# A diffusion-based two-dimensional Empirical Mode Decomposition (EMD) algorithm for image analysis


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# Introduction to Empirical Mode Decomposition (EMD)

Empirical Mode Decomposition (EMD) was introduced in

> *N.E. Huang* et al. *The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis, Proc. Roy. Soc. Lon.* **A 454**, *903-995 (1998),*

EMD is a powerful tool for analyzing linear, nonlinear and nonstationary signals. Its ability to perform local time-frequency analysis for nonstationary signals has been demonstrated for many kinds of real-world signals, e.g., rotating machinery, seismic waves.

- EMD employs a **sifting process** to extract different modes of oscillation of a signal, referred to as **Intrinsic Mode Functions** (IMF). These modes are obtained **from the signal itself** and are not expressed as linear combinations of basis functions.
- Hilbert Transform is applied to each IMF in order to determine its instantaneous local frequencies.
- Repeated application produces a decomposition of the signal into components of decreasing (instantaneous) local frequencies.
- Amplitudes and instantaneous local frequencies produce a local time-frequency analysis of signal.

**Example taken from Huang *et al.*** to illustrate sifting process:

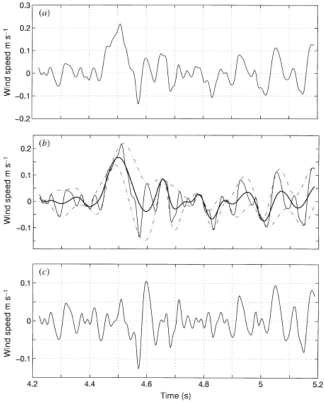



Figure 3. Illustration of the sifting processes: (*a*) the original data; (*b*) the data in thin solid line, with the upper and lower envelopes in dot-dashed lines and the mean in thick solid line; (*c*) the difference between the data and $m_1$. This is still not an IMF, for there are negative local maxima and positive minima suggesting riding waves.

**Example taken from Huang *et al.*** to illustrate sifting process:



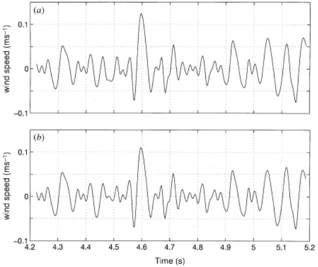



Figure 4. Illustration of the effects of repeated sifting process: (*a*) after one more sifting of the result in figure 3*c*, the result is still asymmetric and still not a IMF; (*b*) after three siftings, the result is much improved, but more sifting needed to eliminate the asymmetry. The final IMF is shown in figure 2 after nine siftings.

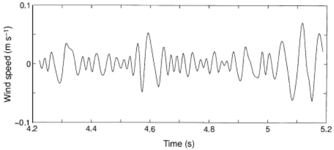



Figure 2. A typical intrinsic mode function with the same numbers of zero crossings and extrema, and symmetry of the upper and lower envelopes with respect to zero.

# Motivation for this work

Until very recently, most work on EMD focussed on algorithms as opposed to theoretical analysis.

Consequently, there has been very little theoretical work, i.e., developing a rigorous mathematical basis for EMD as well as understanding why it fails for certain kinds of signals.

Many EMD methods rely on rather sensitive, if not questionable, procedures in the sifting process. In a previous work (Wang, Mann and V 2017), we proposed a novel, one-dimensional heat/diffusion equation-based sifting method. The method is more stable numerically. It also provides a better mathematical analysis of EMD as well as identifying potential limitations.

There have been a number of efforts to extend EMD to two dimensions, often known as **bidimensional EMD (BEMD)**. Once again, however, much of the work was algorithmic in nature, with little theoretical background. And as in the 1D case, the sifting method involved sensitive procedures which are much more complicated because of the bidimensionalith of the problem.

In this paper, we extend our earlier work to two-dimensions. Our 2D PDE-based BEMD algorithm is more stable numerically and gives rise to mathematical analysis. Its computational time is lower than that of classical BEMD methods.

# A review of standard EMD and BEMD algorithms

For a given signal $S(x)$, the standard 1D EMD method may be summarized as follows:

1. Find all local maxima and minima of $S(x)$.
2. Interpolate between local maxima to obtain an **upper envelope function** $E_{upper}(x)$ and between local minima to obtain a **lower envelope function** $E_{lower}(x)$.
3. Compute the **local mean function**: $m(x) = \frac{1}{2}[E_{upper}(x) + E_{lower}(x)]$.
4. Define $c(x) = S(x) - m(x)$.
5. If $c(x)$ is not an IMF (see note below), iterate $m(x)$ until it is.
6. After finding the IMF, subtract it and repeat Step 2 to obtain the residual.

**Note:** Most EMD procedures employ the following vague definition of an IMF: (i) The number of extrema and zero-crossings of an IMF must differ by at most one and (ii) the mean of the IMF should be close to zero.

**Comment:** Procedures 1) and 2) above are, to say the least, quite *ad hoc* and computationally intensitive. (One might question how/why they would work at all.) How to interpolate between local extrema? Role of noise?

# A review of standard EMD and BEMD algorithms (cont'd)

The result of the above procedure is (hopefully) the following decomposition of the signal $S(x)$:

$$S(x) = \sum_{k=1}^{N} c_k(x) + r(x)\,, \qquad (1)$$

where

1. $c_k(x)$ is the $k$th IMF,
2. $r(x)$ is the residual.

Two-dimensional EMD algorithms follow a similar procedure although the extraction of upper and lower envelopes – in this case, surfaces – is understandably much more complex.

# An earlier theoretical analysis of EMD relevant to this work

The following paper (and others by the same authors),

> *S.D. El Hadji, R. Alexandre and A.O. Boudraa, Analysis of intrinsic mode functions: A PDE approach, IEEE Sig. Proc. Lett.* **17** *(4), 398-401 (2010).*

provides some mathematical analysis and insight into the EMD algorithm.

For a prescribed $\delta > 0$, define upper and lower envelopes of a function $h(x)$ as follows,

$$U_\delta(x) = \sup_{|y|<\delta} h(x+y)\,, \qquad L_\delta(x) = \inf_{|y|<\delta} h(x+y)\,. \tag{2}$$

Now employ Taylor series expansions of $U_\delta$ and $L_\delta$ to obtain – after some quite complicated algebra – the following expression for the mean envelope function,

$$m_\delta(x) = \frac{1}{2}[U_\delta(x) + L_\delta(x)] \approx h(x) + \frac{\delta^2}{2} h''(x)\,. \tag{3}$$

Define the following recursive sifting process applied to a signal $S(x)$,

$$h_{n+1}(x) = h_n(x) - m_{\delta,n}(x)\,, \quad h_0(x) = S(x)\,. \tag{4}$$

# An earlier theoretical analysis of EMD relevant to this work (cont'd)

Consider $h$ as a function of $x$ and $t$, with the goal of converting this discrete process into a continuous one,

$$h(x, n\Delta t) := h_n(x) \,. \tag{5}$$

Now use Taylor expansion in $t$,

$$\begin{aligned} h_{n+1}(x) &= h(x, (n+1)\Delta t) \\ &= h(x, n\Delta t + \Delta t) \\ &= h_n(x) + \Delta t \frac{\partial h(x, n\Delta t)}{\partial t} + O(\Delta t^2) \,. \end{aligned} \tag{6}$$

Substitute into recursive sifting equation given earlier,

$$h_{n+1}(x) = h_n(x) - m_{\delta,n}(x) \,, \quad h_0(x) = S(x) \,. \tag{7}$$

and let $\Delta t \to 0$ to obtain the following PDE,

$$\frac{\partial h}{\partial t} + \frac{1}{\delta^2} h + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} = 0 \,, \quad h(x, 0) = S(x) \,. \tag{8}$$

This PDE is a **backward heat equation**. Why "backward"?

Rewrite it as follows,

$$\frac{\partial h}{\partial t} + \frac{1}{\delta^2} h = \left[ -\frac{1}{2} \right] \frac{\partial^2 h}{\partial x^2} \,, \quad h(x, 0) = S(x) \,. \tag{9}$$

The diffusion constant $D = -\frac{1}{2}$ is **negative**.

This is equivalent to running the usual heat/diffusion equation **backward in time**. In terms of heat, heat will travel from regions of colder temperature to regions of hotter temperature. This is equivalent to the action of a **high-pass filter**.

The term $\frac{1}{\delta^2} h$ may be viewed as a damping term (in time).

**Net result of this analysis:** EMD behaves as a high pass filter.

# Our proposed diffusion-based EMD and BEMD algorithms

As discussed previously, most EMD algorithms obtain the mean function from upper and lower envelopes which, in turn, are obtained by interpolating local maxima and minima of a function $S(x)$. These procedures are **time-consuming** (especially in 2D) and **sensitive to error and noise**.

The backward heat equation method of El Hadji *et al.* also avoids these complications. But, as is well known, backward heat equation is numerically unstable.

Our revised diffusion-based EMD method, on the other hand, avoids these complications. It is based on the intuition that that the mean curve $m(x)$ should pass through inflection points of $S(x)$ (see below).

Let us return to Eq. (3) for the mean curve $m_\delta(x)$ obtained by El Hadji *et al.*,

$$m_\delta(x) = \frac{1}{2}[U_\delta(x) + L_\delta(x)] \approx h(x) + \frac{\delta^2}{2} h''(x) . \tag{10}$$

Note that

- $m_\delta(x) \approx h(x)$ if $x$ is a point of inflection (which verifies above intuition).
- $m_\delta(x) < h(x)$ for $x$ near a relative maximum since $h''(x) < 0$.
- $m_\delta(x) > h(x)$ for $x$ near a relative minimum since $h''(x) > 0$.

# Our proposed diffusion-based EMD and BEMD algorithms (cont'd)

This motivates an iterative procedure that is simply driven by the second derivative term $h''(x)$. For a prescribed $\tau > 0$, let

$$h_n(x) = h(x, n\tau)\,, \qquad n = 0, 1, 2, \cdots\,, \tag{11}$$

as before (we have simply replaced $\Delta t$ with $\tau$), and define the iteration procedure,

$$h_{n+1} = h_n + C\frac{\partial^2 h_n}{\partial x^2}\,. \tag{12}$$

Now apply the following Taylor expansion to $h_{n+1}$,

$$\begin{aligned} h_{n+1}(x) = h(x, n\tau + \tau) &= h(x, n\tau) + \tau\frac{\partial h}{\partial t}(x, n\tau) + O(\tau^2) \\ &= h_n(x) + \tau\frac{\partial h_n}{\partial t} + O(\tau^2)\,. \end{aligned} \tag{13}$$

Comparing Eqs. (12) and (13), we arrive at

$$\tau\frac{\partial h}{\partial t} + O(\tau^2) = C\frac{\partial^2 h}{\partial x^2}\,. \tag{14}$$

Now assume that $C = a\tau$, divide by $\tau$ and let $\tau \to 0$ to obtain

$$\frac{\partial h}{\partial t} = a\frac{\partial^2 h}{\partial x^2}\,, \qquad h(x, 0) = S(x)\,. \tag{15}$$

# Our proposed diffusion-based EMD and BEMD algorithms (cont'd)

$$\frac{\partial h}{\partial t} = a\frac{\partial^2 h}{\partial x^2}\,, \quad h(x,0) = S(x)\,. \tag{16}$$

This leads to the following procedure in 1D:

For prescribed values of the diffusivity constant $a > 0$ and a time $T > 0$ (both of which can be adjusted), we now define the **mean function** of $S(x)$ as

$$m(x) = h(x, T)\,, \tag{17}$$

i.e., the solution of the IVP in Eq. (16) at time $T$.

In other words, the mean function $m(x)$ is obtained from $S(x)$ by **low-pass filtering**.

Our method clearly differs from other EMD algorithms since it bypasses (i) the complicated procedure of extracting local maxima and minima as well as (ii) the interpolations of these extrema to obtain upper and lower envelopes.

# Our proposed diffusion-based EMD and BEMD algorithms (cont'd)

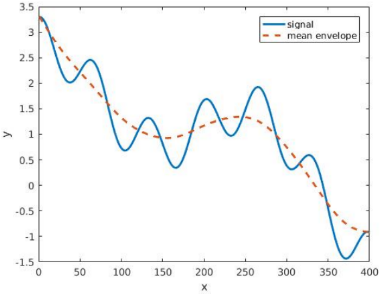



Figure 4.2: Mean Envelope Obtained by Forward Heat Equation

# Our proposed diffusion-based EMD and BEMD algorithms (cont'd)

This leads to the following 1D PDE-based EMD scheme:

Let $S(x)$ be the signal to be analyzed.

1. Initialize: Let $n = 0$ and set $h_0(x, 0) = S(x)$.
2. Find mean of $h_n(x, 0)$: Solve PDE,

$$\frac{\partial h}{\partial t} = a \frac{\partial^2 h}{\partial x^2}, \tag{18}$$

   for $0 \leq t \leq T$. Then define

$$m_n(x) = h_n(x, T). \tag{19}$$

3. Extract mean: Define

$$c_n(x) = h_n(x, 0) - h_n(x, T). \tag{20}$$

4. If $c_n(x)$ is not a BIMF, let $h_{n+1}(x, 0) = c_n(x)$, $n \to n + 1$ and go to Step 2.

**Note:** Since $h_n(x, T)$ is obtained from $h_n(x, 0)$ by **low-pass filtering**, $c_n(x)$ is obtained from $h_n(x, 0)$ by **high-pass filtering**. This is consistent with the analysis of El Hadji *et al.*.

# Our proposed PDE-based 2D EMD/BEMD algorithm for images

We extend the 1D PDE-based method to two dimensions by simply adding another spatial variable to the PDE in Eq. (16). Let $S(x,y)$ be the image function to be analyzed.

1. Initialize: Let $n=0$ and set $h_0(x,y,0) = S(x,y)$.
2. Find mean of $h_n(x,y,0)$: Solve PDE,

$$\frac{\partial h}{\partial t} = D\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right), \tag{21}$$

   for $0 \leq t \leq T$. Then define

$$m_n(x,y) = h_n(x,y,T)\,. \tag{22}$$

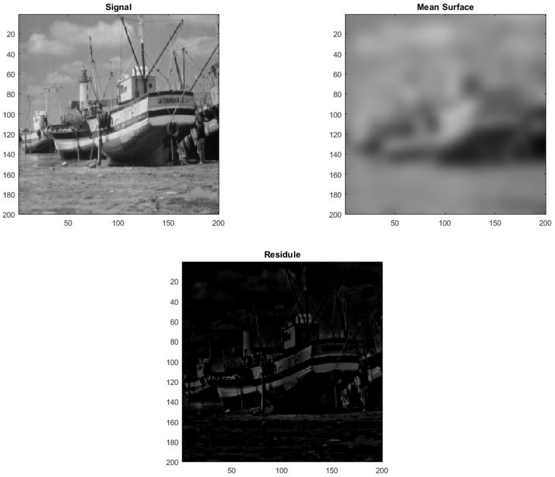
3. Extract mean: Define

$$c_n(x,y) = h_n(x,y,0) - h_n(x,y,T)\,. \tag{23}$$

4. If $c_n(x,y)$ is not a BIMF, let $h_{n+1}(x,y,0) = c_n(x,y)$, $n \to n+1$ and go to Step 2.

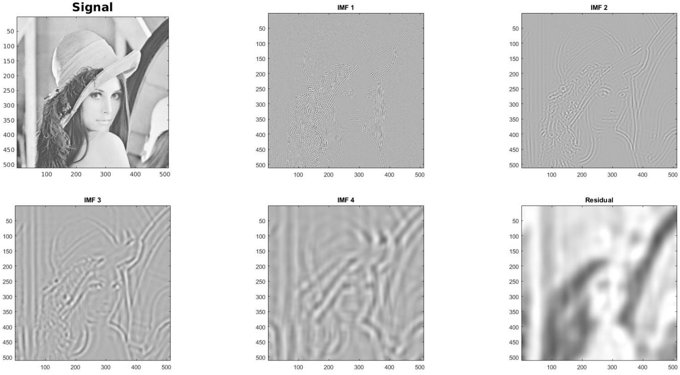
## Some results of our proposed PDE-based 2D EMD/BEMD algorithm

One application of mean surface extraction method to the $512 \times 512$-pixel 8bpp *Boat* image, using parameter values $D = \dfrac{4}{\pi^2}$ and $T = 50$.



# Some results of our proposed PDE-based 2D EMD/BEMD algorithm

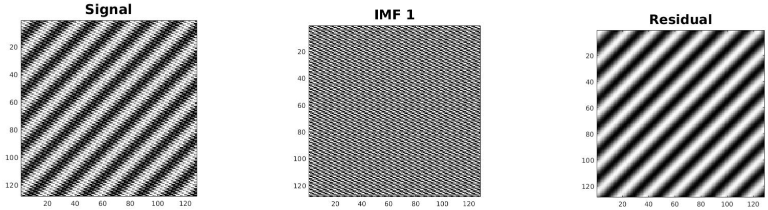
256 $\times$ 256-pixel 8bpp *Lena* image: Original plus first four BIMF's, along with residual.



# Separation of sinusoidal components

512 × 512-pixel 8bpp synthetic image which consists of a mixture of two sine gratings,

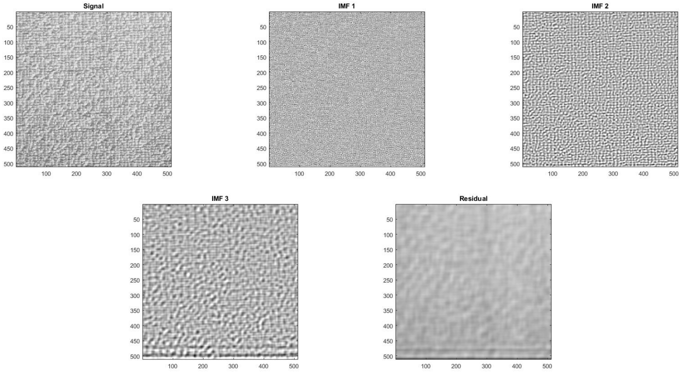
$$S(x,y) = \sin(0.1\pi x + 0.1\pi y) + \sin(-0.4\pi x + 0.8\pi y)\,. \qquad (24)$$




The second (higher frequency) component is extracted as the first BIMF and the first component comprises the residual.

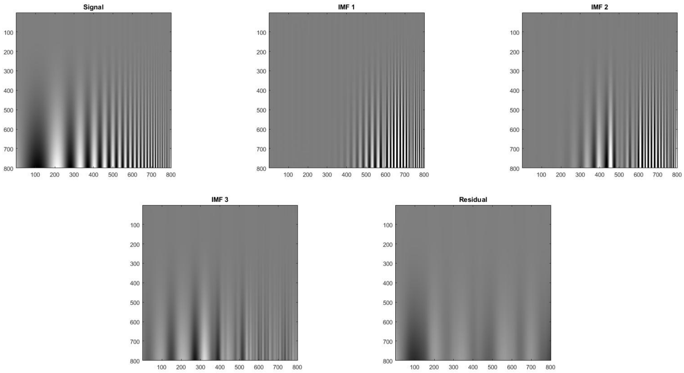
# Texture decomposition

PDE-based 2D EMD algorithm applied to 512 $\times$ 512-pixel 8bpp texture image from Brodatz texture site. Successive BIMFs are comprised of lower frequency components of the texture.



# Contrast-sensitivity function (CSF)

The CSF image demonstrates the sensitivity of an observer to sine wave gratings of differing spatial frequencies.




First BIMF contains highest (horizontal) frequency components from lower right of the CSF. Second BIMF contains slighly lower (horizontal) frequency components. Our method separates (spatially-dependent) frequency components quite well.