The "Waterloo Fractal Analysis and Coding Project:" Generalized fractal transforms, contractive mappings and associated inverse problems

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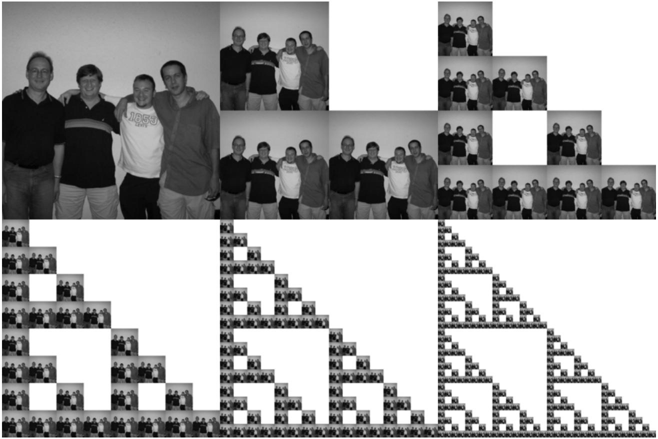
AMMCS 2019
Semi-demi-hemi-quasi-pseudo-virtual-suboptimal-plenary lecture
Special Session on Applied Analysis and Inverse Problems
Tuesday, August 20, 2019

"Waterloo Fractal Analysis and Coding Project"

http://www.links.uwaterloo.ca



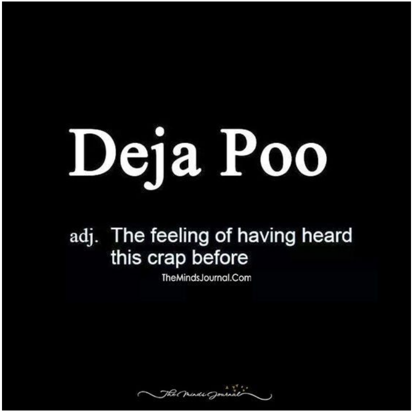
ERV, Herb Kunze (Guelph), Davide La Torre (Milan & ~~Abu Dhabi, Ustana (Kazakhstan), Dubai~~, Sophia Antipolis), Franklin Mendivil (Acadia), ERV, Herb Kunze, Davide La Torre, Franklin Mendivil, ERV, Herb Kunze, Davide La Torre, Franklin Mendivil, ...

The organizers of this session asked me to present a kind of ROMANTIC SURVEY of our work

OK, perhaps more like this

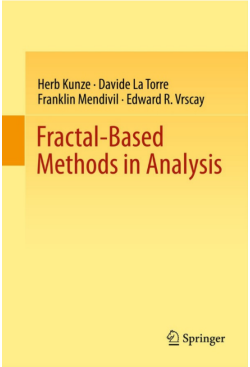
(Wivenhoe Park, J. Constable, 1816, oil on canvas)

To the organizers and the other "cognoscenti": This talk will be



Waterloo Fractal Analysis and Coding Project

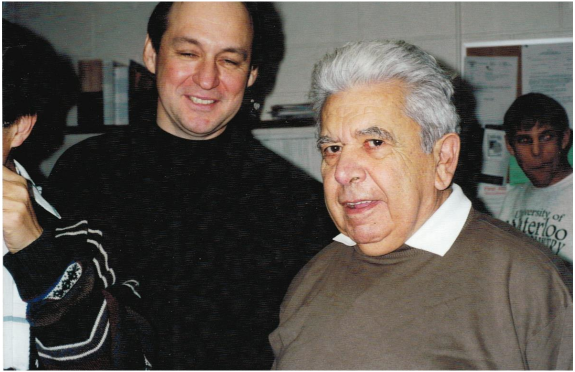
"The Book"



(Springer Verlag 2012)

Dedicated to the memory of:

Bruno Forte (1928-2002)



Bruno Forte †
Department of Applied Mathematics, UW (to 1993)
Universita Degli Studi de Verona, Verona, Italia (1993-2002)

A brief and romantic history

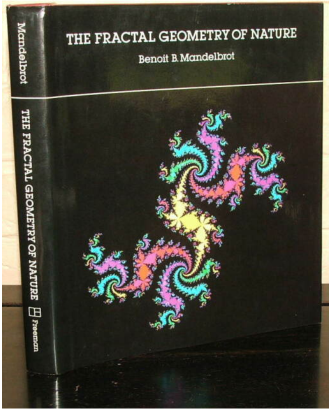
- Dec. 1983: ERV defends his Ph.D. thesis in mathematical physics/quantum mechanics in Dept. of Applied Math, UW.
- 1984-1986: ERV holds an NSERC Postdoctoral Fellowship at the School of Mathematics, Georgia Tech under the supervision of Michael Barnsley. This is an opportunity for him to "retrain" in the area of fractal geometry and dynamical systems.
- 1986: ERV returns to Applied Math Dept., UW as an NSERC University Research Fellow. The Chair of the Department at that time was Bruno Forte, whose expertise included analysis, functional equations and information theory.
- 1987: ERV teaches a graduate course on topics in dynamical systems/ fractal geometry, which includes IFS. Bruno Forte, who originally viewed the work by "the chemist" (ERV) as "$%*@" becomes "hooked" on IFS, especially on the inverse problem. The birth of the Forte-Vrscay collaboration.
- 1991-1993: C. Cabrelli and U. Molter are at UW on leave from U. Buenos Aires. The first GFT: "Iterated Fuzzy Set Systems."
- 1993: Bruno retires from UW and assumes an Emeritus Professorship in Verona, Italy. Collaboration with ERV continues.

Waterloo Fractal Analysis and Coding Project

- 1994: John Kominek, an undergraduate student in Computer Science at Waterloo who is interested in fractal image compression, suggests to ERV that he should consider having a website for the Waterloo fractal work. (This is during the early days of the "WWW.") John sets up such a website, where he also launches his brainchild, the Waterloo "BragZone", a repository of images, papers and software for fractal image compression.
- 1995: Davide La Torre works on his Master's thesis research at University of Milan under Bruno's co-supervision. Subject of research: Inverse problem of measure approximation for IFS with probabilities.
- 1996: The "Waterloo Fractal Compression Project" began as part of a collaborative effort involving the University of Waterloo (ERV), the Ecole Polytechnique in Montreal (C. Tricot) and the Groupe Fractales of the INRIA Labs in Rocquencourt, France (J. Levy-Vehel). This project was made possible with funding from an NSERC Collaborative Research Grant.
- 1996: ERV teaches graduate course on IFS theory. Herb Kunze, a Ph.D. student in Applied Math, takes this course and becomes another victim of the "IFS bug."
- 1997: Franklin Mendivil, a recent Ph.D. graduate in topology from the School of Mathematics, Georgia Tech, comes to Waterloo to work as a postdoctoral researcher. Franklin's position is funded by the NSERC Collaborative Grant.

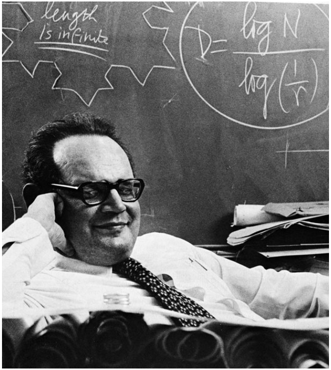
- 1998: Herb Kunze, after finishing his Ph.D. in Applied Math at Waterloo in the area of ordinary differential equations, works as a postdoctoral researcher with ERV. Result: The birth of "collage coding" for differential equations. Herb then moves to the Dept. of Mathematics and Statistics at the University of Guelph, eventually to become a faculty member there.
- 2000: Davide La Torre defends his Ph.D. thesis (in Optimization) at the University of Milan. He already has a position as Assistant Professor in the Department of Economics, University of Milan.
- 2006: Davide La Torre spends a six-month sabbatical leave at UW to work with ERV. This is the start of a beautiful set of friendships and research collaborations with ERV, FM and HK which continue to this day.

In the beginning was “THE Book”



W.H. Freeman and Company, 1982

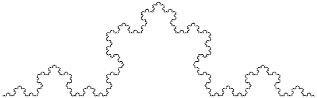
written by the “Father of Fractal Geometry”



Benoit B. Mandelbrot (1924-2010)

The Fractal Geometry of Nature

In *The Fractal Geometry of Nature*, Mandelbrot presented the first description, along with an extensive catalogue, of **self-similar sets**: sets that may be expressed **as unions of contracted copies of themselves**, e.g.,



The celebrated "von Koch curve" C

He called these sets **"fractals"** because their (fractional) Hausdorff dimensions exceeded their (integer-valued) topological dimension.

The von Koch curve C is a bona fide curve in the plane: Topological dimension $= 2$.

Its Hausdorff dimension, however, is $D = \dfrac{\log 4}{\log 3} \sim 1.26$.

C has infinite length (dimension 1) and zero area (dimension 2).

Mandelbrot viewed fractal sets as limits of a recursive procedure that involved **"generators"**. A generator G contains a set of rules – a kind of "grammar" – for operations on sets: Some generators act on line segments, others on areas or volumes.

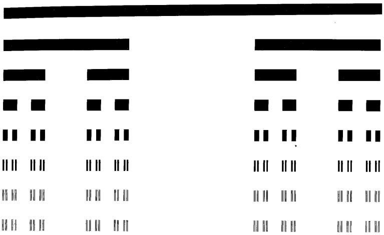
Starting with an appropriate "seed" set S_0 (in $\mathbb{R}^2$ for example), we form the iteration sequence,

$$S_{n+1} = G(S_n)\,, \quad n = 0, 1, 2, \cdots .$$

In the limit $n \to \infty$, the sets S_n approach a limit set S which is typically a fractal.

Fractal construction using "generators"

Celebrated example: Ternary Cantor Set



Plates 80 and 81 ¤ CANTORIAN TRIADIC BAR AND CAKE (HORIZONTAL SECTION DIMENSION D=log 2/log 3=0.6309). SATURN'S RINGS. CANTOR CURTAINS.

The Cantor dust uses [0,1] as initiator, and its generator is

N=2
r=⅓
D=log 2/log 3=.6309

PLATE 80. The Cantor dust is extraordinarily difficult to illustrate, because it is thin and spare to the point of being invisible. To help intuition by giving an idea of its form, thicken it into what may be called a Cantor bar. ◅ In technical terms, this is the Cartesian product of a Cantor dust of length 1 by an

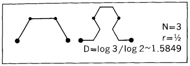
CURDLING. The construction of the Cantor bar results from the process I call *curdling*. It begins with a round bar (seen in projection as a rectangle in which width/length=0.03). It is best to think of it as having a very low density. Then matter "curdles" out of this bar's middle third into the end thirds, so that the positions of the latter remain unchanged. Next matter curdles out of the middle third of each end third into its end thirds, and so on ad infinitum until one is left with an infinitely large number of infinitely thin slugs of infinitely high density. These slugs are spaced along the line in the very specific fashion induced by the generating process. In this illustration, curdling (which eventually requires hammering!) stops when both the printer's press and our eye cease to follow; the last line is indistinguishable from the last but one:

Fractal construction using "generators"

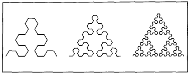
"Sierpinski gasket"

Plate 141, OVERLEAF ¤ SIERPIŃSKI ARROWHEAD (BOUNDARY DIMENSION D~1.5849)

In Sierpiński 1915, the initiator is [0,1], and the generator and second teragon are



This construction's next two stages are

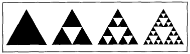


And an advanced stage is shown as the "coastline" of the upper portion of Plate 141 (above the largest solid black triangle).

SELF-CONTACTS. Finite construction stages are free of points of self-contact, as in Chapter 6, but the limit curve *does* self-contact infinitely often.

TILING ARROWHEADS. The arrowhead in Plate 141 (turned sideways, it becomes a tropical fish) is defined as a piece of the Sierpiński curve contained between two successive returns to a point of self-contact, namely the midpoint of [0,1]. Arrowheads tile the plane, with neighboring tiles being linked together by a nightmarish extrapolation of Velcro. (To mix metaphors, one fish's fins fit exactly those of two other fish). Furthermore, by fusing together four appropriately chosen neighboring tiles, one gets a tile increased in the ratio of 2.

THE SIERPIŃSKI GASKET'S TREMAS. I call Sierpiński's curve a *gasket*, because of an alternative construction that relies upon cutting out "tremas," a method used extensively in Chapters 8 and 31 to 35. The Sierpiński gasket is obtained if the initiator, the generator, and next two stages are these closed sets:



This trema generator includes the above stick generator as a proper subset.

WATERSHED. I first encountered the arrowhead curve without being aware of Sierpiński, while studying a certain watershed in Mandelbrot 1975m. ■

Plate 143 ¤ A FRACTAL SKEWED WEB (DIMENSION D=2)

This web obtains recursively, with N=4 and r=½, using a closed tetrahedron as initiator and a collection of tetrahedrons as generator.

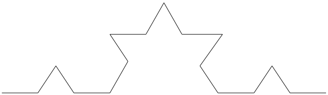
Its dimension is D=2. Let us project it along a direction joining the midpoints of either couple of opposite sides. The initiator tetrahedron projects on a square, to be called initial. Each second-generation tetrahedron projects on a subsquare, namely (¼)th of the initial square, etc. Thus, the web projects on the initial square. The subsquares' boundaries overlap. ■

Fractal construction using “generators”

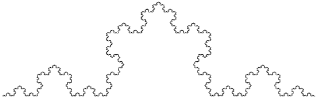
von Koch curve

Consider the following generator G:

After another application:



In the limit:



Each component of the generator produces a contracted copy of the limit set.

From generators to geometric methods of fractal construction

The following seminal works showed how one could construct fractal sets (and measures) geometrically – and therefore study them analytically – using **systems of contractive mappings** which operate in a parallel fashion.

- J. Hutchinson, Fractals and self-similarity, Indiana Univ. J. Math. **30**, 713-747 (1981).
  - Geometric and measure theoretic aspects of systems of contractive maps with associated probabilities, incl. invariant sets and probability measures supported on these sets.
- M.F. Barnsley and S.G. Demko, Iterated function systems and the global construction of fractals, Proc. Roy. Soc. London A **399**, 243-275 (1988).
  - A later, and independent, discovery of such systems of mappings and associated attractors and invariant measures, but in a more probabilistic setting, i.e., random process.
  - Perhaps the first solution of an "inverse problem" of fractal construction (by "moment matching").
  - As is clear from the title, this is the origin of the term "iterated function systems" (IFS).

Notes:

1. Michael Barnsley and colleagues were actually working with systems of mappings much earlier than the 1988 paper would suggest. These mappings, however, were nonlinear and noncontractive – specifically, they were the inverse maps of rational mappings $R(z)$ in the complex plane. The attractors of these systems were the Julia sets of $R(z)$. The idea of IFS with contractive maps came from playing with linearizations of these maps (and ERV saw him playing with them around 1984).
2. B. Mandelbrot would eventually refer to systems of contractive mappings not as IFS but as **"Map Bags"**.

From generators to geometric methods of fractal construction

Example: Consider the following two maps $f_i : [0,1] \to [0,1]$:

$$f_1(x) = \frac{1}{3}x\,,\;\; f_1(0) = 0\,, \qquad f_2(x) = \frac{1}{3}x + \frac{2}{3}\,,\;\; f_2(1) = 1\,.$$

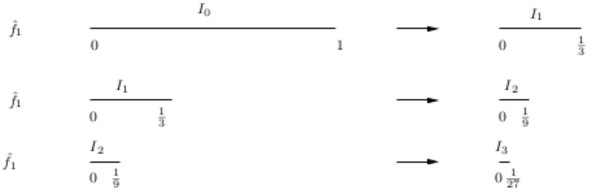
Each map f_i is contractive on [0,1] and has a unique fixed point $\bar{x}_i \in [0,1]$. Each fixed point $\bar{x}_i$ is attractive under iteration of f_i.

For each map, f_i, define its set-valued counterpart $\hat{f}_i$ as follows: For $S \subseteq [0,1]$, denote

$$\hat{f}_i(S) = \{ f_i(x) \,|\, x \in S \}\,.$$

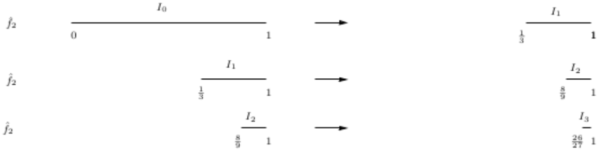
Action of $\hat{f}_1$

$$\hat{f}_1([0,1]) = \left[0, \frac{1}{3}\right]$$



Action of $\hat{f}_2$

$$\hat{f}_2([0,1]) = \left[\frac{2}{3}, 1\right] .$$



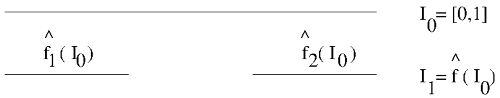
From generators to geometric methods of fractal construction

Now define the following “parallel machine” set-valued mapping,

$$\hat{\mathbf{f}}(S) = \hat{f}_1(S) \cup \hat{f}_2(S)\,, \quad \forall S \in [0,1].$$

Then the action of $\hat{\mathbf{f}}$ on the set $I_0 = [0,1]$ is

$$\begin{aligned} I_1 = \hat{\mathbf{f}}([0,1]) &= \hat{f}_1([0,1]) \cup \hat{f}_2([0,1]) \\ &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]. \end{aligned}$$

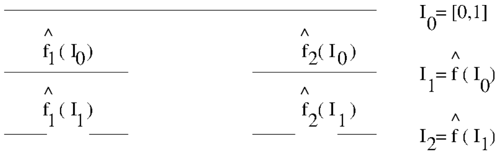


Is this starting to look like something we’ve seen before?

From generators to geometric methods of fractal construction

Now apply $\hat{\mathbf{f}}$ to the set I_1:

$$
\begin{aligned}
I_2 = \hat{\mathbf{f}}(I_1) &= \hat{f}_1(I_1) \cup \hat{f}_2(I_1) \\
&= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{1}{9}, 1\right] .
\end{aligned}
$$



Voilà! We see that repeated application of the "parallel machine" $\hat{\mathbf{f}}$ performs the "middle-thirds dissection procedure" that was employed in the construction of the ternary Cantor set C in $[0, 1]$.

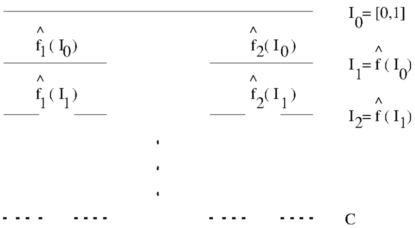
From generators to geometric methods of fractal construction

If we now consider the following iteration process involving the parallel map $\hat{\mathbf{f}}$ acting on sets:

$$I_{n+1} = \hat{\mathbf{f}}(I_n)\,,$$

then it appears that

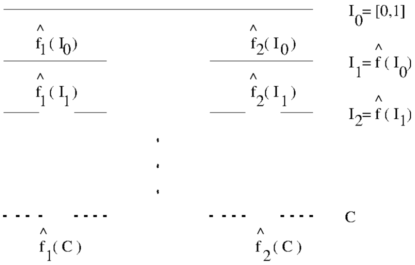
$$\text{``}\lim_{n\to\infty} I_n = C\text{''}\,, \qquad \text{ternary Cantor set in } [0,1]\,.$$



From generators to geometric methods of fractal construction

But the story is not over! **The ternary Cantor set is the unique "fixed point" of the parallel operator $\hat{\mathbf{f}}$**, i.e.,

$$C = \hat{\mathbf{f}}(C) = \hat{f}_1(C) \cup \hat{f}_2(C)\,.$$



Recalling that the set-valued maps $\hat{f}_i$ "shrink" sets, we see that

the ternary Cantor set C is a union of contracted copies of itself.

The maps f_1 and f_2 comprise an **iterated function system (IFS)**.

Systems of contraction mappings

Before we go on, the idea of contraction maps which operate in a parallel fashion had been around – in some way, shape or form – for quite some time, but with no idea of fractal construction, e.g.,

- R.F. Williams, Composition of contractions, Bol. Soc. Brasil Mat. **2**, 55-59 (1971). Fixed points of finite compositions of contraction maps.
- S. Nadler, Multi-valued contraction mappings, Pacific J. Math. **30**, 475-488 (1969). Systems of contraction maps considered as defining "multifunctions."

However,

- S. Karlin, Some random walks arising in learning models, I, Pacific J. Math. **3**, 725-756 (1953). Random walks over Cantor-like sets on $[0,1]$ and associated measures - essentially the "Chaos Game" of Barnsley-Demko.

Iterated Function Systems (IFS)

Ingredients:

- (X, d): A complete metric space (e.g., $[0, 1]^n$ with Euclidean metric)
- $(\mathcal{H}(X), h)$: Complete metric space of non-empty compact subsets of X with Hausdorff metric h
- $w_i : X \to X$, $1 \leq i \leq N$: Set of contraction maps on X with contraction factors $c_i \in [0, 1)$.

Associated with each w_i is a set-valued mapping $\hat{w}_i : \mathcal{H}(X) \to \mathcal{H}(X)$, where

$$\hat{w}_i(S) = \{w_i(x), \, x \in S\} \;\; \forall \; S \in \mathcal{H}(X) \,.$$

IFS operator $\hat{\mathbf{w}}$ associated with N-map IFS $\mathbf{w}$ defined as follows:

$$\hat{\mathbf{w}}(S) = \bigcup_{i=1}^{N} \hat{w}_i(S) \,, \quad S \in \mathcal{H}(X) \,.$$

Theorem (Hutchinson): $\hat{\mathbf{w}} : \mathcal{H}(X) \to \mathcal{H}(X)$ is contractive,

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \leq c \, h(A, B) \;\; \forall \; A, B \in \mathcal{H}(X) \,,$$

where

$$c = \max_{1 \leq i \leq N} c_i < 1 \,.$$

Important consequence:

From Banach's Fixed Point Theorem, there exists a unique compact set $A \in \mathcal{H}(X)$ which is the fixed point of $\hat{\mathbf{w}}$, i.e., $\hat{\mathbf{w}}(A) = A$, i.e.,

$$A = \bigcup_{i=1}^{N} A_i \quad \text{where} \quad \hat{w}_i(A)\,, 1 \leq i \leq N.$$

A is "self-similar," i.e., a union of contracted copies of itself.

Furthermore: For any $S_0 \in \mathcal{H}(X)$, define the iteration sequence

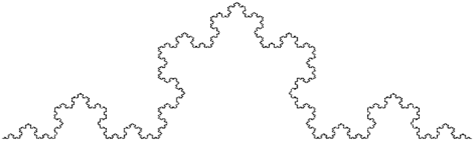
$$S_{n+1} = \hat{w}(S_n) = \bigcup_{i=1}^{N} \hat{w}_i(S_n)\,.$$

Then

$$\lim_{n \to \infty} h(S_n, A) = 0\,.$$

A is the unique (global) *attractor* of the IFS $\hat{\mathbf{w}}$.

von Koch curve



Attractor of a 4-map affine IFS in $\mathbb{R}^2$:

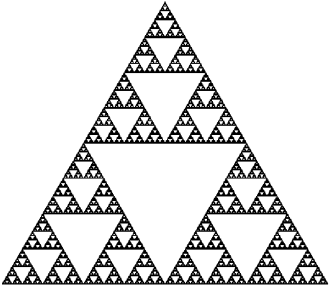
$$w_1(x,y) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$w_2(x,y) = \begin{pmatrix} \frac{1}{6} & -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$$

$$w_3(x,y) = \begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$$w_4(x,y) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

"Sierpinski triangle (gasket)"



Attractor of a 3-map affine IFS in $\mathbb{R}^2$: $w_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$,

$w_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$, $w_3(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)$.

Another celebrated example: Barnsley's spleenwort fern

Attractor of a 4-map affine IFS in $\mathbb{R}^2$.

Of course, this leads to the question, "Can we use IFS to generate other interesting sets? Plants? Trees? Faces? ... Anything?"

This is an inverse problem

Random iteration algorithm or "Chaos Game" to generate pictures of IFS attractors

- Let (X, d) be a compact metric space (e.g., $[0,1]^n$ with Euclidean metric)
- Let $w_i : X \to X$, $1 \le i \le N$ be a set of contraction maps
- Associated with each IFS map w_i is a (nonzero) probability $p_i \in [0,1]$ such that
$$\sum_{i=1}^{N} p_i = 1 \ .$$

Given a "seed" point $x_0 \in X$, construct the random iteration sequence $\{x_n\}$ as follows,

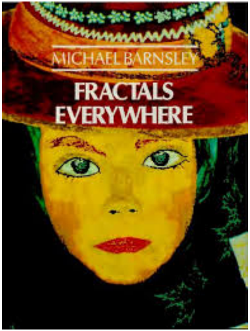
$$x_{n+1} = w_{\sigma_n}(x_n)\,, \quad n = 0, 1, 2, \cdots\,,$$

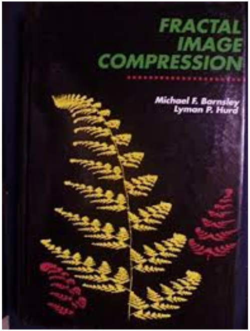
where σ_n is chosen randomly from the index set $\{1, \cdots, N\}$ with $P(\sigma_n = i) = p_i$, $1 \le i \le N$.

Because of the contractivity of the w_i maps, the iterates x_n approach the attractor A of the IFS $\mathbf{w}$. (If $x_k \in A$ for some $k \ge 0$, then $x_n \in A$ for all $n \ge k$.)

For some suitably large integer $M > 0$, plot iterates $\{x_n\}$ for $n > M$ to obtain a picture of the attractor A of the IFS $\mathbf{w}$.

Michael Barnsley, the "Father of Fractal Image Compression"

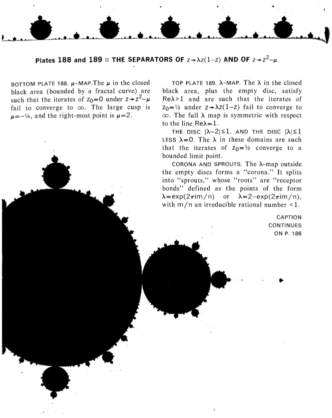




- Ph.D. in Theoretical Chemistry, University of Wisconsin, 1972.
- 1973-76: Postdoc at U. Bradford. Research work in Padé approximants and the moment problem.
- 1976-1979: Postdoc at Centre d'Etudes Nucléaires, Saclay, France. Further work on Padé approximants, moment problems in theoretical physics (with Daniel Bessis).
- Professor, School of Mathematics, Georgia Institute of Technology (1979-1991)
- Co-founder (with Alan Sloan) of Iterated Systems, Inc., 1989

Back to Mandelbrot's *Fractal Geometry of Nature* for a moment

The generators shown earlier correspond to **linear** geometric maps. A large portion of Mandelbrot's book is also devoted to **nonlinear** maps. After all, Mandelbrot discovered the so-called **Mandelbrot set** associated with quadratic complex maps of the form $f_c(z) = z^2 + c$.



Plates 188 and 189 ⌑ THE SEPARATORS OF $z \to \lambda z(1-z)$ AND OF $z \to z^2 - \mu$

BOTTOM PLATE 188. μ-MAP. The μ in the closed black area (bounded by a fractal curve) are such that the iterates of $z_0=0$ under $z \to z^2-\mu$ fail to converge to ∞. The large cusp is $\mu=-1/4$, and the right-most point is $\mu=2$.

TOP PLATE 189. λ-MAP. The λ in the closed black area, plus the empty disc, satisfy $\mathrm{Re}\lambda>1$ and are such that the iterates of $z_0=1/2$ under $z \to \lambda z(1-z)$ fail to converge to ∞. The full λ map is symmetric with respect to the line $\mathrm{Re}\lambda=1$.

THE DISC $|\lambda-2|\leq 1$, AND THE DISC $|\lambda|\leq 1$ LESS $\lambda=0$. The λ in these domains are such that the iterates of $z_0=1/2$ converge to a bounded limit point.

CORONA AND SPROUTS. The λ-map outside the empty discs forms a "corona." It splits into "sprouts," whose "roots" are "receptor bonds" defined as the points of the form $\lambda=\exp(2\pi i m/n)$ or $\lambda=2-\exp(2\pi i m/n)$, with m/n an irreducible rational number <1.

CAPTION CONTINUES ON P. 186

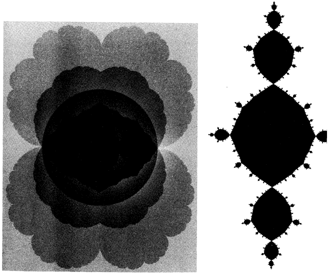


Plate 185 ⌑ SELF-SQUARED FRACTAL CURVES FOR REAL λ

The shapes in Plates 185 to 192 are presented here for the first time, except for a few that are reproduced from Mandelbrot 1980n.

The left side of this plate represents the maximal bounded self-squared domains for λ = 1, 1.5, 2.0, 2.5 and 3.0. The central black shape spans the segment [0,1].

$\lambda=1$: SCALLOP SHELL.

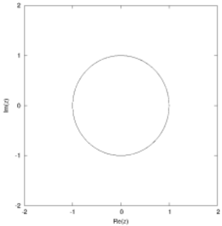
$\lambda=3$: SAN MARCO DRAGON CURVE. This is a mathematician's wild extrapolation of the skyline of the Basilica in Venice, together with its reflection in a flooded Piazza; I nicknamed it the *San Marco dragon*.

The right side of this plate is relative to $\lambda=3.3260680$. This is the nuclear λ (as defined on p. 184) corresponding to $w=2$. The corresponding self-squared shape is turned by 90° to make it fit in. ■

Julia sets and Mandelbrot sets - a very brief look (less than a primer)

Definition: Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function. Then the **Julia set** of f, to be denoted as J_f, is the closure of the set of all repulsive k-cycles of f, $k = 1, 2, \cdots$.

Example: Consider the simple quadratic map $f(z) = z^2$. Then $J_f = C = \{z \in \mathbb{C} \mid |z| = 1\}$, the unit circle in the complex plane $\mathbb{C}$ centered at 0.



Julia sets and Mandelbrot sets - a very brief look (less than a primer)

A few properties of the Julia set J_f of a rational map $f : \mathbb{C} \to \mathbb{C}$.

- J_f is an invariant set, i.e., $f : J_f \to J_f$.
- Consequence of the above, $f_k^{-1} : J_f \to J_f$ for any inverse of f.
- J_f is a **repeller set** under the action of f: points close to J_f but not on it are mapped farther away from J_f.
- A consequence of the previous property: J_f is an **attractor set** under the action of f_k^{-1}.

Julia sets and Mandelbrot sets - a very brief look (less than a primer)

Return to Example: $f(z) = z^2$ with Julia set $J_f = C$, unit circle centered at 0. Inverse maps of f are: $f_1^{-1}(z) = \sqrt{z}$ and $f_2^{-1}(z) = -\sqrt{z}$.

(For convenience) let us define $\sqrt{z}$ as follows: For $z = re^{i\theta}$, $\theta \in [0, 2\pi)$, then $\sqrt{z} = r^{1/2}e^{i\theta/2}$ (branch cut on non-negative real axis). Then,

- $f_1^{-1} : C \to C_+ = \{z \in \mathbb{C} \mid |z| = 1\,, \mathrm{Re}(z) \geq 1\}$ (upper semicircle)
- $f_2^{-1} : C \to C_- = \{z \in \mathbb{C} \mid |z| = 1\,, \mathrm{Re}(z) \leq 1\}$ (lower semicircle)

Note that $C = C_- \cup C_+$. Consequently,

$$C = \hat{f}_1^{-1}(C) \cup \hat{f}_2^{-1}(C)\,,$$

The Julia set C is the unique **fixed point** of the "IFS" composed of the two (noncontractive) maps f_1^{-1} and f_2^{-1}!

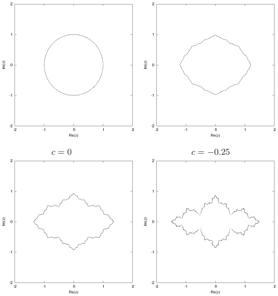
It is also the **attractor set** of this IFS.

This result generalized to the general one-parameter family of quadratic complex maps $f_c(z) = z^2 + c$ with inverse maps $f_1^{-1}(z) = \sqrt{z - c}$, $f_2^{-1}(z) = -\sqrt{z - c}$.

These invariance properties for rational complex maps were studied by Barnsley and coworkers before their work on (contractive) IFS. Reason: Application to theoretical physics/condensed matter physics ("Renormalization Theory").

Julia sets and Mandelbrot sets - a very brief look (less than a primer)

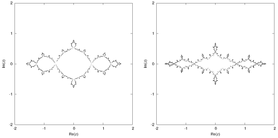
Some Julia sets J_c for $g_c(z) = z^2 + c$



$c = 0$ $c = -0.25$

$c = -0.5$ $c = -0.75$

At $c = -0.75$, second fixed point, $\bar{x}_2$ ceases to be attractive, hence J_c touches real axis to include it.

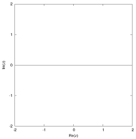


$c = -1.0$ $c = -1.25$

At $c = -1.25$, two-cycle ceases to be attractive, hence J_c touches real axis to include them.

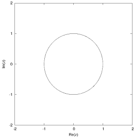
Julia sets and Mandelbrot sets - a very brief look (less than a primer)

Some Julia sets J_c for $g_c(z) = z^2 + c$

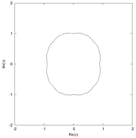


$c = -2$ $J_c = [-2, 2]$

Now, the "other direction" on $\mathbb{R}$ from $c = 0$:

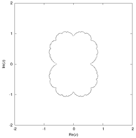


$c = 0.0$

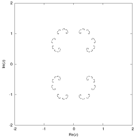


$c = 0.1$

For $c > 0$, repulsive fixed point $\bar{x}_1$ moves from 1 inward.

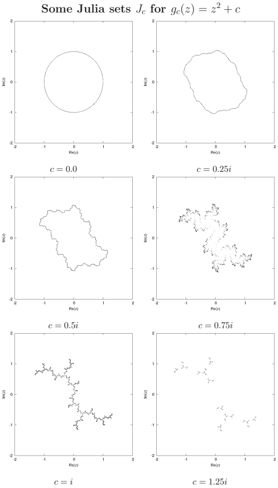


$c = 0.25$



$c = 0.5$

Julia sets and Mandelbrot sets - a very brief look (less than a primer)

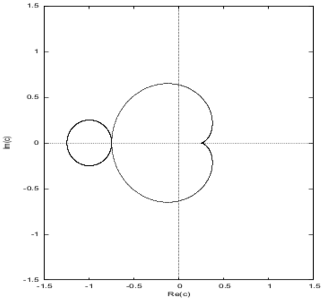


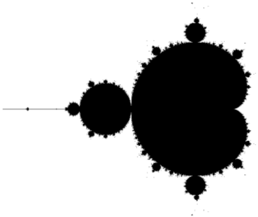
For $c = i$, J_c is dendritic. For $c = 1.25i$, J_c is totally disconnected.

Figure 3:

Julia sets and Mandelbrot sets - a very brief look (less than a primer)

And finally, the Mandelbrot set M **for the one-parameter family of complex maps** $f_c = z^2 + c$





M is the set of all parameter values $c \in \mathbb{C}$ for which the Julia set J_c of f_c is **connected**.

Inverse problem of fractal approximation

And now back to: Barnsley's spleenwort fern

Attractor of a 4-map affine IFS in $\mathbb{R}^2$.

Of course, this leads to the question, "Can we use IFS to generate other interesting sets? Plants? Trees? Faces? ... Anything?"

This is an inverse problem

First thoughts on how to solve such an inverse problem

Suppose we have a (bounded) set $S \subset \mathbb{R}^2$, for example, another leaf-like set: Do we just start playing around with (affine) contraction maps in the plane, perturbing them, generating attractors, etc.?

Perhaps a more clever approach: We're trying to approximate S by the attractor A of an N-map IFS, i.e.,

$$S \approx A = \bigcup_{i=1}^{N} \hat{w}_i(A) = \hat{\mathbf{w}}(A)\,.$$

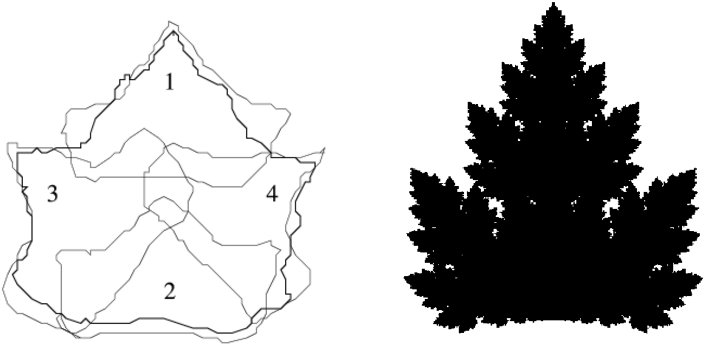
If $S \approx A$, then $\hat{w}_i(A) \approx \hat{w}_i(S)$, which implies that

$$S \approx \bigcup_{i=1}^{N} \hat{w}_i(S) = \hat{\mathbf{w}}(S)\,.$$

In other words: **We try to approximate S as a union of contracted copies of itself.**

Presumably, the closer S is to $\hat{\mathbf{w}}(S)$, a union , or "collage," of contracted copies of itself, the closer it is to the attractor A of the IFS $\hat{\mathbf{w}}$, i.e., the better it is approximated by A. Maybe we can express this more mathematically ... a little later.

Inverse problem by "collaging"



Left: Approximating a leaf S (darker boundary) with four contracted copies $\hat{w}_i(S)$, $1 \leq i \leq 1$ of itself. **Right:** The attractor of the resulting IFS $\hat{\mathbf{w}}$.

Banach's Fixed Point Theorem*

also known as the "Contraction Mapping Theorem", "Contraction Mapping Principle," etc.

is central to most (almost all?) fractal-based methods:

Theorem: Let (Y, d_Y) be a complete metric space and $T : Y \to Y$ a contraction mapping, i.e., $d_Y(Ty_1, Ty_2) \leq c d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$ where $c \in [0, 1)$. Then there exists a unique $\bar{y} \in Y$ such that

- $T\bar{y} = \bar{y}$ (fixed point of T)
- $d_Y(T^n y_0, \bar{y}) \to 0$ as $n \to \infty$ (attractive fixed point)

*S. Banach, Sur les opérations dans les ensembles abstraites et leurs applications aux équations integrales, Fund. Math. **3** 133-181 (1922). The CMT is in an appendix to this paper, which is based on Banach's Ph.D. thesis.

We now formulate the following

Inverse problem of approximation by fixed points of contraction mappings

Let (Y, d_Y) be a complete metric space and $Con(Y)$ the set of all contraction maps $T : Y \to Y$. Now let $Con'(Y) \subset Con(Y)$ be a particular class of contraction maps that we wish to consider. (For example, in $\mathbb{R}^2$, the set of all N-map affine IFS, $N = 1, 2, \cdots$).

Then given a $y \in Y$ (our "target") and an an $\epsilon > 0$, can we find a $T \in Con'(Y)$ with fixed point $\bar{y}$ such that

$$d_Y(y, \bar{y}) < \epsilon\,?$$

In other words, can we approximate y with the fixed point $\bar{y}$ to ϵ-accuracy?

In general, especially for fractal transforms, this problem is intractable. The following "collaging" result simplifies the problem.

The "Collage Theorem"

(That's what it's called in the fractal coding literature.)

Theorem: Let (Y, d_Y) be a complete metric space and T a contraction map on Y with contraction factor $c_T \in [0,1)$ and fixed point $\bar{y}$. Then for any $y \in Y$,

$$d_Y(y, \bar{y}) \leq \frac{1}{1 - c_T}\, d_Y(Ty, y)\,.$$

$$\left[\begin{array}{c} \text{Error in approximating} \\ y \text{ with } \bar{y} \end{array} \right] \leq K(T) \left[\begin{array}{c} \text{"Collage error" in} \\ \text{approximating } Ty \text{ with } y \end{array} \right]$$

"Collage coding:" Try to make $d_Y(y, \bar{y})$ by finding a $T \in Con'(Y)$ that makes the collage error $d_y(y, Ty)$ as small as possible. Or rephrase as: Given a $y \in Y$ and a $\delta > 0$, find T so that

$$d_Y(Ty, y) < \delta\,.$$

Note: T does NOT have to be a fractal-type operator. More on this later.

The "Collage Theorem"

The "Collage Theorem" was proved in an IFS setting in

- M.F. Barnsley, V. Ervin, D. Hardin and J. Lancaster, Solution of an inverse problem for fractals and other sets, Proc. Nat. Acad. Sci. USA **83**, 1975-1977 (1985).

(There is an entry for it, once again in an IFS setting, in Wikipedia.)

The general result given earlier is presented as a Remark to Banach's Theorem in

- D. Smart, *Fixed Point Theorems*, Cambridge University Press, London (1974).

Simple proof: Just play around – in the right way – with y, $\bar{y}$ and Ty, using – what else? – the triangle inequality:

$$\begin{aligned} d_Y(y,\bar{y}) &\leq d_Y(y,Ty) + d_Y(Ty,\bar{y}) \\ &= d_Y(y,Ty) + d_Y(Ty,T\bar{y}) \\ &\leq d_Y(y,Ty) + c_T d_Y(y,\bar{y}), \end{aligned}$$

and the desired result follows, i.e.,

$$d_Y(y,\bar{y}) \leq \frac{1}{1-c_T}\, d_Y(Ty,y).$$

The "Collage Theorem"

But, like, what if you play with y, Ty and $\bar{y}$ in the "wrong way", i.e., start with y and Ty:

$$\begin{aligned} d_Y(y, Ty) &\leq d_Y(y, \bar{y}) + d_Y(\bar{y}, Ty) \\ &= d_Y(y, \bar{y}) + d_Y(T\bar{y}, Ty) \\ &\leq d_Y(y, \bar{y}) + c_T d_Y(\bar{y}, y)\,, \end{aligned}$$

which yields

The "Anti-Collage Theorem"

$$d_Y(y, \bar{y}) \geq \frac{1}{1 + c_T} d_Y(Ty, y)\,.$$

Net result:

$$\frac{1}{1 + c_T} d_Y(Ty, y) \leq d_Y(y, \bar{y}) \leq \frac{1}{1 - c_T} d_Y(Ty, y)\,.$$

The Collage Theorem

The Collage Theorem provides a systematic method for producing fixed-point approximations to a "target" element $y \in Y$.

In practice, a contraction map of interest, $T \in Con'(Y)$, will be defined uniquely by a number of coefficients, say, $\mathbf{c} = \{c_1, c_2, \cdots, c_M\} \in \Pi_M$. Here, $\Pi_M \subset \mathbb{R}^M$ is a **feasible parameter space** which guarantees the contractivity of the T_M.

Given a $y \in Y$, and a fixed M, we find the "best" contraction map $T_M \in Con'(Y)$ with M parameters by solving the following optimization problem,

$$\min_{\mathbf{c} \in \Pi_M} d_Y(T(\mathbf{c})y, y)\,.$$

Then increase M as desired (or until your program "blows up.")

Usually, d_Y is some kind of Eucldean metric. Minimization of the squared collage distance yields a quadratic optimization problem for the c_k with constraints.

Continuity of fixed points of contraction maps*

This property, generally ignored in the fractal coding literature, makes "collage coding" possible.

The idea: If you "tweak" a contraction map T_1 slighly to produce a new contraction map T_2, you expect their respective fixed points to be close to each other.

Theorem: Let (Y, d_Y) be a **compact** metric space. Define the following metric on $Con(Y)$: For $T_1, T_2 \in Con(Y)$,

$$d_{Con(Y)}(T_1, T_2) = \max_{y \in Y} d_Y(T_1 y, T_2 y) .$$

If $\bar{x}_1$ and $\bar{x}_2$ are the unique fixed points of T_1 and T_2, respectively, then

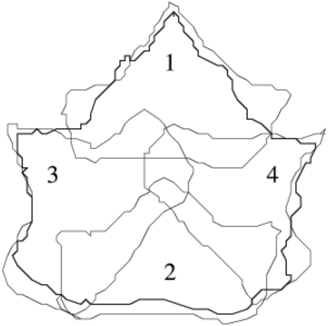
$$d_Y(\bar{x}_1, \bar{x}_2) \leq \frac{1}{1-c} \, d_{Con(Y)}(T_1, T_2) .$$

Here, $c = \min\{c_1, c_2\}$ where c_1 and c_2 are the contraction factors of T_1 and T_2, respectively.

Remark: In practice, $T(\mathbf{c})$ is continuous with respect to its parameters c_i. In fact, it's usually differentiable as well.

*P. Centore and ERV, Continuity of attractors and invariant measures of iterated function systems, Can. Math. Bull. **37** 315-329 (1994).

Back to the inverse problem:



If we wish to get more "realistic": Leaves – and scenes in general – are not just black or white. They have **shading**. (And colour, but let's not worry about this for now.)

We have to think about having some kind of variable values associated with points/regions on the attractor sets ....

Iterated Function Systems with Probabilities (IFSP)

Ingredients:

The "IFS" part:

- (X, d): A compact metric space (e.g., $[0, 1]^n$ with Euclidean metric)
- $(\mathcal{H}(X), h)$: Complete metric space of non-empty compact subsets of X with Hausdorff metric h
- $w_i : X \to X$, $1 \leq i \leq N$: Set of contraction maps on X with contraction factors $c_i \in [0, 1)$.

The "P" part:

- Associated with each IFS map w_i, $1 \leq i \leq N$, is a probability $p_i \in [0, 1]$, such that $\sum_{i=1}^{N} p_i = 1$.

The resulting IFSP will now operate on **measures** on X:

Iterated Function Systems with Probabilities (IFSP)

Let $\mathcal{M}(X)$ denote the space of probability measures on (the Borel sigma field of) X with the following (Monge-Kantorovich) metric,

$$d_M(\mu, \nu) = \sup_{f \in Lip_1(X)} \left| \int_X f \, d\mu - \int_X f \, d\nu \right| ,$$

where

$$Lip_1(X) = \{ f : X \to \mathbb{R} , \; |f(x_1) - f(x_2)| \leq |x_1 - x_2| \; \forall x_1, x_2 \in X \} .$$

Theorem (Hutchinson): The metric space $\mathcal{M}(X), d_M)$ is complete.

Iterated Function Systems with Probabilities (IFSP)

Associated with an N-map IFSP $(\mathbf{w}, \mathbf{p})$ is an operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$ defined as follows: For a $\mu \in \mathcal{M}(X)$, let $\nu = M\mu$ such that for each Borel set $S \in X$:

$$\nu(S) = (M\mu)(S) = \sum_{i=1} p_i \mu(\hat{w}_i^{-1}(S)) \, .$$

Theorem (Hutchinson): M is contractive in $(\mathcal{M}(X), d_M)$, i.e.,

$$d_M(M\mu, M\nu) \leq c d_M(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{M}(X) \, .$$

(Recall that $c = \max_{1 \leq i \leq N} c_i \leq 1$.)

Corollary: There exists a unique measure $\bar{\mu} \in \mathcal{M}(X)$ such that $\mu = M\mu$. This **invariant measure** of the IFSP $(\mathbf{w}, \mathbf{p})$ satisfies the following,

$$\bar{\mu}(S) = \sum_{i=1}^{N} p_i \bar{\mu}(\hat{w}_i^{-1}(S)) \;\; \forall S \in X \, .$$

In other words, μ may be expressed as a combination of spatially-contracted and translated (via the w_i) and range-altered (via the p_i) copies of itself.

Inverse problem of measure approximation using IFSP

Given a (target) measure $\mu \in \mathcal{M}(X)$, and an $\epsilon > 0$, can we find an N-map IFSP with invariant measure $\bar{\mu}$ such that

$$d_M(\mu, \bar{\mu}) < \epsilon \quad ?$$

Once again, we may resort to the "Collage Theorem" to consider the following inverse problem: Given a (target) measure μ and a $\delta > 0$, can we find an N-map IFSP with associated operator M so that

$$d_M(\mu, M\mu) < \delta \quad ?$$

Problem: It's difficult to work with measures directly. But we can work with their **moments**

"Collage Theorem for Moments:" B. Forte and ERV, Solving the inverse problem for measures using iterated function systems, Adv. Appl. Prob. **27**, 800-820 (1995).

D. La Torre, E. Maki, F. Mendivil and ERV, Iterated function systems with place-dependent probabilities and the inverse problem of measure approximation using moments, Fractals **26** (5) 1850076 (2018).

But that didn't stop the "Founders of fractal image compression":

- M.F. Barnsley and A. Sloan, A better way to compress images, BYTE Magazine, January 1988.
- A. Jacquin, Image coding based on a fractal theory of iterated contractive image transformations, IEEE Trans. Image Proc. **1**, 18-30 (1992).
- A. Jacquin, A novel fractal block-coding technique for digital images, Proc. ICASSP'90, pp. 2225-2228.

The last two papers were based on Jacquin's Ph.D. thesis in the School of Mathematics, Georgia Institute of Technology (supervised by Michael Barnsley):

- A.E. Jacquin, A fractal theory of iterated Markov operators with applications to digital image coding, Ph.D. Dissertation, Georgia Tech, 1989.

In these works, the greyscale value at a pixel of an image was treated as the measure μ of the set S represented by that pixel. **The action of the operator M on these greyscale values/measures was essentially equivalent to the action of an operator that acts on functions defined on the lattice of pixels.**

This naturally leads to the formulation of **fractal transforms over function spaces.**

A brief recap:

- Action of an N-map IFS operator $\hat{\mathbf{w}}$ on a set $S \in \mathcal{H}(X)$:

$$\hat{\mathbf{w}}(S) = \bigcup_{i=1}^{N} \hat{w}_i(S)\,.$$

- Action of the operator M associated with an N-map IFSP on a measure $\mu \in \mathcal{M}(X)$:

$$(M\mu)(S) = \sum_{i=1}^{N} p_i \mu(\hat{w}_i^{-1}(S)) \quad \forall S \in \mathcal{B}(X)\,.$$

Moral of the story: In both cases, an IFS-type operator "T" acts on "something" – call it "y" – by producing spatially-contracted and translated (and in the case of measures, range-modified) copies of that "something" and appropriately combining these copies to produce "something else" – call it "Ty".

This basic idea can be applied to functions, multifunctions, inclusions, multimeasures, etc.. We call such an operator T a **generalized fractal transform**.

For example, in the case of functions:

Iterated Function Systems with Greyscale Maps (IFSM)

B. Forte and ERV, Solving the inverse problem for function and image approximation using iterated function systems, Dyn. Cont. Impul. Sys. **1(2)**, 177-231 (1995).

Ingredients:

- The **base space** (or pixel space) (X, d) on which our functions will be supported: A compact metric space (e.g., $[0, 1]^n$ with Euclidean metric).
- The **(image) function space** $\mathcal{F}(X) = \{u : X \to R_g\}$ where $R_g \subset \mathbb{R}$ denotes the (greyscale) range space. (In applications, $R_g \subset \mathbb{R}^+$.)
- The IFS contraction maps $w_i : X \to X$, $1 \leq i \leq N$ with contraction factors $c_i \in [0, 1)$, also assumed to be one-to-one. These maps will produce the spatially-contracted and translated copies of our functions. In applications, we use affine IFS maps, e.g., $w_i(x) = s_i x + a_i$.
- The **greyscale maps**: Associated with each IFS map w_i is a greyscale map $\phi_i : R_g \to R_g$. We may also wish to consider place-dependent greyscale maps, i.e., $\phi_i : R_g \times X \to \mathbb{R}$.
  We'll assume that the ϕ_i are Lipschitz, i.e., for each $1 \leq i \leq N$, there exists a $K_i \geq 0$, such that

  $$|\phi_i(t_1) - \phi_i(t_2)| \leq K_i |t_1 - t_2| \quad \forall t_1, t_2 \in R_g \, .$$

  In applications, we use affine ϕ-maps, i.e., $\phi_i = \alpha_i t + \beta_i$.

Iterated Function Systems with Greyscale Maps (IFSM)

Associated with an N-map IFSM $(\mathbf{w}, \mathbf{\Phi})$, is the **fractal transform** $T : \mathcal{F}(X) \to \mathcal{F}(X)$ defined as follows: For a $u \in \mathcal{F}(X)$, $v = Tu$ is given by

$$v(x) = (Tu)(x) = \sum_{i=1}^{N}{}' \phi_i(u((w_i^{-1}(x))) ,$$

where the prime denotes summation over those $i \in \{1, \cdots, N\}$ for which $x \in w_i(X)$. In order that $v(x)$ be defined for all $x \in X$ we must have the additional condition that

$$\bigcup_{i=1}^{N} \hat{w}_i(X) = X ,$$

i.e., the sets $\hat{w}_i(X)$ cover X.

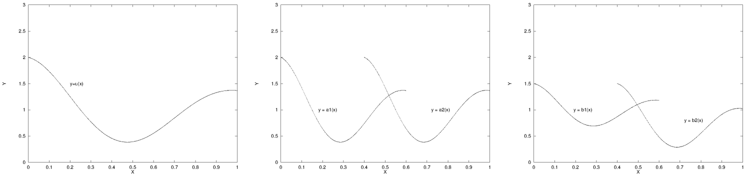
Action of fractal transform T on u:

1. It produces N spatially-contracted and translated copies $u_i = u \circ w_i^{-1}$ of u.
2. It modifies the range values each of these copies: $v_i = \phi_i \circ u_i$.
3. It combines the v_i – by simple addition – to produce a new function $v = Tu$.

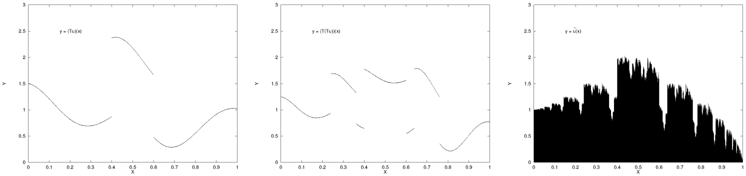
Iterated Function Systems with Greyscale Maps (IFSM)

Example: 2-map IFSM on $X = [0, 1]$, $R_g = \mathbb{R}^+$.

$$\begin{aligned} w_1(x) &= 0.6x & \phi_1(t) &= 0.5t + 0.5 \\ w_2(x) &= 0.6x + 0.4 & \phi_2(t) &= 0.75t \end{aligned}$$



Left: The function $u(x)$. **Middle:** The spatially-contracted and translated copies $u_i(x) = u(w_i^{-1}(x))$. **Right:** The range-modified copies $v_i(x) = \phi_i(u_i(x))$.

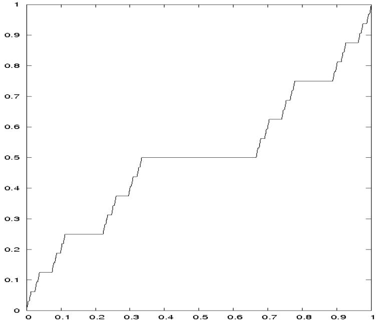


Left: The function $v = Tu$ obtained by adding the v_i above. **Middle:** The function $w = Tv = T^2u$ **Right:** The fixed point $\bar{u} = T\bar{u}$.

Iterated Function Systems with Greyscale Maps (IFSM)

Example: The following 3-map IFSM on $X = [0,1]$, $R_g = \mathbb{R}^+$.

$$\begin{array}{ll} w_1(x) = \frac{1}{3}x & \phi_1(t) = \frac{1}{2}t \\ w_2(x) = \frac{1}{3}x + \frac{1}{3} & \phi_2(t) = \frac{1}{2} \\ w_3(x) = \frac{1}{3}x + \frac{2}{3} & \phi_2(t) = \frac{1}{2}t + \frac{1}{2} \end{array}$$



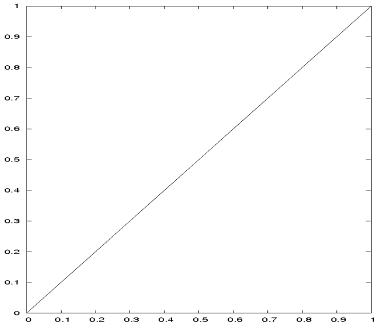
The attractor $\bar{u}$ of this IFSM is the famous **"Devil's staircase function."** It is self-similar, i.e., a union of three contracted copies of itself.

Iterated Function Systems with Greyscale Maps (IFSM)

But not all fixed point functions $\bar{u}$ need to be "fractal-like"

Example: The following 2-map IFSM on $X = [0,1]$, $R_g = \mathbb{R}^+$.

$$\begin{aligned} w_1(x) &= \tfrac{1}{2}x & \phi_1(t) &= \tfrac{1}{2}t \\ w_2(x) &= \tfrac{1}{2}x + \tfrac{1}{2} & \phi_2(t) &= \tfrac{1}{2}t + \tfrac{1}{2} \end{aligned}$$



The attractor $\bar{u}$ of this IFSM: $\bar{u}(x) = x$.

Iterated Function Systems with Greyscale Maps (IFSM)

We haven't said anything about contractivity of IFSM operators. For $\mathcal{F}(X) = L^p(X)$,

$$
\begin{aligned}
\|Tu - Tv\|_p &\leq \left\| \sum_{i=1}^{N} \left[\phi_i \circ u \circ w_i^{-1} - \phi_i \circ v \circ w_i^{-1} \right] \right\|_p \\
&\leq \sum_{i=1}^{N} \|\phi_i \circ u \circ w_i^{-1} - \phi_i \circ v \circ w_i^{-1}\|_p \\
&\leq \sum_{i=1}^{N} \left[\int_{\hat{w}_i(X)} \left| \phi_i(u(w_i^{-1}(x))) - \phi_i(v(w_i^{-1}(x))) \right|^p \, dx \right]^{1/p} \\
&\leq \sum_{i=1}^{N} K_i \left[\int_{\hat{w}_i(X)} \left| u(w_i^{-1}(x)) - v(w_i^{-1}(x)) \right|^p \, dx \right]^{1/p} \\
&\leq \sum_{i=1}^{N} K_i c_i^{1/p} \left[\int_X |u(y) - v(y)|^p \, dy \right]^{1/p} \qquad (\, y = w_i^{-1}(x) \implies x = w_i(y)\,) \\
&= \left[\sum_{i=1}^{N} K_i c_i^{1/p} \right] \|u - v\|_p \,.
\end{aligned}
$$

OK, so, like, is this just a "get promotion and/or tenure" project?

At this point, a nonbeliever might interject: "OK, fine, great. Nice job. But how many functions – wait, useful ones – are self-similar?"

"Aha, Monsieur/Madame!" would be the reply. "Many more than you might think!"

For example, multiresolution analysis using wavelets relies on the concept of self-similarity ...

Scaling equation of multiresolution analysis

Let $\phi \in L^2(\mathbb{R})$ be a **scaling function** which defines a multiresolution analysis on $L^2(\mathbb{R})$. Then we may define the following,

$$V_0 = \overline{\text{span}\{\phi(x-n)\}_{n\in\mathbb{Z}}} \cap L^2(\mathbb{R}) \qquad V_1 = \overline{\text{span}\{\phi(2x-n)\}_{n\in\mathbb{Z}}} \cap L^2(\mathbb{R})\,.$$

so that

$$V_0 \subset V_1 \subset L^2(\mathbb{R})\,.$$

This implies that ϕ satisfies the following **scaling** or **dilatation equation**,

$$\phi(x) = \sum_{k\in\mathbb{Z}} \sqrt{2} h_k \phi(2x-k)\,,$$

where not all of the coefficients h_k are zero. ϕ is **self-similar** – it may be expressed as a linear combination of translations of spatially-contracted copies of itself.

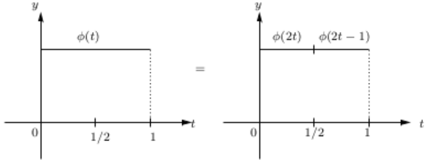
The scaling function $\phi(x)$ may be viewed as the fixed point of the fractal transform associated with the following IFSM,

$$w_k(x) = \frac{1}{2}x + \frac{k}{2}\,, \quad \phi_k(t) = \sqrt{2} h_k t\,, \quad k \in \mathbb{Z} \;\text{ such that }\; h_k \neq 0\,.$$

Example: Haar scaling function $\phi(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

$h_0 = h_1 = \frac{1}{\sqrt{2}}$, all other $h_k = 0$ so that scaling equation becomes

$$\phi(x) = \phi(2x) + \phi(2x - 1)\,.$$



Induced fractal transform operators

Let ϕ be a 1-1 mapping of our space of functions $\mathcal{F}(X)$ to a representation space $\mathcal{G}$, e.g.,

- Fourier transforms
- Orthogonal expansion (e.g., Fourier, wavelet)
- (in the case of probability measures, the moment space $(1, g_1, g_2, \cdots)$)

A fractal transform $T : \mathcal{F}(X) \to \mathcal{F}(X)$ induces an operator $M : \mathcal{G} \to \mathcal{G}$:

$$\begin{array}{ccccc} & & T & & \\ & u & \longrightarrow & v & \\ \phi & \downarrow & & \uparrow & \phi^{-1} \\ & U & \longrightarrow & V & \\ & & M & & \end{array}$$

Action of M on an element $U \in \mathcal{G}$: It produces N copies of U and recombines them to produce a $V = MU$ in $\mathcal{G}$.

Apart from the mathematical beauty of this result, it may actually be desirable/beneficial to work in the alternate space $\mathcal{G}$.

Example: Fourier transforms

Let T be the fractal transform associated with an N-map affine IFSM on $\mathbb{R}$:

$$w_i(x) = s_i x + a_i\,, \qquad \phi_i(t) = \alpha_i t + \beta_i\,.$$

so that

$$v(x) = (Tu)(x) = \sum_{i=1}^{N}{}' \left[\alpha_i u(w_i^{-1}(x)) + \beta_i\right]\,.$$

Once again, $v = Tu$ is a linear combination of N **spatially-contracted**, translated and range-modified copies of v.

Operator M induced on the space of Fourier transforms:

$$V(\omega) = (MU)(\omega) = \sum_{i=1}^{N} a_k s_k e^{-a_k\omega} U(s_k\omega) + \sum_{k=1}^{N} \beta_k s_k e^{-ia_k\omega} \mathcal{F}_X(s_k\omega)\,,$$

where

$$\mathcal{F}_X = \int_X e^{-i\omega x}\,dx\,.$$

$V = MU$ is a linear combination of N phase-shifted, **frequency-expanded**, translated and range-modified copies of U (plus some sinc functions).

Wavelet transforms

Unfortunately, there is no time to cover this beautiful topic. See “The Book”, Section 3.3, pp. 102-110.

- F. Mendivil and ERV, Correspondence between fractal-wavelet transforms and iterated function systems with grey-level maps, in *Fractals in Engineering: From Theory to Industrial Applications*, J. Levy-Vehel, E. Lutton and C. Tricot, editors. Springer Verlag, London (1997). pp. 54-64.
- ERV, A generalized class of fractal-wavelet transforms for image representation and compression, Can. J. Elect. Comput. Eng. **23**, 69-84 (1998).

In a nutshell: Lower wavelet subtrees (representing higher frequency components) are replaced by scaled copies of higher wavelet subtrees (representing lower frequency components).

Inverse problem for IFSM

Let's go directly to "Collage coding": Given a $u \in L^p(X)$, can we find an N-map IFSM $(\mathbf{w}, \mathbf{\Phi})$ with contractive fractal transform T so that for a desired $\delta > 0$,

$$d_p(u, Tu) \leq \delta \quad ?$$

Usual strategy: Fix the IFS maps w_i, $1 \leq i \leq N$, and find the best ϕ_i maps, i.e., the ϕ_i maps that minimize the collage distance $d_p(u, Tu)$.

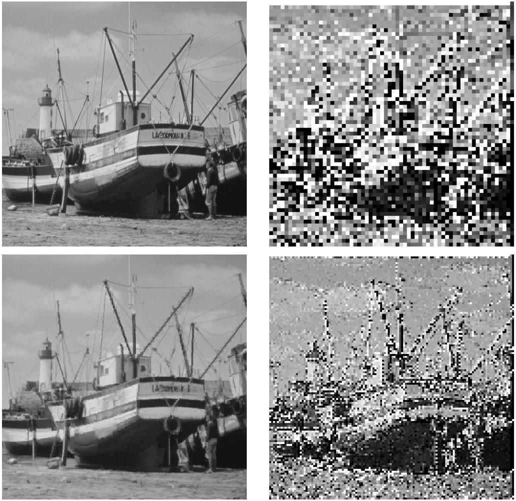
And to make life even simpler: Use "nonoverlapping" IFS maps w_i, i.e., the sets $\hat{w}_i(X)$ overlap on sets of Lebesgue measure zero.

Example: On $X = [0, 1]$,

$$w_1(x) = \frac{1}{2}x\,, \quad w_2(x) = \frac{1}{2}x + \frac{1}{2}\,.$$

The inverse problem "separates" into inverse problems over each each subset $X_i = \hat{w}_i(X)$. Over each subset X_i, you find the best ϕ_i map.

The “Holy Grail:” Fractal image coding!



Clockwise, starting from top left: Original *Boat* image. The iterates u_1 and u_2 and fixed point approximation $\bar{u}$ obtained by iteration of fractal transform operator. ($u_0 = \mathbf{0}$.) 8×8-pixel range blocks. 16×16-pixel domain blocks.

Inverse problem for Fourier transforms

Recall: A fractal transform $T : \mathcal{F}(X) \to \mathcal{F}(X)$ induces an operator $M : \mathcal{G} \to \mathcal{G}$:

$$\begin{array}{ccccc} & & T & & \\ & u & \longrightarrow & v & \\ \phi & \downarrow & & \uparrow & \phi^{-1} \\ & U & \longrightarrow & V & \\ & & M & & \end{array}$$

- G. Mayer and ERV, Iterated Fourier transform systems: A method for frequency extrapolation, in *Image Analysis and Recognition*, LNCS **4633** (Springer, Berlin-Heidelberg), p. 728-739.

Motivation: The signal from an MRI is a Fourier transform $U(\omega)$. It must be inverted to produce an image $u(x)$ of the subject. Instead of working on the image u, e.g., denoising, superresolution, we work on the original signal U, and then invert.

“Generalized fractal transforms” (GFTs)

Given a mathematical object “u” in some complete metric space (Y, d_Y), e.g.,

- a subset $S \subset \mathbb{R}^n$,
- a probability measure $\mu \in \mathcal{M}([0,1])$,
- a function $f \in \mathcal{F}([0,1])$, where $\mathcal{F}$ is an appropriate space of functions,
- a function $u : [0,1]^2 \to \mathbb{R}$ which represents a greyscale image,
- a function $u : [0,1]^2 \to \mathbb{R}^n$ which represents a colour or multispectral image,
- a function-valued mapping $u : [0,1]^2 \to \mathcal{G}$ ($\mathcal{G}$ is an appropriate space of functions) which represents a **hyperspectral image**,
- a set-valued function $u : [0,1]^2 \to \mathbb{S}^2$ which represents a **diffusion MRI image**.

From u, we produce a “union” or assembly of geometrically contracted and modified copies, u_k, of itself, i.e.,

$$v = \mathcal{O}(u_1, u_2, \cdots, u_N) = \text{“}Tu\text{”},$$

where the operator $\mathcal{O}$ performs some kind of “putting together” of the components u_k which is appropriate for the space in which we are working so that $v \in Y$. The net result is a **generalized fractal transform** operator $T : Y \to Y$.

“Generalized fractal transforms” (GFTs)

Under appropriate conditions on the geometric contractions and other modifications, the mapping $T : Y \to Y$ will be contractive – we'll let $Con(Y)$ denote this set of contractive (generalized) fractal transform operators.

Now given an element $u \in Y$, can we find a fractal transform T (which may also involve finding N) such that

$$u = Tu = \mathcal{O}(u_1, u_2, \cdots, u_N)\,?$$

In other words, can we express u as a union of copies of itself?

In general, the answer will be “no”, so we'll have to settle with the following: Perhaps we can approximate u as such a union to some degree of accuracy, i.e.,

$$d_Y(u, Tu) < \delta\,, \text{for some acceptable } \delta > 0\,.$$

But this is just “Collage Coding”!!!!!!!

In other words, we're just trying to approximate u with the fixed point $\bar{y}$ of a contractive generalized fractal transform T.

“Generalized fractal transforms” (GFTs)

This has led to a large number of “iterated somethings” and “fractal other things” over the years, e.g.,

- Iterated fuzzy set systems (C. Cabrelli, B. Forte, U. Molter and ERV 1992).
- Iterated function systems with grey-level maps (B. Forte and ERV 1994)
- Fractal-wavelet transforms (F. Mendivil and ERV 1997, ERV 1998)
- IFS operators on integral transforms (B. Forte, F. Mendivil and ERV 1999)
- IFS over vector-valued measures (F. Mendivil and ERV 2002)
- Fractal-wavelet denoising of images (M. Ghazel, G. Freeman and ERV 2006)
- Fractal image coding as projections onto convex sets (M. Ebrahimi and ERV 2006)
- Iterated Fourier transform systems (G. Mayer and ERV 2007)
- Contractive multifunctions, fixed point inclusions and iterated multifunction systems (H. Kunze, D. La Torre and ERV 2007)
- Measure-valued images, associated fractal transforms and the self-similarity of images (D. La Torre, ERV, M. Ebrahimi and M.F. Barnsley 2009)
- Union-additive multimeasures and self-similarity (D. La Torre and F. Mendivil 2009)
- Random measure-valued image functions, fractal transforms and self similarity (D. La Torre and ERV 2011).
- Generalized fractal transforms and self-similar objects in cone metric spaces (H. Kunze, D. La Torre, F. Mendivil and ERV 2012)

“Generalized fractal transforms” (GFTs)

All of the above can be found in “the Book”

Since then:

- Transfer operator associated with an IFS over “flow space” of stream functions in $\mathbb{R}^2$ (J. Vass, Ph.D. Thesis 2013)
- Fractal transforms for hyperspectral images (ERV, D. Otero and D. Latorre 2014)
- Iterated function systems on functions of bounded variation (D. La Torre, F. Mendivil and ERV 2016)
- Iterated function systems with place-dependent probabilities (D. La Torre, E. Maki, F. Mendivil and ERV 2018)

$$\vdots$$

?

Inverse problems of approximation by fixed points of "nonfractal" contraction mapppings

Recall: Let (Y, d_Y) be a complete metric space and $Con(Y)$ the set of all contraction maps $T : X \to X$. Now let $Con'(Y) \subset Con(Y)$ be a particular class of contraction maps that we wish to consider.

Then given a $y \in Y$ (our "target") and an an $\epsilon > 0$, can we find a $T \in Con'(Y)$ with fixed point $\bar{y}$ such that

$$d_Y(y, \bar{y}) < \epsilon\,?$$

In other words, can we approximate y with the fixed point $\bar{y}$ to ϵ-accuracy?

Recast this problem as follows:

"Collage coding:" Given a $y \in Y$ and a $\delta > 0$, find T so that

$$d_Y(Ty, y) < \delta\,.$$

So, like, can we find an interesting set of "nonfractal" contraction mappings?

Like, yes!

Warmup:

ODE initial value problem (IVP) for $x(t)$ on $\mathbb{R}$:

$$x' = f(x,t) \quad x(t_0) = x_0 \,. \tag{1}$$

For the moment, simply assume that $f(x,t)$ is continuous in x and t.

IVP in Eq. (1) is equivalent to the following integral equation,

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s),s)\,ds \,.$$

The solution $x(t)$ may be viewed as the fixed point of the following (Picard) integral operator $T : C(I) \to C^1(I)$, where I is an interval containing t_0:

$$v(t) = (Tu)(t) = x_0 + \int_{t_0}^{t} f(u(s),s)\,ds \,.$$

Continuity of f is sufficient to guarantee the existence of a solution to IVP in Eq. (1) (Peano's Theorem).

“Nonfractal” mappings

But if f satisfies the following (uniform) Lipschitz property: For all $t \in I$,

$$|f(x_2, t) = f(x_1, t)| \leq K|x_2 - x_1| \quad \forall x_1, x_2 \in J \,,$$

where J is an interval containing x_0, the solution to (1) is **unique** .

Reason: $T : S \to S$ is a contraction mapping, where S is a space of continuous functions.

From Banach's Fixed Point Theorem, there exists a unique $x \in S$ such that

$$x = Tx \,.$$

We have the ingredients for an inverse problem! Another set of “nails” for our “hammer”!

“Nonfractal” mappings

Given a function $u(t)$, the evolution of which we suspect to be described by a DE of the form,

$$x' = f(x, t)\,,$$

can we find a function $g(x, t)$ which defines a Picard integral operator T which, in turn, maps x close to itself so that, at least,

$$x' \approx g(x, t)\,.$$

Strategy: We consider a limited class of functions g, e.g. polynomials in x (and perhaps t) and optimize over the coefficients.

Easily extended to systems of ODEs.

IN THE BEGINNING OF THE “COLLAGE CODING ERA” WAS

H. Kunze and ERV, Solving inverse problems for ordinary differential equations using the Picard contraction mapping, Inverse Problems **20**, 3, 977-991 (1999).

“Nonfractal” mappings

Example 1(a): (The second example considered in the 1999 paper.) Let $x(t) = t^2$ be the target solution on $I = [0, 1]$. Find the best ODE of the form

$$\frac{dx}{dt} = c_0 + c_1 x \,, \quad x(0) = x_0 \,,$$

with c_0, c_1 and x_0 to be determined.

Result: The IVP

$$\frac{dx}{dt} = \frac{5}{12} + \frac{35}{18} x \,, \quad x(0) = -\frac{1}{27} \,. \tag{2}$$

The solution to this IVP is

$$\bar{x}(t) = \frac{67}{378} \exp\left(\frac{35}{18} t\right) - \frac{3}{14} \,.$$

L^2 distance between target x and $\bar{x}$ is

$$\|x - \bar{x}\|_2 = 0.0123 \,.$$

Example 1(b): If we impose the condition that $x(0) = x_0 = 0$, then only c_0 and c_1 are to be determined. The resulting DE is

$$\frac{dx}{dt} = \frac{5}{12} + \frac{35}{16} x \,, \quad x(0) = -\frac{1}{27} \,.$$

The soluton to this IVP is

$$\bar{x}(t) = \frac{1}{7} \exp\left(\frac{35}{16} t\right) - \frac{1}{7} \,.$$

L^2 distance between target x and $\bar{x}$ is

$$\|x - \bar{x}\|_2 = 0.0463 \,.$$

“Nonfractal” mappings

Our “collage coding” approach provides a method of performing “parameter estimation” for systems of ODEs

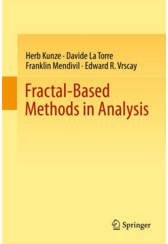
This is important in a number of scientific applications, including

- chemical kinetics, especially “pharmacokinetics,”

Since the 1999 paper, we have built up our “Collage Industry”, extending our approach to treat a variety of inverse problems in:

- PDEs (hyperbolic, elliptic, parabolic)
- Random ODEs,
- Stochastic ODEs,
- Random PDEs,
- Stochastic PDEs.

Some of these results (up to 2012) are presented in



“Nonfractal” mappings - The inverse problem for PDEs

Q: So, like, how would you, like, treat PDEs?

A: Like, by means of a **generalized collage theorem** which is associated with the classical Lax-Milgram representation theorem, at least for linear PDEs.

Recall the classical treatment of solving a linear PDE:

Multiply both sides of the PDE with an element $v \in H$, where H is a suitable Hilbert space of functions (e.g., the Sobolev space $H_0^1(\Omega)$) and integrate over Ω to obtain (by integration of parts and the concept of weak derivative) the equation

$$a(u,v) = b(v)\,, \quad v \in H\,. \qquad (3)$$

Here, $a(u,v)$ is a bilinear form on $H \times H$ and $b(v)$ is a linear functional on H.

Under appropriate conditions satisfied by the bilinear form a, there is a unique $u \in H$ which satisfies Eq. (3) for all $v \in H$. This is the (weak) solution to the PDE.

The inverse problem:

Suppose that you are given a “target solution” $u(\mathbf{x}, t)$. You want to find the “best” PDE – e.g., a nonhomogeneous diffusion equation – which admits u as an approximate solution. This “best” PDE is to be determined from an n-parameter family of PDEs – e.g. the parameters are the multinomial expansion coefficients of the diffusivity function $\kappa(\mathbf{x})$.

Associated with this family of PDEs is a family of bilinear functionals $a_\lambda(u, v)$, $\lambda \in \mathbb{R}^n$. We seek to find “optimal” values of the parameters λ, e.g., those that minimize the function

$$F(\lambda) = \sup_{v \in H, \|v\|=1} |\, a_\lambda(u, v) - b(v) \,| \,.$$

Now suppose that

$$\lambda^* = \operatorname{argmin} F(\lambda) \,.$$

From the Lax-Milgram Theorem, there exists a unique function u_{λ^*} such that

$$a(u_{\lambda^*}, v) = b(v) \quad \text{for all} \quad v \in H \,.$$

The big question: How close (or how far) is u_{λ^*} to u?

Generalized Collage Theorem:

$$\|u - u_{\lambda^*}\| \leq \frac{1}{m_{\lambda^*}} F(\lambda^*)\,,$$

where the constant m_λ characterizes the expansivity of the bilinear form.

We view

$$F(\lambda) = \sup_{v \in H, \|v\|=1} |\, a_\lambda(u, v) - b(v)\,|$$

as a **generalized collage distance** and the minimization of $F(\lambda)$ as a **generalized collage method**.

H. Kunze, D. La Torre and ERV, A generalized collage method based upon the Lax-Milgram functional for solving boundary value inverse problems, Nonlinear Analysis: Theory, Methods and Applications **71** (12), e1337-e1343 (2009).

and many more papers by H. Kunze, D. La Torre, students and co-workers.

THE FUTURE?

1. Continuation of present work on
   - Generalized fractal transforms and associated inverse problems on more esoteric spaces, e.g.,
     - "Superfractals" and "V-fractals"
     - Iterated deep neural networks?
   - Differential calculus on fractal measures, e.g.,

     KLMV, Self-similarity of solutions to integral and differential equations with respect to a fractal measure, Fractals **27** *(3), 1950014 (2019), 13 pp.*

     Schrödinger equation on an IFS attractor A with invariant measure μ.

2. More connections with the "inverse problems community" (e.g., Prof. F. Cakoni)!

   We need to expand our "gene pool"! Otherwise ...

   Actually, we did try, e.g., past connection with Prof. C. Groetsch. And we (Herb) did talk about "collage coding" at ICIPE 17 (U Waterloo, 2017). There were a lot of "parameter estimators" there, but our ideas seemed to travel like the proverbial "Pb balloon"! Keep trying.

THE FUTURE?

Can/should we climb out of our "collage rut"? Out of our "Contraction mapping rut"?

What about the following:

Relaxing the restriction of contractive (and mostly affine) IFS maps and working with systems of nonlinear and noncontractive maps with attractive fixed points, i.e.,

Nonlinear IFS or "Nonlinear Map Bags"?

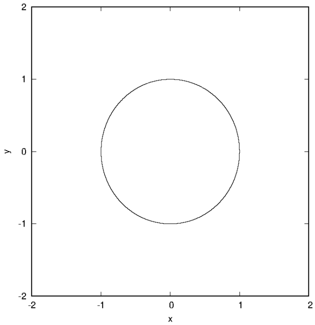
Nonlinear IFS/"Map Bags"

Example 1: The following two-map nonlinear IFS in $\mathbb{C}$:

$$w_1(z) = \sqrt{z}\,, \qquad w_2(z) = -\sqrt{z}\,.$$

- Fixed points of $w_1(z)$: $\bar{z} = 0$ repulsive, $\bar{z} = 1$ attractive.
- Fixed point of $w_2(z)$: $\bar{z} = 0$ repulsive.

Recall that the attractor set of the IFS $\{w_1, w_2\}$ is the unit circle C, the Julia set of the rational map $f(z) = z^2$.

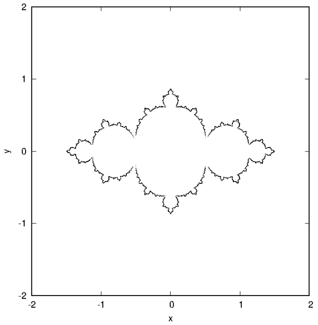


Nonlinear IFS/"Map Bags"

Example 2: The following two-map nonlinear IFS in $\mathbb{C}$:

$$w_1(z) = \sqrt{z + \frac{3}{4}}, \qquad w_2(z) = -\sqrt{z + \frac{3}{4}}.$$

The attractor set of the IFS $\{w_1, w_2\}$ is the unit circle C, the Julia set of the rational map $f(z) = z^2 - \dfrac{3}{4}$.

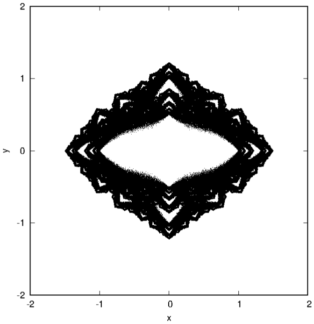


Nonlinear IFS/"Map Bags"

Example: The following four-map nonlinear IFS in $\mathbb{C}$:

$$w_1(z) = \sqrt{z}\,, \quad w_2(z) = -\sqrt{z} \quad w_1(z) = \sqrt{z + \frac{3}{4}}\,, \quad w_2(z) = -\sqrt{z + \frac{3}{4}}\,.$$

A "mixture" of inverse maps of $f_0(z) = z^2$ and $f_{-3/4}(z) = z^2 - \frac{3}{4}$.
Approximation of attractor set (5×10^6 points generated by Chaos Game).



This attractor should contain the Julia sets J_0 and $J_{-3/4}$.

THANK YOU!

May you be rewarded generously for your patience.

"... but thou has ordered all things in measure and number and weight." *Book of Wisdom 11:20.*

"May everything that lives and that breathes give praise to the Lord." *Psalm 150:6.*