Function-valued mappings, total variation and compressed sensing for diffusion MRI

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Outline

Ongoing work on the development of **multifunction representations** of images, in particular,

- Measure-valued image functions: D. La Torre, E.R.V., M. Ebrahimi, M.F. Barnsley, SIAM J Imaging Sciences, **2** (2), 470-507 (2009)
- Function-valued image functions: the subject of this talk

Primary motivation for this study: Multispectral/hyperspectral imaging

Our work represents a deviation from the usual mathematical representation of a (greyscale/colour) image, i.e.,

$$u : X \to R_g,$$

where

- X: **base** or **pixel space**, the support of the image, $X \subset \mathbb{R}^n$, $n = 1, 2, 3$,
- $R_g \subset \mathbb{R}$ (or $\mathbb{R}^3$): the **greyscale** (or **colour**) **range**.

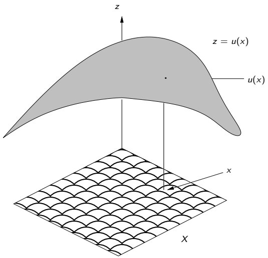
Representations of image functions

Greyscale-valued image function

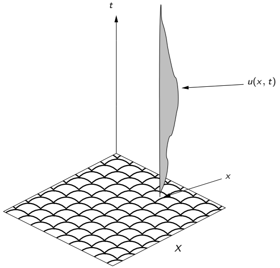
At each pixel $x \in X$, $u(x)$ is a **real value** (or a vector of real values, i.e., "RGB")

Function-valued image function

At each pixel $x \in X$, $u(x)$ is a **real-valued function**, i.e., $u(x; t)$



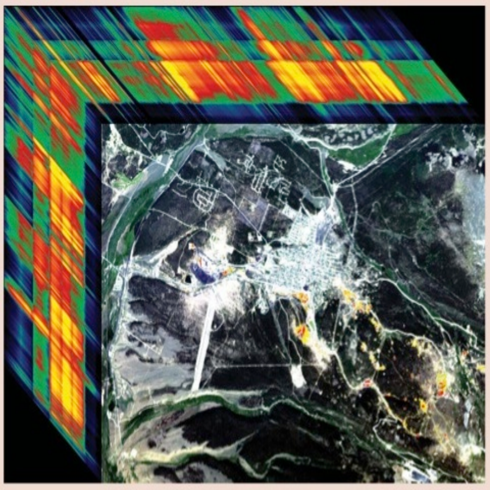
(a) Greyscale-valued image function



(b) Function-valued image function

Example: In multispectral/hyperspectral imaging, u represents the **spectral density function**. The values $u(x, t_k)$, $t_1 < t_2 < \cdots < t_N$ represent intensities of reflected radiation from point x on ground, as captured by satellite reading.

Hyperspectral imaging



A natural question: Why not simply consider these spectral readings to define a **vector-valued function**,

$$u : X \to \mathbb{R}^N,$$

i.e.,

$$u(x) = (u_1(x), u_2(x), \cdots, u_N(x)),$$

where

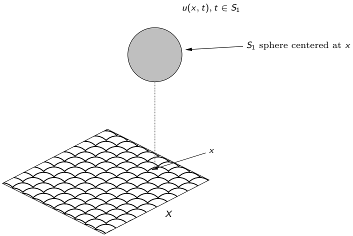
$$u_k : X \to \mathbb{R}, \quad 1 \leq k \leq N$$

are **greyscale image functions**, as we do for **colour (RGB)** images?

Answer: We certainly can, which has been the traditional way of dealing with such images, i.e., process images $u_1(x), u_2(x), \cdots, u_N(x)$ separately, without necessarily taking their correlation into consideration.

More on this later.

High angular resolution diffusion imaging (HARDI)



At each pixel $x \in X$, the HARDI signal $u(x, t)$ is related to probability of diffusion of water molecules in the direction $t \in S^2$ (unit sphere).

u is a **function-valued image function**:

$$u : X \to L^2(\mathbb{S}^2).$$

“q-space imaging”

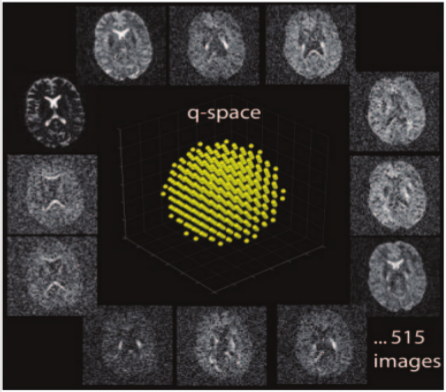


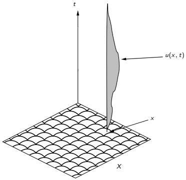
Figure 12. Series of diffusion-weighted MR brain images obtained with variations in the direction and strength of the diffusion gradient in the pulsed gradient SE sequence. Each image shows the signal sampled at one point in q-space (one yellow dot). Every sampling point in q-space corresponds to a specific direction and strength of the diffusion gradient.

From “Understanding Diffusion MR Imaging Techniques: From Scalar Diffusion-weighted imaging to Diffusion Tensor Imaging and Beyond,” P. Hagmann, L. Jonasson, P. Maeder, J.-P. Thiran, V.J. Weeden, R. Meuli, Radiographics 2006, 26:S205-S223, Published online 10.1148rg.26si065510.

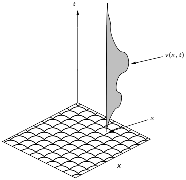
"Connectome"

In the brain, neuronal tissue is highly fibrillar, consisting of tightly packed and aligned axons that are surrounded by glial cells and organized in bundles. Diffusion of water parallel to fiber is typically greater than perpendicular to it.
In "tractography", the streamlines associated with the anisotropic diffusion vector fields are constructed in order to give an idea of the connectivity of neurons.

Outline



(a)



(b)

Linear space: For $u, v : X \to L^2(R_g)$, define

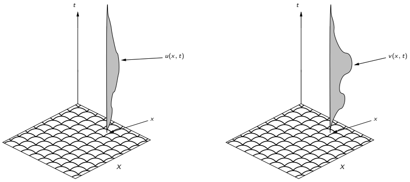
$$(c_1 u + c_2 v)(x,t) = c_1 u(x,t) + c_2 v(x,t), \quad \text{etc. (linear space)}$$

Normed linear space Y: For $u : X \to L^2(R_g)$, norm of $u(x)$ is given by

$$\|u(x)\|^2_{L^2(R_g)} = \int_{R_g} u(x,t)^2 \, dt.$$

Integrate over all $x \in X$ to define norm of u:

$$\|u\|^2_Y = \int_X \|u(x)\|^2_{L^2(R_g)} \, dx.$$



(a) (b)

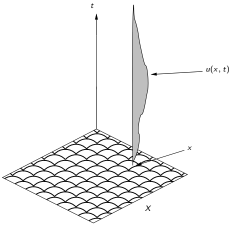
Complete metric space (Y, d_Y):

1. At each $x \in X$, compute L^2 distance between functions $u(x)$ and $v(x)$:

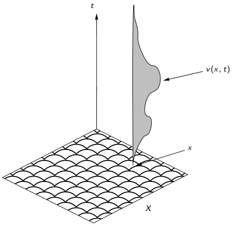
$$\|u(x) - v(x)\|^2_{L^2(R_g)} = \int_{R_g} [u(x,t) - v(x,t)]^2 \, dt$$

2. Integrate over all $x \in X$:

$$[\, d_Y(u, v)\,]^2 = \int_X \|u(x) - v(x)\|^2_{L^2(R_g)} \, dx.$$



(a)



(b)

Hilbert space:
Since $u(x), v(x) \in L^2(R_g)$, we may compute their inner product $\langle u(x), v(x) \rangle_{L^2(R_g)}$.
Integrate over all $x \in X$ to define inner product between function-valued image mappings,

$$\langle u, v \rangle_Y = \int_X \langle u(x), v(x) \rangle_{L^2(R_g)} \, dx, \quad u, v \in Y.$$

Complete metric space (Y, d_Y) of function-valued image mappings

$$Y = \{u : X \to L^2(R_g) \mid \|u\|_Y < \infty\}$$

where

$$\|u\|_Y^2 = \int_X \|u(x)\|_{L^2(R_g)}^2 \, dx$$

In our applications to diffusion MRI,

$$R_g = \mathbb{S}^2 = \{(\theta, \phi),\ \phi \in [0, 2\pi], \theta \in [0, \pi]\} \quad \text{(unit sphere in } \mathbb{R}^3\text{)}$$

Outline

Approximation theory

Orthonormal basis: Let $\{\phi_k\}$ be a complete orthonormal basis in $L^2(R_g)$. Then at each pixel $x \in X$, we may expand the function-valued mapping $u \in Y$ as follows,

$$u(x) = \sum_k c_k(x)\,\phi_k, \quad \forall x \in X,$$

where

$$c_k(x) = \langle u(x), \phi_k \rangle = \int_{R_g} u(x,t)\,\phi_k(t)\,dt.$$

Approximation theory (cont'd)

Frames: Let $\{\psi_k\}$ be a frame for $L^2(R_g)$, i.e., there exist $0 < A < B < \infty$ such that

$$A\|f\|_{L^2(R_g)} \leq \sum_k |\langle f, \psi_k \rangle_{L^2(R_g)}|^2 \leq B\|f\|_{L^2(R_g)}, \quad \forall f \in L^2(R_g).$$

Then for any $u \in Y$,

$$u(x) = \sum_{k \in \mathcal{I}} c_k(x) \psi_k,$$

where

$$c_k(x) = \langle u(x), \phi_k \rangle = \int_{R_g} u(x,t)\, \tilde{\psi}_k(t)\, dt\,,$$

$\mathcal{I}$ denotes the infinite-dimensional set of frame indices and $\{\tilde{\psi}_k\}$ denotes the canonical **dual frame** of $\{\phi_k\}$.

This framework is important for **compressed sensing**.

Total variation (TV)

Total variation (TV) is important in image processing, e.g., denoising, as a constraint to *smoothen* images. Our goal is to employ TV in the study and processing of function-valued image mappings. It is therefore necessary to mathematically formulate TV over the space (Y, d_Y). To do so, it is necessary to return to "first principles."

TV for functions $u : \mathbb{R} \to \mathbb{R}$: Consider sampling of u over a regular grid of points x_k, $1 \leq k \leq N$, with $\Delta x = x_{k+1} - x_k$. Then define

$$\begin{aligned} TV[u](\Delta x) &= \sum_{k=0}^{N-1} |u(x_{k+1}) - u(x_k)| \\ &= \sum_{k=0}^{N-1} \left| \frac{u(x_{k+1}) - u(x_k)}{\Delta x} \right| \Delta x. \end{aligned}$$

Define TV of u to be

$$TV[u] = \lim_{\Delta x \to 0} TV[u](\Delta x) = \int_X \|\vec{\nabla} u(x)\|_1 \, dx.$$

Total variation

TV for functions $u : \mathbb{R}^n \to \mathbb{R}$: One usually employs the n-dimensional extension of the above, i.e.,

$$TV[u] = \int_X \|\vec{\nabla} u(\mathbf{x})\|_1 \, d\mathbf{x}.$$

Total variation

Recall that for C^1 real-valued functions $u : \mathbb{R}^n \to \mathbb{R}$, directional derivative of u at $x \in \mathbb{R}^n$ in direction $\mathbf{d} \in \mathbb{R}^n$, with $\|\mathbf{d}\| = 1$, is defined as

$$Du(\mathbf{x}, \mathbf{d}) = \lim_{t \to 0^+} \frac{u(\mathbf{x} + t\mathbf{d}) - u(\mathbf{x})}{t} = \vec{\nabla} u(\mathbf{x}) \cdot \mathbf{d},$$

For smooth functions $u : \mathbb{R}^n \to \mathbb{R}^m$, directional derivative of u at $x \in \mathbb{R}^n$ in direction $\mathbf{d} \in \mathbb{R}^n$, with $\|\mathbf{d}\| = 1$, is defined as

$$Du(\mathbf{x}, \mathbf{d}) = \lim_{t \to 0^+} \frac{u(\mathbf{x} + t\mathbf{d}) - u(\mathbf{x})}{t} = \mathbf{J}(\mathbf{x})\mathbf{d},$$

where $\mathbf{J}(\mathbf{x})$ is usual $m \times n$ **Jacobian matrix** of u at $\mathbf{x}$, with elements

$$J_{ij} = \frac{\partial u_i}{\partial x_j}(\mathbf{x}).$$

We must now extend these ideas to function-valued image mappings.

Total variation

For mappings, $u : X \to Y$, where X and Y are Banach spaces, we say that u is **Gateaux differentiable** at $x \in X$ if the following limit exists for any $d \in X$,

$$Du(x,d) = \lim_{t \to 0^+} \frac{u(x+td) - u(x)}{t}.$$

We say that $Du(x,d)$ is the **Gateaux derivative** of u at x (in direction d). But this limit exists in terms of the distance in Y, i.e.,

$$\lim_{t \to 0^+} \left\| \frac{u(x+td) - u(x)}{t} - Du(x,d) \right\|_Y = 0.$$

In our case, $Y = L^2(R_g)$, i.e.,

$$\lim_{t \to 0^+} \int_{R_g} \left[\frac{u(x+td)(\tau) - u(x)(\tau)}{t} - D(u,d)(\tau) \right]^2 d\tau = 0.$$

We consider the three canonical directions in $\mathbb{R}^3$, i.e., $d = d_k = \hat{e}_k$, $k = 1,2,3$, and define the **total variation semi-norm** of $u : X \to L^2(R_g)$ as

$$\|u\|_{TV} = \sum_{k=1}^{3} \int_X \|Du(x,d_k)\|_{L^2(R_g)}\, dx.$$

Outline

From previous slide, **total variation semi-norm** of $u : X \to L^2(R_g)$ as

$$\|u\|_{TV} = \sum_{k=1}^{3} \|u\|_{TV,d_k} = \sum_{k=1}^{3} \int_X \|Du(x, d_k)\|_{L^2(R_g)}\, dx,$$

where d_k, $k = 1, 2, 3$, are canonical directions in $\mathbb{R}^3$.
In practice, signals are discretized. Suppose that $u(x, y, z)(t)$ is sampled on a regular grid of points (pixels), (x_k, y_l, z_m), and t_n, $1 \le n \le N$, with $\Delta x = \Delta y = \Delta z = 1$.
Then for d_1 direction,

$$\|D(x_k, y_l, z_m; d_1)\|_{L^2(R_g)} = \left[\sum_{n=1}^{N} [u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)]^2 \right]^{1/2}.$$

Integrate over X (sum over all pixels) to obtain TV in direction d_1:

$$\begin{aligned} \|u\|_{TV,d_1} &= \sum_{k,l,m} \|D(x_k, y_l, z_m; d_1)\|_{L^2(R_g)} \\ &= \sum_{k,l,m} \left[\sum_{n=1}^{N} [u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)]^2 \right]^{1/2}. \end{aligned}$$

From previous slide, TV in direction d_1 is

$$\begin{aligned}\|u\|_{TV,d_1} &= \sum_{k,l,m} \|D(x_k, y_l, z_m; d_1)\|_{L^2(R_g)} \\ &= \sum_{k,l,m} \left[\sum_{n=1}^{N} [u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)]^2 \right]^{1/2}\end{aligned}$$

Contrast the above approach with "traditional" method in which the N (RGB...) levels are treated separately and then combined in some manner for final processing:

TV in direction d_1 of nth level:

$$\|u\|_{TV',t_n,d_1} = \sum_{k,l,m} |u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)|$$

Then sum over all levels, t_n, $1 \leq n \leq N$,

$$\|u\|_{TV',d_1} = \sum_{n}^{N} \left[\sum_{k,l,m} |u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)| \right]$$

Summary of TV methods

Total variation for function-valued image mappings

$$\begin{aligned} \|u\|_{TV,d_1} &= \sum_{k,l,m} \|D(x_k, y_l, z_m; d_1)\|_{L^2(R_g)} \\ &= \sum_{k,l,m} \left[\sum_{n=1}^{N} [u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)]^2 \right]^{1/2} \end{aligned} \quad (1)$$

Total variation for vector-valued functions

$$\|u\|_{TV',d_1} = \sum_{n}^{N} \left[\sum_{k,l,m} |u(x_{k+1}, y_l, z_m)(t_n) - u(x_k, y_l, z_m)(t_n)| \right] \quad (2)$$

Claim: The minimization of (1) and (2) represent different constraints/mathematical problems.

A final mathematical detail

We shall restrict signals of interest $u \in Y$ to be members of the following subspace of (Y, d_Y):

$$\mathcal{B} = \{u : X \to L^2(R_g) \mid Du(x, d_k) \in L^2(R_g) \; \forall x, k, \; \|u\|_Y + \|u\|_{TV} < \infty\}$$

Theorem: $\mathcal{B}$ is a Banach space.

Outline

In what follows, let

- $u(x)$ denote HARDI signal at spatial location $x \in X$.
- $\{s_k\}_{k=1}^K$ denote a set of K diffusion-encoding directions which define the associated sampling functions,

$$\phi_k(s) = \delta(1 - s_k \cdot s), \quad s \in \mathbb{S}^2.$$

The HARDI measurements may be expressed as

$$y_k(x) = \langle u(x), \phi_k \rangle_{L^2} + e_k(x), \quad k = 1, 2, \cdots, K, \ \forall x \in X,$$

where $e_k(x)$ accounts for measurement imperfections and instrument noise.

For each $k \in \{1, 2, \cdots, K\}$, $y_k(x)$ is a scalar-valued mapping from X to $\mathbb{R}$, commonly referred to as a **diffusion-encoded image associated with direction** s_k.

In our formalism, we shall consider the representation of the HARDI signals in terms of their **frame coefficients**, $c_k(x)$, recalling

$$u(x) = \sum_{i \in \mathcal{I}} c_i(x) \phi_i,$$

where $\mathcal{I}$ denotes the infinite-dimensional set of frame indices.

In practice, we must work with a finite-dimensional subset $\mathcal{I}_{\mathcal{M}} \subset \mathcal{I}$ of frame indices. Define the **synthesis operator**, $\mathcal{U} : \ell_2(\mathcal{I}_M) \to Y$, as follows,

$$c(x) \mapsto \mathcal{U}\{c\}(x) = \sum_{i \in \mathcal{I}_M} c_i(x)\psi_i, \forall x \in X.$$

Given a noisy observation y of u, i.e., $y = u + e$, where $\epsilon = \|e\|_Y^2 < \|u\|_Y^2$, the **optimal frame coefficients** c may be found as a solution to

$$\inf_c \left\{ \frac{1}{2}\|y - \mathcal{U}\{c\}\|_Y^2 + \lambda\|\mathcal{U}\{c\}\|_{TV} + \mu\|c\|_1 \right\} \quad (3)$$

Here, $\lambda, \mu > 0$ are **regularization parameters** and the ℓ_1 norm of c is defined as

$$\|c\|_1 = \int_X \sum_{i \in \mathcal{I}_{\mathcal{M}}} |\, c_i(x) \,|\, dx < \infty.$$

HARDI minimization problem

$$\inf_{c}\left\{\frac{1}{2}\|y-\mathcal{U}\{c\}\|_Y^2+\lambda\|\mathcal{U}\{c\}\|_{TV}+\mu\|c\|_1\right\} \tag{4}$$

Brief explanation of terms in (4):

- $\|y-\mathcal{U}\{c\}\|_Y^2 = \int_X \| y(x) - \sum_{i\in\mathcal{I}_\mathcal{M}} c_i(x)\psi_i \|_{L^2(\mathbb{S})}^2 dx$ – approximation of data
- $\|\mathcal{U}\{c\}\|_{TV}$ – smoothness constraint
- $\|c\|_1$ – sparseness constraint, a fundamental premise of the theory of compressed sensing.

Outline

Results

In this study, we employed simulated HARDI signals. **Upper left** of next figure shows such a signal. It mimics the structure of two crossing fibres.

Simulated signals were sampled using $K = 16$ diffusion orientations.

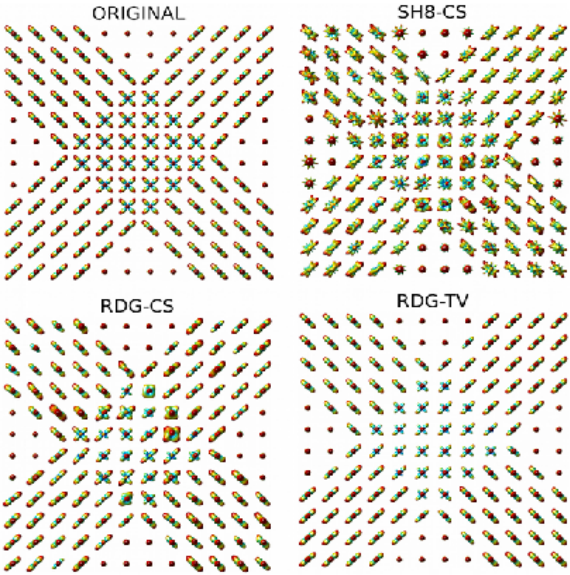
They were also contaminated by Rician noise, producing a signal-to-noise ratio of 18 dB.

Unless otherwise indicated, **spherical ridgelets**, $\{\psi_i\}_{i \in \mathcal{I}_{\mathcal{M}}}$, constructed in O. Michailovich and Y. Rathi, IEEE Trans. Image Proc. **19** (2) 461-477 (2010), were used for the frame basis functions.

Lower right, denoted "RDG-TV", shows results obtained with *lambda* $= 0.03$ (TV) and $\mu = 0.05$ (sparsity constraint/compressed sensing).

On average, only 6-8 ridgelet functions are needed at each point, exceeding the precision obtained by 45 spherical harmonics.

Results (cont'd)



Results (cont'd)

Lower left, denoted "RDG-CS", obtained by setting $\lambda = 0$, i.e., impose compressed sensing and ignore TV.

Upper right, denoted "SH8-CS", obtained by using **spherical harmonics** as basis functions to eighth order and setting $\lambda = 0$.

Conclusion: RDG-TV method yields best reconstruction.