen" and $u \in B(X)$ will be an "image function", e.g.,

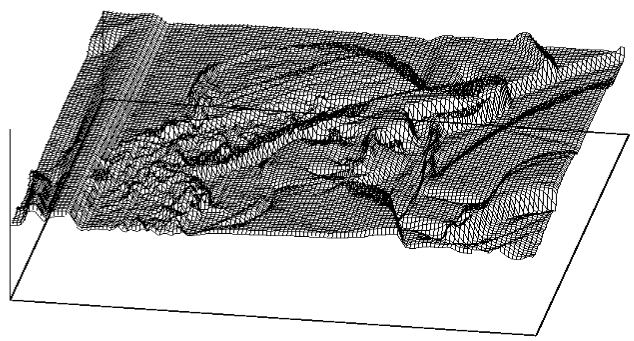


Image function $u(x, y)$ associated with 8-bit (256 level) "Lena" image.

Now suppose that $T : B(X) \to B(X)$ is a contraction mapping, i.e., for all $u, v, \in B(X)$,

$$\| Tu - Tv \| < c_T \| u - v \|, \quad (1)$$

for some $c_T \in [0, 1)$.

Banach Contraction Mapping Theorem (1922)

There exists a unique $\bar{u} \in B(X)$ such that

1. $T\bar{u} = \bar{u}$
2. For any $u_0 \in B(X)$, define the sequence $u_{n+1} = Tu_n$, $n = 0, 1, 2, \cdots$. Then

$$\| u_n - u \| \to 0 \quad \text{as} \quad n \to \infty. \quad (2)$$

T possesses a unique, globally attractive fixed point.

Example: Consider $X = [0, 1]$ and let $B(X)$ be your favourite Banach space of functions on $[0, 1]$, e.g., $C(X)$ (continuous functions on X) or $L^2(X)$ (square integrable functions on X).

Consider the mapping $T : X \to X$ given by

$$T : u \to \frac{1}{2}u + \frac{1}{2}.$$

In other words, if $v = Tu$, then

$$v(x) = (Tu)(x) = \frac{1}{2}u(x) + \frac{1}{2}, \quad \text{for all} \quad x \in [0, 1]. \tag{3}$$

Note that for any $u, v \in B(X)$:

$$\begin{aligned} \| Tu - Tv \| &= \| \left(\frac{1}{2}u + \frac{1}{2}\right) - \left(\frac{1}{2}v + \frac{1}{2}\right) \| \\ &= \| \frac{1}{2}u - \frac{1}{2}v \| \\ &= \frac{1}{2} \| u - v \| . \end{aligned}$$

Therefore, T is a contraction mapping on $B(X)$. The "fixed point" of T is the function

$$\bar{u}(x) = 1, \qquad x \in [0, 1].$$

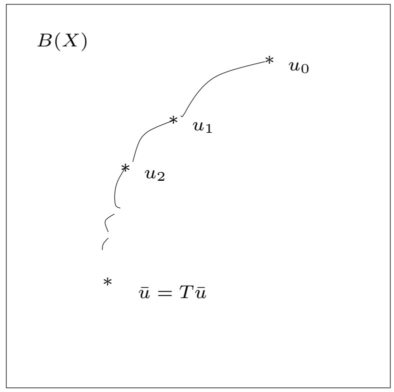
Discrete evolution under iteration of T

This means that if we start with any function $u_0 \in B(X)$, and form the sequence

$$u_{n+1} = \frac{1}{2}u_n + \frac{1}{2},$$

then "$u_n \to \bar{u}$," i.e., $\| u_n - \bar{u} \| \to 0$.

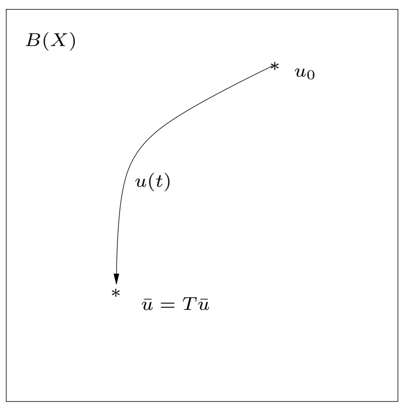
Discrete evolution under iteration of T



We can consider the u_n as functions $u(t)$, where $t \in \{0, 1, 2, \cdots\}$.

The goal is to produce a *continuous evolution* of functions $u(t)$, $t \in [0, \infty)$ such that $u(0) = u_0$ and

$$\| u(t) - \bar{u} \| \to 0 \quad \text{as} \quad t \to \infty. \tag{4}$$



(Note that we do **not** demand that $u(t)$ interpolates the $u_n = T^{\circ n} u_0$.)

2. The evolution equation (quite simple!):

(With J. Bona, UI at Chicago) Given a $T \in Con(B(X))$, consider the following evolution equation

$$\frac{\partial u}{\partial t} = Tu - u, \tag{5}$$

where $u = u(x,t)$, $x \in X$.

Clearly the fixed point function $\bar{u} = T\bar{u}$ is an equilibrium solution of this equation. Is this solution globally asymptotically stable, i.e., do all solutions $u(t)$ converge to $\bar{u}$ as $t \to \infty$? The answer is **yes**:

Main result:

For any initial value $u(x,0) = u_0(x) \in B(X)$, the solution $u(t)$ to $u_t = Tu - u$ converges (exponentially rapidly) to $\bar{u}$ as $t \to \infty$.

Example: Let us return to previous example with contraction mapping $T : X \to X$ given by

$$T : u \to \frac{1}{2}u + \frac{1}{2}$$

and fixed point $\bar{u}(x) = 1$.

Continuous evolution via equation $u_t = Tu - u$

Now let $u = u(x,t)$, $x \in [0,1]$ and $t \in \mathbf{R}$. We now use the contraction mapping T in our evolution equation:

$$\begin{aligned}\frac{\partial u}{\partial t} &= Tu - u \\ &= \left(\frac{1}{2}u + \frac{1}{2}\right) - u \\ &= -\frac{1}{2}u + \frac{1}{2}.\end{aligned}$$

For each $x \in [0,1]$, this becomes a simple first-order linear ODE in t of the form

$$\frac{dv}{dt} + \frac{1}{2}v = \frac{1}{2},$$

where $v(t) = u(x,t)$. Solution is

$$v(t) = 1 + (v(0) - 1)e^{-\frac{1}{2}t}.$$

Consider any starting function u_0 and let $u(x,0) = u_0(x)$. Then continuous evolution of u under T is given by

$$u(x,t) = 1 + [u(x,0) - 1]e^{-\frac{1}{2}t}, \qquad t \geq 00.$$

Note that

$$u(x,t) \to 1 \quad \text{as} \ \ t \to \infty \ \ \text{for all} \ \ x \in [0,1].$$

4. The principal motivation: fractal image coding

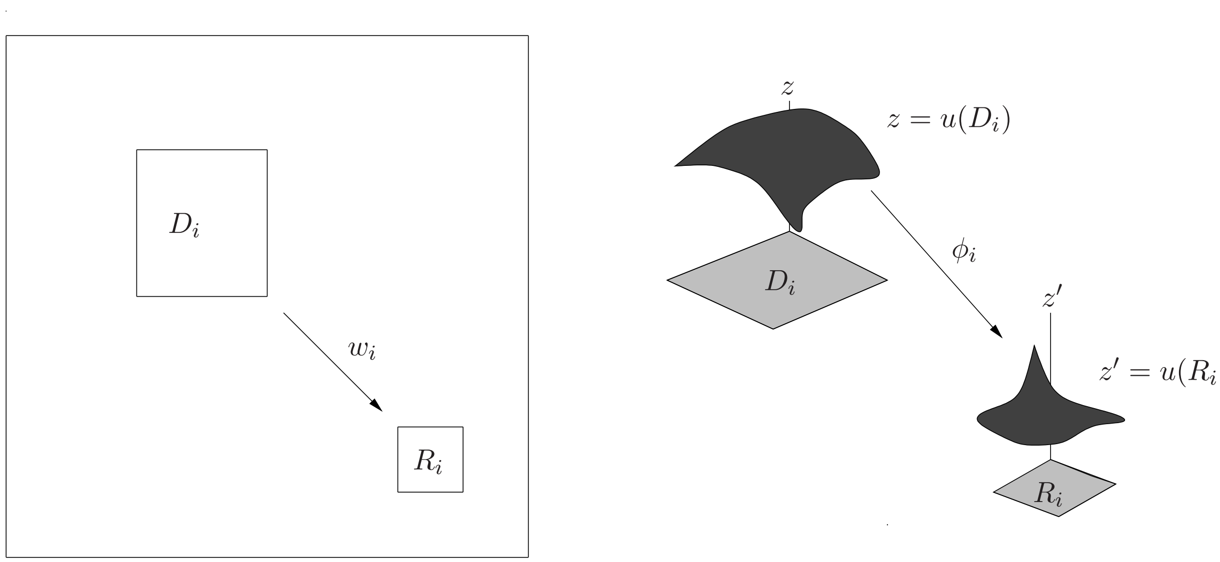


Illustration of the fractal transform. Left: Range block R_i and associated domain block D_i. Right: Greyscale mapping ϕ_i from $u(D_i)$ to $u(R_i)$.

Fractal image coding seeks to express an image u as a union of spatially-contracted and greyscale-modified copies of subsets of itself:

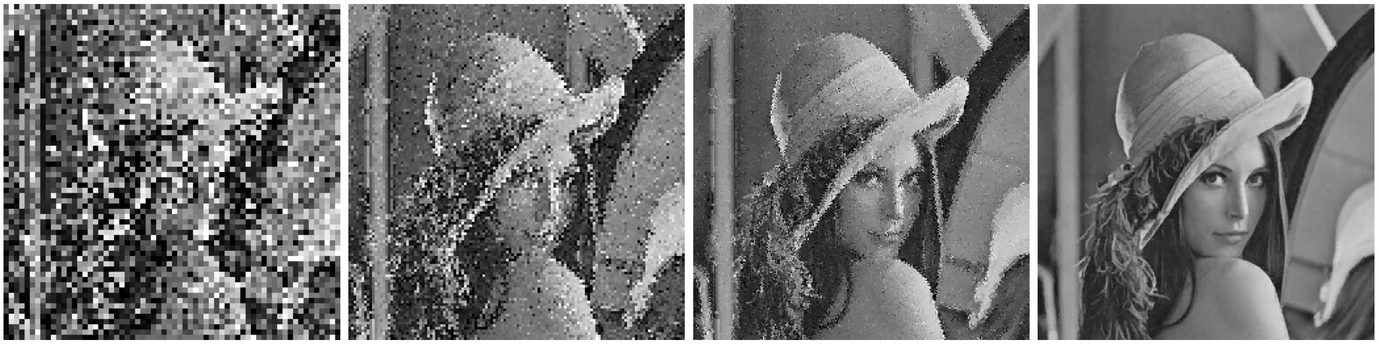
$$u(R_i) \cong \phi_i(u(D_i)) = \phi_i(u(w_i^{-1}(R_i))), \quad i = 1, 2, \cdots N,$$

where the $\phi_i : \mathbf{R} \to \mathbf{R}$ are *greyscale* maps that operate on pixel intensities.
Assuming that the partition blocks R_i are nonoverlapping, we may write

$$u(x, y) \cong (Tu)(x, y) = \sum_i \phi_i(u(w_i^{-1}(x, y))), \qquad (x, y) \in R_i.$$

We may consider this union of modified copies as defining a special kind of operator T, the **fractal transform** operator. Under appropriate conditions, T is a **contraction mapping**.

Iteration of contractive fractal transform operator T to produce the fixed point image function $\bar{u}$



Left to right: The iterates u_1, u_2 and u_3 along with the fixed point $\bar{u}$ of the fractal transform operator T designed to approximate the standard 512×512 (8bpp) "Lena" image. The "seed" image was $u_0(x) = 255$ (plain white). The fractal transform T was obtained by "collage coding" using 4096 8×8 nonoverlapping pixel range blocks. The domain pool consisted of the set of 1024 nonoverlapping 16×16 pixel blocks.

"Given a target image u, how do we determine T?

This is the "inverse problem of fractal-based approximation"

“Collage coding” in fractal image coding:

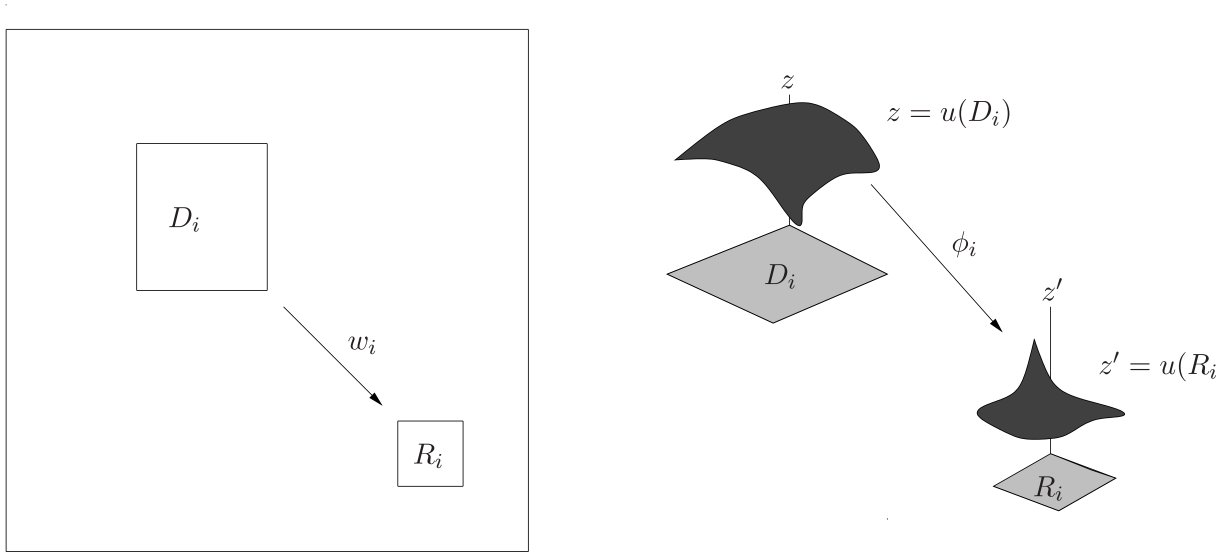


Illustration of the fractal transform. Left: Range block R_i and associated domain block D_i. Right: Greyscale mapping ϕ_i from $u(D_i)$ to $u(R_i)$.

The “collage distance” associated with each range block R_i is

$$\Delta_i = \| u(R_i) - \phi_i(u(D_i)) \|, \quad i = 1, 2, \cdots N,$$

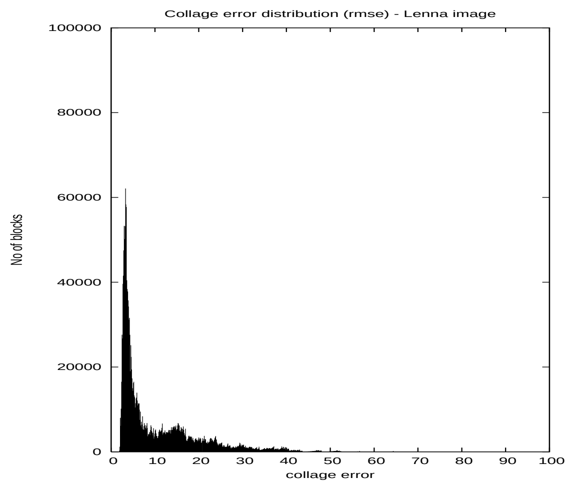
You minimize this distance by finding the “best” greyscale maps ϕ_i.

Generally, we assume affine greyscale maps:

$$\phi_i(t) = \alpha_i t + \beta_i$$

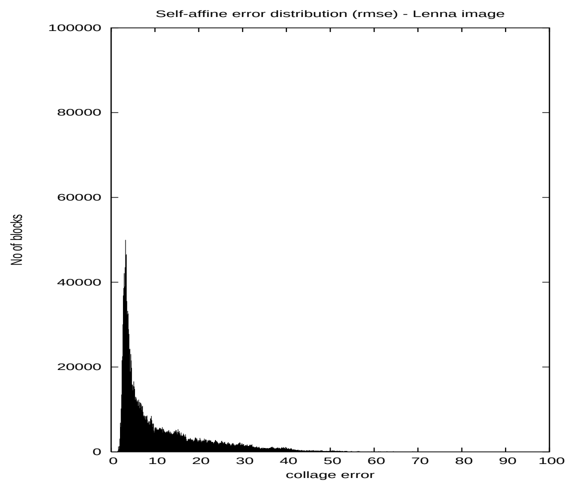
In L^2, this leads to “least squares” determination of α_i, β_i.

Is fractal image coding such an outlandish idea? Not at all. In general, images are locally quite self-similar



A plot of collage distances for the “Lena” image for all possible domain-range pairings. Domain pool: $32^2 = 1024$ nonoverlapping 16×16 pixel blocks. Range pool: $64^2 = 4096$ nonoverlapping 8×8 pixel blocks. Affine greyscale maps used: $\phi(t) = \alpha t + \beta$.

Actually, images are also quite translationally invariant – up to affine greyscale transformations



A plot of collage distances for the "Lena" image for all possible domain-range pairings. Domain and range pool: $64^2 = 4096$ nonoverlapping 8×8 pixel blocks. Affine greyscale maps used: $\phi(t) = \alpha t + \beta$.

The fractal transform operator T is a discrete, non-local operator:

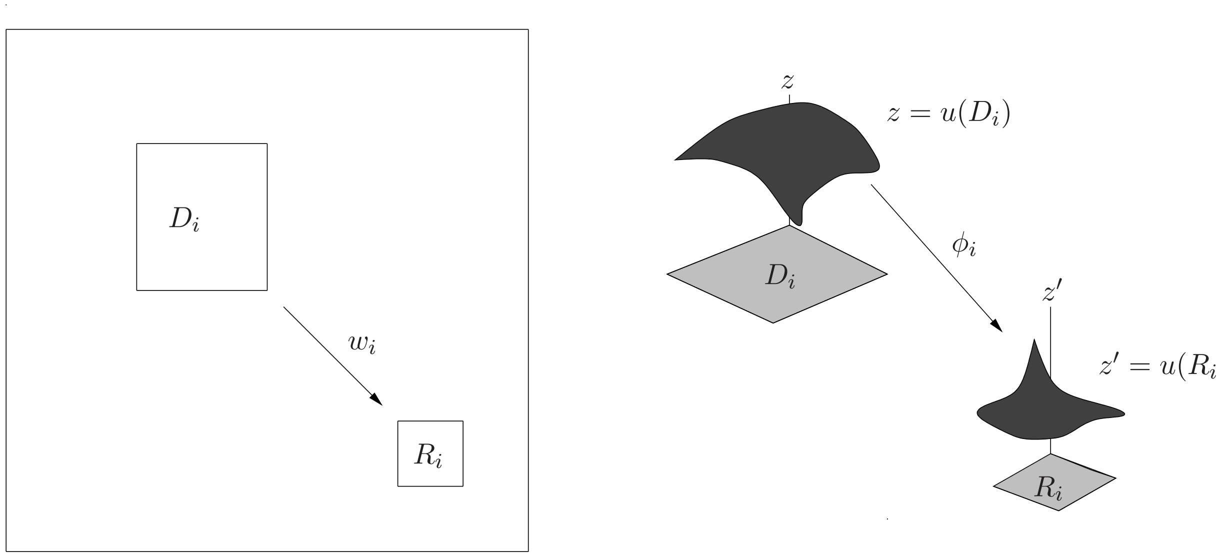


Illustration of the fractal transform. Left: Range block R_i and associated domain block D_i. Right: Greyscale mapping ϕ_i from $u(D_i)$ to $u(R_i)$.

This is in stark contrast to local discrete operators, e.g.,

- blurring – linear (local weighted averaging) and nonlinear operators
- sharpening – local masks
- denoising, e.g. Lee filter mask

It is also in stark contrast to PDE imaging methods, e.g.,

- blurring – evolution under heat/diffusion equation
- denoising – evolution under anisotropic diffusion equation(s)

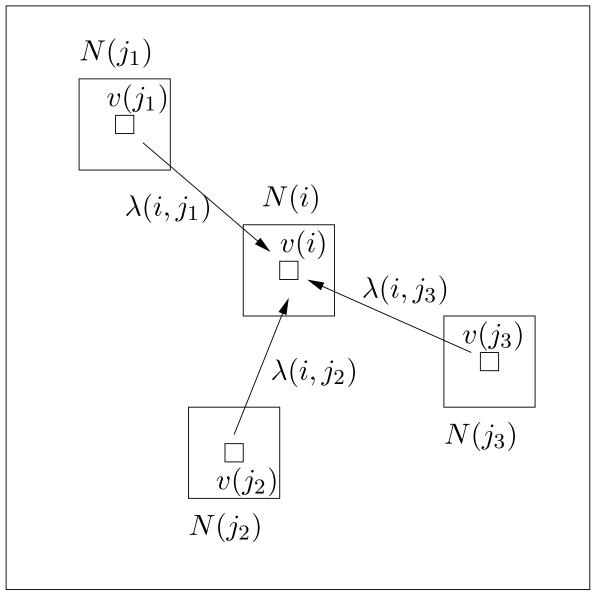
Until perhaps more recently ...

“A nonlocal algorithm for image denoising,”

by A. Buades, B. Coll and J.-M. Morel, CVPR (2) 2005, pp. 60-65.

“**NL-means algorithm:** Given noisy image $v = \{v(i), i \in I\}$, replace each pixel value $v(i)$ by estimated value $NL[v](i)$ where

$$NL[v](i) = \sum_{j \in I} \lambda(i,j) v(j). \tag{6}$$



The weights $\lambda(i,j)$ depend upon the “similarity” between two pixels i and j which, in turn, depends upon the “similarity” of greyscale pixel blocks $v(N_i)$ and $v(N_j)$.

Here, N_k denotes a square neighbourhood of fixed size and centered at pixel k.

$$\lambda(i,j) = \frac{1}{Z(i)} e^{-A\|v(N_i) - v(N_j)\|^2} \tag{7}$$

where $A > 0$ is a constant (related to filtering parameter) and $Z(i)$ is the normalization constant

$$Z(i) = \sum_j e^{-A\|v(N_i) - v(N_j)\|^2}. \tag{8}$$

Fractal image denoising

M. Ghazel, G.H. Freeman and E.R.V., IEEE Trans. I.P. **12**, 1560-1578 (2003).

It is well known that lossy compression schemes (e.g., JPEG, wavelet) can denoise images. The same is true for fractal image coding.

Noisy "Lena" image - independent, zero mean, Gaussian noise, $\sigma = 25$. RMSE = 25.01, PSNR = 20.17

Left: Straightforward fractal coding of noisy image produces denoising, RMSE = 11.56, PSNR = 26.87.

Right: Improved fractal denoising procedure, RMSE = 10.10, PSNR = 28.05

Continuous evolution of fractal transform operator

Using the evolution equation

$$u_t = Tu - u \tag{9}$$

we have

$$\frac{\partial u(x,t)}{\partial t} = \phi_i(u(w_i^{-1}(x))) - u(x), \quad x \in R_i. \tag{10}$$

In most applications, greyscale maps are affine, i.e.

$$\phi_i(t) = \alpha_i t + \beta_i, \tag{11}$$

so that evolution equation becomes

$$\frac{\partial u(x,t)}{\partial t} = \left[\alpha_i u(w_i^{-1}(x)) + \beta\right] - u(x), \quad x \in R_i. \tag{12}$$

Important point: NONLOCALITY!

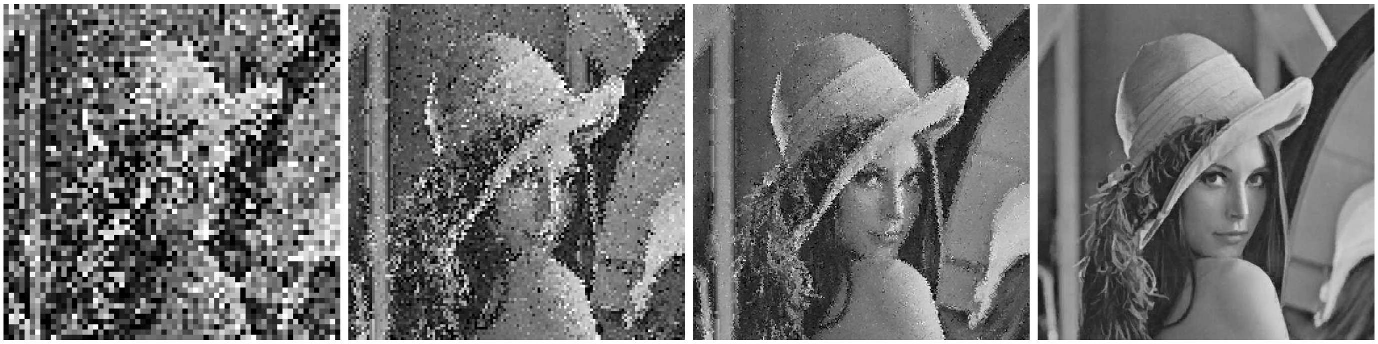
Since the fractal transform T does not contain any differential operators, Eqs. (10) and (12) are ordinary differential equations in $u(x,t)$, involving only time derivatives. Nevertheless, because of the terms $w_i^{-1}(x)$, these DE's are **nonlocal** in that the time evolution of $u(x,t)$ is determined by values of u generally **not** at x. This can lead to rather complicated evolution.

Original goal: to develop continuous, yet fractal-like, touch-up operations on images.

Application to fractal block image coding

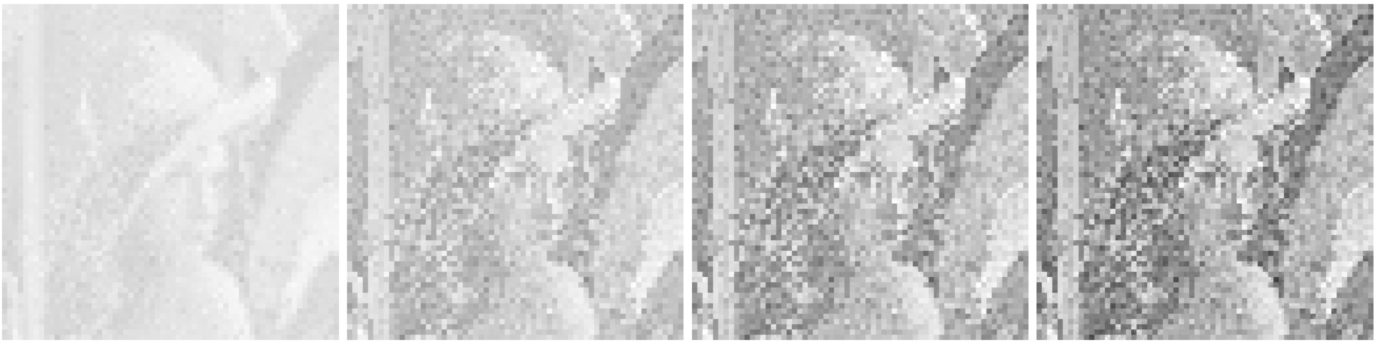
Discrete iteration of fractal transform $u_{n+1} = Tu_n$

(This figure was shown earlier)



Left to right: The iterates u_1, u_2 and u_3 along with the fixed point $\bar{u}$ of the fractal transform operator T designed to approximate the standard 512×512 (8bpp) "Lena" image. The "seed" image was $u_0(x) = 255$ (plain white). The fractal transform T was obtained by "collage coding" using 4096 8×8 nonoverlapping pixel range blocks. The domain pool consisted of the set of 1024 nonoverlapping 16×16 pixel blocks.

Continuous evolution $u_t = Tu - u$



Left to right: The images $u(x,t)$ at times 0.2, 0.4, 0.6, 0.8 produced from $u(x,0) = 255$ (plain white) under evolution by $u_t = Tu - u$ where T is the fractal transform whose discrete iteration was shown earlier. Euler method, step-size $h = 0.1$.

In the limit $t \to \infty$ (in this practical case, $t = 20$) the $u(x,t) \to \bar{u}$, the fixed point of the fractal transform T.

Evolution in the presence of diffusion

Now modify the evolution equation (5) by adding a small diffusion term, e.g.,

$$\frac{\partial y}{\partial t} = \epsilon \triangle y + Ty - y, \tag{13}$$

Case 1: $\epsilon > 0$ (positive diffusion)

Left to right: The limiting images $u(x,t), t \to \infty$ produced by integrating Eq. (13) for ϵ values of $0.5, 5.0, 10.0$ and 20.0, respectively. Euler method, step-size $h = 0.01$, integrated to $t = 50$.

As ϵ increases, the blurring effects of the diffusion operator are more pronounced.

Relevance to historical development of fractal image coding

1. Fixed points $\bar{u}$ of fractal transform operators T generally exhibit blockiness.

2. In most fractal schemes, much of this blockiness is due to the partitions used to produce the child blocks – lack of effort in "patching" neighbouring regions.

3. Among the various methods devised to reduce such blockiness:

   (a) "Postprocessing" – blur the final fixed point image, either over its entirety or selectively across the boundaries of the child blocks.

   (b) "Intermediate processing" - process the image after each application of the fractal transform operator T.

   (c) Better fractal transform operators – e.g., more attention paid to matching at boundaries.

The diffusion operator in Eq. (13) essentially performs such an intermediate processing but in a continuous manner.

The asymptotic images are no longer fixed points of the fractal transform operator T but are (steady state) solutions of the partial differential equation

$$\epsilon \triangle y + Ty - y = 0. \tag{14}$$

Case 2: $\epsilon < 0$ (negative diffusion)

Below is presented the limiting image for $\epsilon = -0.1$.

The limiting image $u(x,t), t \to \infty$ produced by integrating Eq. (13) for $\epsilon - 0.1$, corresponding to negative diffusion. Euler method, step-size $h = 0.001$, integrated to $t = 50$.

This image is a somewhat sharpened version of the fixed-point Lena image $\bar{u}$. Unfortunately, the sharpening enhances not only the edges present in the Lena image but also those that lie along the boundaries of the 8×8 child blocks.

Return to steady-state diffusion equation

$$\epsilon \triangle y + Ty - y = 0. \tag{15}$$

Now consider $\epsilon \to \infty$ and rewrite as

$$\triangle y + \eta(Ty - y) = 0, \qquad \eta \to 0, \tag{16}$$

i.e., perturbation of Laplace's equation on Ω.

Of course, we can consider other differential operators, e.g.,

$$u_t = \nabla^p u + Tu - u, ... \tag{17}$$

Evolution under a convex combination of contraction mappings

Let T_i, $i = 1, 2, \cdots, n$ be a set of contraction maps on $\mathcal{B}(X)$ with contraction factors $c_i \in [0, 1)$ and fixed points $\bar{y}_i \in \mathcal{B}(X)$. Now let λ_i, $i = 1, 2, \cdots, n$, be a partition of unity, i.e., $\lambda_i \in (0, 1)$ with $\sum_i^n \lambda_i = 1$, and consider the evolution equation

$$\frac{\partial y}{\partial t} = \sum_{i=1}^{n} \lambda_i (T_i y - y). \tag{18}$$

This equation may be rewritten as

$$\frac{\partial y}{\partial t} = Ty - y, \tag{19}$$

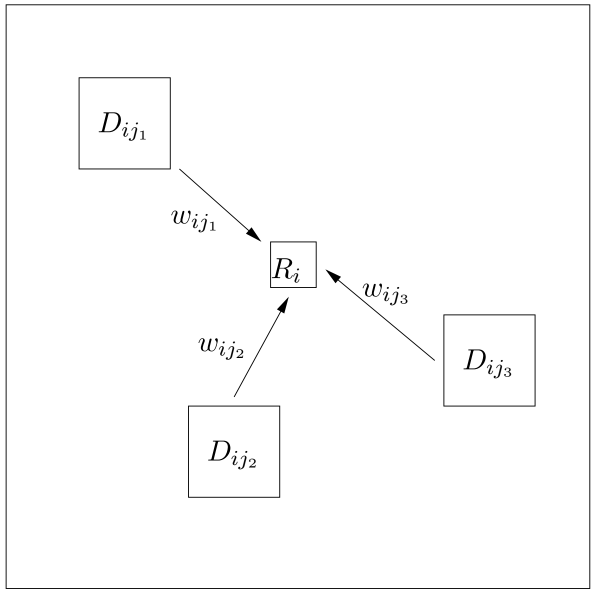
where

$$T = \sum_{i=1}^{n} \lambda_i T_i. \tag{20}$$

T is a contraction mapping with a unique fixed point $\bar{y} \in \mathcal{B}(X)$. (See paper for proof.) From our earlier result associated with Eq. (5), it also follows that $\bar{y}$ is a globally asymptotically stable solution of Eq. (18).

Special case: Fractal image coding/denoising with multiple parent blocks

(S. Alexander, Ph.D. Thesis, University of Waterloo, 2005)



Each image range block $u(R_i)$ is expressed as a weighted sum of spatially-contracted and greyscale modified copies of a number of image domain blocks $u(D_{ij})$:

$$u(R_i) \cong \sum_j \lambda_{ij}\phi_{ij}(u(D_{ij}) = \sum_j \lambda_{ij}\phi_{ij}(u(w_{ij}^{-1}(R_{ij}))), \quad i = 1, 2, \cdots N,$$

where

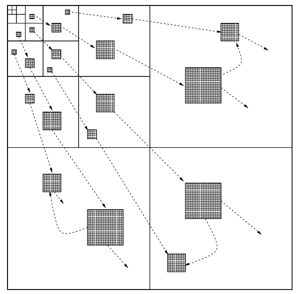
$$\phi_{ij}(t) = \alpha_{ij}t + \beta_{ij}$$

and

$$\sum_j \lambda_{ij} = 1.$$

Application to fractal-wavelet image coding

Fractal-wavelet transforms involving mapping modified wavelet coefficient subblocks onto lower subblocks. (E.R.V., Can. J. Elect & Comp. Eng. **23**, 70-83 (1998).)



A schematic illustration of the of the fractal-wavelet transform operator on wavelet coefficient subtrees.

Continuous evolution of wavelet coefficient tree will assume the form

$$c_t = Mc - c, \tag{21}$$

where c denotes the matrix of wavelet expansion coefficients of an image u and M is contractive fractal-wavelet transform operator whose action is depicted above.

Continuous evolution of wavelet coefficient tree

$$c_t = Mc - c$$

Left to right: The images $u(x,t)$ at times $t = 0, 0.2, 0.4$ and 0.6 under evolution by $c_t = Mc - c$, where M is the fractal-wavelet transform with parent/child levels $(k_1, k_2) = (5, 6)$ in the Coifman-6 generalized wavelet basis. Euler method, step-size $h = 0.1$