Inverse Problem Methods for Generalized Fractal Transforms

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Chapter 1

Introduction

In a previous paper in this volume [15], henceforth to be referred to as Paper I, we presented a unified treatment of IFS/Fractal Transform methods in terms of "generalized fractal transforms". A theory of Fractal Transforms on Distributions was developed so that the various IFS methods on function spaces, e.g. IFS, IFZS and IFSM (including "Fractal Transforms" and the "Bath Fractal Transform") could be united with IFSP on measure spaces.

In this paper we examine various methods to attack the inverse problem of function/measure approximation using generalized fractal transforms, which involve the use of IFS maps whose "range blocks" may overlap. As such, these results may not appear to be greatly effective in the problem of fractal image compression since overlapping blocks are viewed as redundant in the coding of an image. However, our philosophy has been to develop a systematic theory of approximation of functions, measures and distributions using a complete "basis set" of IFS maps.

As in Paper I, we consider such target functions or images to be elements of an appropriate complete metric space (Y, d_Y). The underlying idea in fractal compression is the approximation, to some suitable accuracy, of a target $y \in Y$ by the fixed point $\bar{y}$ of a contraction mapping $f \in Con(Y)$, i.e. $f(\bar{y}) = \bar{y}$. It is then f which is stored in computer memory. By Banach's celebrated Fixed Point Theorem or Contraction Mapping Principle (CMP), the unique fixed point $\bar{y}$ may be readily generated by iteration of f, using an arbitrary "seed" image $y_0 \in Y$. We have been interested [11, 12, 13, 14] in a complete mathematical solution to the following formal inverse problem:

Given a "target" $y \in Y$ and an $\epsilon > 0$, find a map $f_\epsilon \in Con(Y)$ such that $d_Y(y, \bar{y}_\epsilon) < \epsilon$, where $f_\epsilon(\bar{y}_\epsilon) = \bar{y}_\epsilon$. As is well known, the inverse problem is somewhat simplified by the following corollary to the CMP, which is now referred to in the IFS literature as the "Collage Theorem":

Theorem 1 *Let (Y, d_Y) be a complete metric space. Given a $y \in Y$ suppose that there exists a map $f \in Con(Y)$ with contraction factor $c_f \in [0, 1)$ such that $d_Y(y, f(y)) < \delta$. Then*

$$d_Y(y, \bar{y}) < \frac{\delta}{1 - c_f}, \tag{1.1}$$

where $\bar{y} = f(\bar{y})$ is the fixed point of f.

In other words, the closer that f maps a target point y to itself, the closer that y is to the fixed point $\bar{y}$ of f. By making δ sufficiently small (if possible), $\bar{y}$ may become a suitable approximation to y. Rigorous solutions to the inverse problem of approximation using IFS-type methods (i.e. for arbitary $\delta > 0$) have been provided in [12] (for measures) and [13, 14, 11] (for functions) as well as algorithms for the construction of these approximations. Our basic strategy has been to work with an infinite set $\mathcal{W} = \{w_1, w_2, \ldots\}$ of *fixed* affine contraction maps which satisfy density conditions appropriate to the metric space Y being employed. From this set, sequences of N-map IFS $\mathbf{w}^N$ may then be chosen to produce approximations of arbitrary accuracy. As such, our formal solution establishes that the set of fixed points generated from this infinite set of IFS maps $\mathcal{W}$ is dense in (Y, d_Y).

The layout of this paper is as follows. In Section 2 we briefly review our solution of the inverse problem for IFSM since it is a direct method. Section 3 is devoted to measure approximation using IFSP. This is an indirect method, since it is the moments of the target measure μ that we seek to approximate as closely as possible. In Section 4 we formulate a general fractal transform method on orthogonal function expansions. In the specific case of the local IFSM method applied to wavelet expansions, the method leads to "discrete wavelet fractal compression" In the Appendix, a Collage Theorem for Fourier transforms of measures is given. Much of the notation used in this paper follows the notation of Paper I.

Chapter 2

Inverse Problem for IFSM

The inverse problem for IFSM is a "direct method" since we may work on the target functions/images directly with IFS operators.

2.1 Collage Theorem for IFSM in $\mathcal{L}^2(X, \mu)$

From the Collage Theorem, the inverse problem for the approximation of functions in $\mathcal{L}^p(X, \mu)$ by IFSM may be posed as follows:

Given a target function $v \in \mathcal{L}^p(X, \mu)$ and a $\delta > 0$, find an IFSM $(\mathbf{w}, \Phi)$ with associated operator T such that $d_p(v, Tv) < \delta$.

For an N-map contractive IFSM $(\mathbf{w}, \Phi)$ on (X, d) with associated operator T, the squared $\mathcal{L}^2$ collage distance is given by

$$\begin{aligned} \Delta^2 &= \| v - Tv \|_2^2 \\ &= \int_X [\sum_{k=1}^{N}{}' \phi_k(v(w_k^{-1}(x))) - v(x)]^2 d\mu. \end{aligned} \tag{2.1}$$

Following our discussion in the previous section (and our strategy in [12]), we consider the IFS maps w_i to be fixed. The problem reduces to the determination of grey level maps ϕ_i which minimize the collage distance Δ^2. In the special "μ-nonoverlapping case", i.e.,

1. $\cup_{k=1}^N X_k = \cup_{k=1}^N w_i(X) = X$, i.e. the sets $X_k = w_k(X)$ "tile" X, and

2. $\mu(w_i(X) \cap w_j(X)) = 0$ for $i \neq j$, then

the squared collage distance Δ^2 becomes

$$\begin{aligned}\Delta^2 &= \sum_{k=1}^{N} \int_{X_k} [\phi_k(v(w_k^{-1}(x)) - v(x)]^2 d\mu \\ &= \sum_{k=1}^{N} \Delta_k^2, \end{aligned} \tag{2.2}$$

i.e. the sum of collage distances over the μ-nonoverlapping subsets X_k. The minimization of each integral is a continuous version of "least squares" with respect to the measure μ: For each subset X_k, find the $\phi_k : R_g \to R_g$ which provides the best $\mathcal{L}^2(X,\mu)$ approximation to the graph of $v(x)$ vs. $v(w_k^{-1}(x))$ for $x \in X_k$.

Most, if not all, applications in the literature assume the μ-nonoverlapping property, with $\mu = m$ (Lebesgue measure on X) and $w_k \in Aff_1(X)$. In the following discussion, however, we consider the more general case where the sets $w_i(X)$ can overlap on sets of nonzero μ-measure. We also assume the following:

3. $\cup_{k=1}^{N} w_k(X) = X$, i.e. the sets $w_i(X)$ "tile" X. Note that $w_i \in Aff_1(X)$ implies that $|J_i| > 0$ for $1 \le i \le N$, where $|J_i|$ denotes the Jacobian associated with the mapping $y = w_i(x)$.

4. affine grey level maps, i.e. $\phi_i : \mathbf{R}^+ \to \mathbf{R}^+$, where $\phi_i(t) = \alpha_i t + \beta_i, \ \ t \in \mathbf{R}^+$. Thus, $\alpha_i, \beta_i \ge 0$ for $1 \le i \le N$.

The squared $\mathcal{L}^2$ collage distance then becomes

$$\begin{aligned}\Delta^2 &= < v - Tv, v - Tv > \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} [< \psi_k, \psi_l > \alpha_k\alpha_l \ + \ 2 < \psi_k, \chi_l > \alpha_k\beta_l \ + \ < \chi_k, \chi_l > \beta_k\beta_l] \\ &\quad - 2\sum_{k=1}^{N} [< v, \psi_k > \alpha_k + < v, \chi_k > \beta_k] + < v, v >, \end{aligned} \tag{2.3}$$

where

$$\psi_k(x) = u(w_k^{-1}(x)), \quad \chi_k(x) = I_{w_k(X)}(x), \quad x \in w_k(X). \tag{2.4}$$

Note that Δ^2 is a quadratic form in the ϕ-map parameters α_i and β_i, i.e.

$$\Delta^2 = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + d_0, \tag{2.5}$$

where $\mathbf{x}^T = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N}$. The elements of the symmetric matrix $\mathbf{A}$ are given by

$$\begin{aligned} a_{i,j} &= < \psi_i, \psi_j > \\ a_{N+i,N+j} &= < \chi_i, \chi_j > \\ a_{i,N+j} &= < \psi_i, \chi_j >, \quad 1 \le i \le N, \ 1 \le j \le N. \end{aligned} \tag{2.6}$$

As well,

$$\begin{aligned} b_i &= -2 < v, \psi_i > \\ b_{N+i} &= -2 < v, \chi_i >, \quad 1 \le i \le N \end{aligned} \tag{2.7}$$

and $d_0 =< v, v >=\| v \|_2^2$.

For a given target $v \in \mathcal{L}^p(X,\mu)$, assuming $\| v \|_1 \neq 0$, we denote the feasible set of N-map IFSM grey-level parameters as

$$\Pi_v^{2N} = \{(\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N} \mid \| Tv \|_1 \le \| v \|_1, \ \alpha_i, \beta_i \ge 0\}. \tag{2.8}$$

(Note that Π_v^{2N}, which is compact in the natural topology on $\mathbf{R}^{2N}$, depends on the target function v.) In terms of the grey level map parameters, the condition $\| Tv \|_1 \le \| v \|_1$ is a linear inequality constraint, i.e.

$$\sum_{k=1}^{N} (\alpha_k \| v \circ w_k^{-1} \|_1 + \beta_k \mu(X_k)) \le \| v \|_1 . \tag{2.9}$$

For the case $X \subset \mathbf{R}^D$, $\mu = m$ and $w_i \in Aff_1(X)$, $1 \le i \le N$, which will be used in all applications, the above linear inequality constraint becomes

$$\sum_{k=1}^{N} |J_k| (\alpha_k \| v \|_1 + m(X)\beta_k) \le \| v \|_1, \tag{2.10}$$

The minimization of Δ^2 may now be written as the following QP problem with linear constraints:

$$\text{minimize} \quad \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + d_0, \quad \mathbf{x}^T \in \Pi_v^{2N}. \tag{2.11}$$

The advantages of QP problems have been discussed in [12]. Briefly,

1. QP algorithms locate an absolute minimum of the objective function Δ^2 in the feasible region Π_v^{2N} in a finite number of steps and

2. in many problems, the minimum value $\Delta^2_{\min}$ is achieved on a boundary point of the feasible region. In such cases, if $(\alpha_k, \beta_k) = (0,0)$ then $\phi_k(t) = 0$ which implies that the associated IFS map w_k is superfluous. QP (as opposed to gradient-type schemes) will locate such boundary points, essentially discarding such superfluous maps. The elimination of such maps represents an increase in the data compression factor. (This feature was observed with minimization of the collage distance involving IFS with probabilities [12].)

The following result guarantees that, with the exception of a degenerate case, the IFSM operator T corresponding to a feasible N-map IFSM grey-level parameter $\mathbf{x}^T \in \Pi_v^{2N}$ is contractive in $\mathcal{L}^1(X, \mu)$.

Proposition 1 *Let $X \subset \mathbf{R}^D$, $\mu = m^{(D)}$ and $v \in \mathcal{L}^1(X, \mu)$, $\| v \|_1 \neq 0$. Assume that $w_i \in Aff_1(X)$ for $1 \leq i \leq N$ and $\mathbf{x}^T = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \Pi_v^{2N}$. Then the operator T corresponding to the N-map IFSM $(\mathbf{w}, \Phi)$ is contractive in $(\mathcal{L}^1(X, \mu), d_1)$ except possibly when $\beta_1 = \beta_2 = ... = \beta_N = 0$. In this special case $\overline{u} \equiv 0$ is a fixed point of T.*

Note that $\mathbf{x}^T \in \Pi_v^{2N}$ does *not* guarantee that T is contractive in $(\mathcal{L}^2(X, \mu), d_2)$. Hence, the Collage Theorem does not apply in $\mathcal{L}^2(X, \mu)$. Nevertheless, our algorithm to approximate functions in $\mathcal{L}^2(X, \mu)$ exploits the contractivity of T in $(\mathcal{L}^1(X, \mu), d_1)$.

2.2 Formal Solution to the IFSM Inverse Problem

In this section, we outline the basic ideas behind a formal solution to the inverse problem posed above. Detailed proofs appear in [11]. These results would not be of as much interest to fractal compression as they would be to the approximation theory of functions using fractal transforms.

One intuitively expects that the $\mathcal{L}^2$ collage distance $\| v - Tv \|_2$ can be made arbitrarily small by adding IFS maps with increasing degrees of refinement, i.e. by increasing N. A trivial yet practical way of doing this is by simply dividing up the base space X into smaller regions with minimal overlap. This is essentially the approach adopted in image compression, e.g. quadtrees using nonoverlapping IFS maps. However, in the spirit of our earlier work on measures, we would like to consider a *complete* and infinite set of IFS maps which can provide various degrees of refinement. Therefore, in our formal solution to the inverse problem, we construct sequences of N-map IFSM, denoted as $(\mathbf{w}^N, \Phi^N)$, $N = 1, 2, \ldots$,

$$\mathbf{w}^N = (w_1, w_2, \ldots, w_N), \quad \Phi^N = (\phi_1, \phi_2, \ldots, \phi_N), \tag{2.12}$$

where the IFS maps w_i are chosen from a *fixed* and infinite set $\mathcal{W}$ of contraction maps. The (contractive) operators T^N associated with these N-map IFSM will play the role of f in the Collage Theorem. In order for the collage distance to become arbitrarily small with increasing N, a set of conditions will have to be imposed on the set of maps $\mathcal{W}$, according to the following definitions.

Definition 1 *Let (X, d) be a compact metric space and $\mu \in \mathcal{M}(X)$. A family $\mathcal{A}$ of subsets $A = \{A_i\}$ of X is "μ-dense" in a family $\mathcal{B}$ of subsets B of X if for every $\epsilon > 0$ and any $B \in \mathcal{B}$ there exists a collection $A \in \mathcal{A}$ such that $A \subseteq B$ and $\mu(B \setminus A) < \epsilon$.*

Definition 2 *Let $\mathcal{W} = \{w_1, w_2, ...\}$, $w_i \in Con(X)$ be an infinite set of contraction maps on X. We say that $\mathcal{W}$ generates a "μ-dense and nonoverlapping" - to be abbreviated as "μ-d-n" - family $\mathcal{A}$ of subsets of X if for every $\epsilon > 0$ and every $B \subseteq X$ there exists a finite set of integers $i_k \geq 1, 1 \leq k \leq N$, such that*

1. *$A \equiv \cup_{k=1}^{N} w_{i_k}(X) \subseteq B$,*
2. *$\mu(B \setminus A) < \epsilon$ and*
3. *$\mu(w_{i_k}(X) \cap w_{i_l}(X)) = 0$ if $k \neq l$.*

A useful set of affine maps satisfying such a condition on $X = [0,1]$ with respect to Lebesgue measure is given by the following "wavelet-type" functions:

$$w_{ij}(x) = 2^{-i}(x + j - 1),\ i = 1, 2, ...,\ j = 1, 2, ..., 2^i. \tag{2.13}$$

For each $i^* \geq 1$, the set of maps $\{w_{i^*j}, j = 1, 2, ..., 2^{i^*}\}$ provides a set of 2^{-i^*}-contractions of [0,1] which tile [0,1]. As such, the set $\mathcal{W}$ provides N-map IFS with arbitrarily small degrees of refinement on (X, d).

We now describe our algorithm. Let $\mathcal{W}$ be an infinite set of fixed one-to-one affine contraction maps on $X \subset \mathbf{R}^D$ which generates a μ-dense and nonoverlapping family of subsets of X and let $\mathbf{w}^N$ denote N-map truncations of $\mathcal{W}$. Given a target function $v \in \mathcal{L}^2(X, m)$, the region Π_v^{2N}, as defined in Eq. (2.8), contains all feasible points $\mathbf{x}^N = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N}$, each of which defines a unique N-vector of affine grey level maps Φ^N,

$$\Phi^N = \{\alpha_1 t + \beta_1, ..., \alpha_N t + \beta_N\}. \tag{2.14}$$

For an $\mathbf{x}^N \in \Pi_v^{2N}$, let $T^N : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$ be the operator associated with the N-map IFSM $(\mathbf{w}^N, \Phi^N)$ and

$$\Delta_N^2 = \| v - T^N v \|_2^2 \tag{2.15}$$

denote the corresponding squared $\mathcal{L}^2$ collage distance. Since $\Delta_N^2 : \Pi_v^{2N} \to \mathbf{R}^+$ is continuous in the natural topology on $\mathbf{R}^{2N}$, it attains an absolute minimum value, $\Delta_{N,\min}^2$ on Π_v^{2N}. For each N, we may determine this minimum value using QP. The following result confirms that our procedure provides a solution to the formal inverse problem posed earlier.

Theorem 2 *$\Delta_{N,\min}^2 \to 0$ as $N \to \infty$.*

This theorem implies that the set of all attractors $\bar{u}$ of N-map IFSM constructed from $\mathcal{W}$ is dense in $\mathcal{L}^p(X)$.

In Figure 2.1 are given some results of this method as applied to the approximation of $v(x) = \sin(\pi x)$ on $X = [0,1]$. The wavelet maps of Eq. (2.13) were used. The truncated IFS map vectors $\mathbf{w}^N$ in Eq. (2.12) were formed by arranging the wavelet w_{ij} maps in the following manner:

$$w_{1,1}, w_{1,2}, w_{2,1}, \ldots, w_{2,4}, w_{3,1}, \ldots, w_{3,8}, \ldots. \tag{2.16}$$

2.3 "Local IFSM" on $\mathcal{L}^p(X, m))$

Our IFSM method can easily be generalized to incorporate the strategy originally described by Jacquin [19], namely, that we consider the actions of contractive maps w_i on *subsets* of X (the "parent blocks") to produce smaller subsets of X (the "child blocks"). (This is also referred to as a "local IFS" (LIFS) in [3].) Rather than trying to approximate a target as a union of contracted copies of itself as in the IFS method, the local IFS method tries to express the target as a union of copies of *subsets* of itself.

Here we formulate a simple "local IFSM" (LIFSM) - in fact, the usual "Fractal Transform" employed in the literature - on $\mathcal{L}^2(X, \mu)$, where $\mu = m$. Let $R_k \subset X,\ k = 1, 2, ..., N$, with $N \geq 1$, such that

1. $\cup_{k=1}^N R_k = X$ (tiling condition) and
2. $\mu(R_j \cap R_k) = 0$ for $j \neq k$ (μ-nonoverlapping condition).

Also suppose that for each $R_k,\ 1 \leq k \leq N$, there exists an $D_{j(k)} \subseteq X$ and a map $w_{i(k),k} \in Aff_1(X)$, with contractivity factor $c_{i(k),k}$, such that $w_{i(k),k}(D_{i(k)}) = R_k$. In other words, for each *range* or *child* block R_k, there is a corresponding *domain* or *parent* block $D_{i(k)}$. For each map $w_{i(k)} : D_{i(k)} \to R_k$, let there be a grey level map $\phi_k : \mathbf{R} \to \mathbf{R}$. The vectors $\mathbf{w}_{loc} = \{w_{i(1),1}, ..., w_{i(N),N}\}$ and Φ comprise an N-map LIFSM $(\mathbf{w}_{loc}, \Phi)$. Now define an associated operator $T_{loc} : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$ as follows: For $u \in \mathcal{L}^p(X, \mu)$ and $x \in R_k,\ k \in \{1, 2, ..., N\}$,

$$(T_{loc}u)(x) \equiv \begin{cases} \phi_k(u(w_{i(k),k}^{-1}(x))), & x \in R_k - \cup_{l \neq k} R_k \cap R_l, \\ 0, & x \in \cup_{l \neq k} R_k \cap R_l. \end{cases} \tag{2.17}$$

Proposition 2 *Let $X \subset \mathbf{R}^D$ and $\mu = m$. Let $(\mathbf{w}_{loc}, \Phi)$ be a local IFSM defined as above, with $\phi_k \in Lip(\mathbf{R}; \mathbf{R})$ for $1 \leq k \leq N$. Then for $u, v \in \mathcal{L}^p(X, m)$,*

$$d_p(T_{loc}u, T_{loc}v) \leq C_{loc,p} d_p(u, v), \quad C_{loc,p} = [\sum_{k=1}^N |J_k| K_k^p]^{1/p}, \tag{2.18}$$

where $|J_k|$ denotes the Jacobian of the transformation $x = w_{i(k),k}(y)$.

Remarks:

1. If $C_{loc,p} < 1$, then T_{loc} is contractive over the space $(\mathcal{L}^p(X, m), d_p)$ and possesses a unique fixed point $\overline{u}$.
2. The factor $C_{loc,p}$ is similar in form to the "optimal" factor $\bar{C}_p$ in Eq. (3.32) of Paper I, due to the nonoverlapping property of the R_k. It is not necessary to impose the restriction that all ϕ_k maps be contractive. As before, it

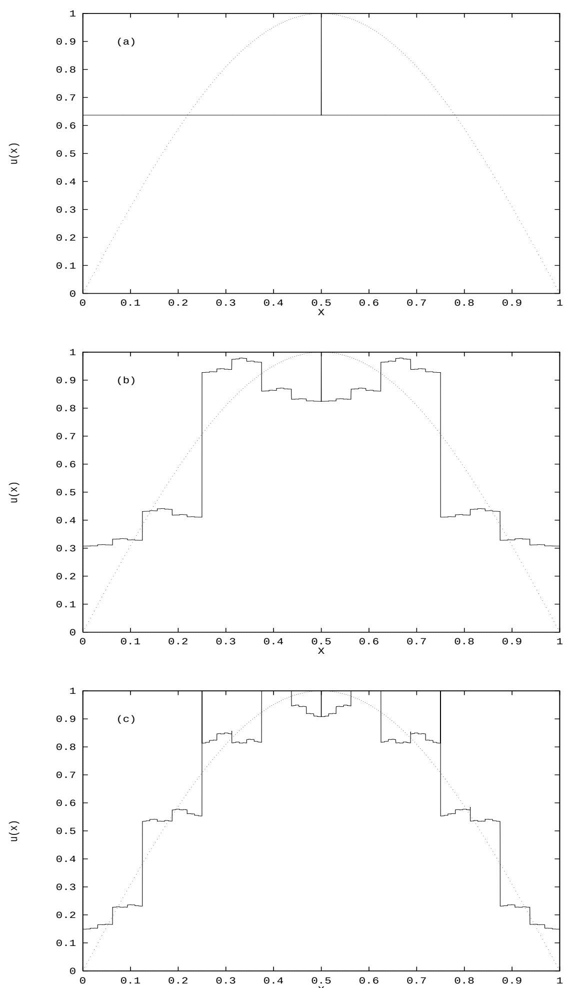

Figure 2.1: Approximations to the target set $v(x) = \sin(\pi x)$ on $X = [0, 1]$ yielded by the "normal" IFSM method of Section 2.2, using the wavelet-type basis of Eq. (2.13), with $N = 2, 6$ and 14 maps, respectively.

follows that

$$C_{loc,p}(D,p) \leq K, \qquad K = \max_{1 \leq k \leq N} K_k. \tag{2.19}$$

The weaker upper bound, K, which is independent of p, is identical to the result for the "Fractal Grey-Scale Transform" [3].

Given the above N-map LIFSM, we now compute the squared $\mathcal{L}^2$ collage distance, i.e.

$$\begin{aligned} \Delta^2 &= \| T_{loc}v - v \|_2^2 \\ &= \sum_{k=1}^{N} \int_{R_k} [\phi_i(v(w_{i(k),k}^{-1}(x)) - v(x)]^2 dx \\ &= \sum_{k=1}^{N} \Delta_k^2. \end{aligned} \tag{2.20}$$

Again, because the range blocks are conveniently nonoverlapping, the problem reduces to the minimization of each squared collage distance Δ_k^2 over the block R_k, a "least squares" determination of ϕ_k. In the special case that the ϕ_k maps are affine, the minimization of each Δ_k^2 is a quadratic programming problem in the two parameters α_k and β_k.

Given a target set v, a formal solution of the inverse problem for the nonoverlapping LIFSM case is straightforward, following the ideas of Section 2.2. The formal construction is outlined in [11].

In Figure 2.2 are some approximations to the target $v(x) = \sin(\pi x)$ using the nonoverlapping local IFSM method. As expected, the accuracy of approximation has improved. Some caution must be employed, however, as seen in Figure 2.2(c), where two range blocks and eight domain blocks have been used. The approximation is rather poor since the "halves" of the function $\sin(\pi x)$ provide poor collages of the rather straight portions R_k, $k = 1, 2, 3, 6, 7, 8$. As a result, it is necessary to employ more refined partitions for the parent cells.

2.4 Inverse Problem With Place-Dependent IFSM

The above methods for a formal solution to the inverse problem can be applied to place-dependent IFSM (introduced in Paper I). The expression for the squared $\mathcal{L}^2$ collage distance will depend upon the functional form assumed for the ϕ_k maps. We consider the following "nonoverlapping IFS" case:

1. $X \subset \mathbf{R}^D$ and $\mu = m$. We consider only the case $D = 1$ here, since the expressions involving the variable $s \in X$ become quite complicated.

2. $w_i \in Aff_1(X)$. As well, $X = \cup_{i=1}^{N} X_i$, where $X_i = w_i(X)$; in other words, the X_i "tile" the space X.

3. $\mu(X_i \cap X_j) = 0$ for $i \neq j$ (μ-nonoverlapping condition).

We assume that the grey-level maps ϕ_i assume the following functional form:

$$\phi_i(t, s) = \alpha_i(s)t + \beta_i(s), \quad t \in \mathbf{R}, \ s \in X, \tag{2.21}$$

where

$$\alpha_i(s) = \sum_{j=0}^{n_1} a_{ij} s^j, \quad \beta_i(s) = \sum_{j=0}^{n_2} b_{ij} s^j. \tag{2.22}$$

(The special case $n_1 = n_2 = 0$ corresponds to the "normal" IFSM affine maps.) Each squared collage distance Δ_i^2 over X_i becomes

$$\begin{aligned}
\Delta_i^2 &= \int_{X_i} [\alpha_i(w_i^{-1}(x))v((w_i^{-1}(x))) + \beta_i(w_i^{-1}(x)) - v(x)]^2 dx \\
&= c_i \int_X [\alpha_i(x)v(x) + \beta_i(x) - v(w_i(x))]^2 dx \\
&= c_i \int_X [v(x) \sum_{j=0}^{n_1} a_{ij} x^j + \sum_{k=0}^{n_2} b_{ik} x^k - v(w_i(x))]^2 dx. \qquad (2.23)
\end{aligned}$$

Δ_i^2 is a quadratic form in the coefficients a_{ij}, b_{ij}. The coefficients of this quadratic form involve power moments of the functions v, v^2 and $v \circ w_i$ as well as moments over X. The minimization of Δ_1^2 is a quadratic programming (QP) problem subject to suitable constraints.

The method of "least squares" can be applied to this problem. By imposing the stationarity conditions,

$$\frac{\partial \Delta_i^2}{\partial a_{ij}} = \frac{\partial \Delta_i^2}{\partial b_{ij}} = 0, \qquad j = 1, 2, ..., n, \tag{2.24}$$

one obtains a set of linear equations in the place-dependent polynomial coefficients α_{ij}, β_{ij}.

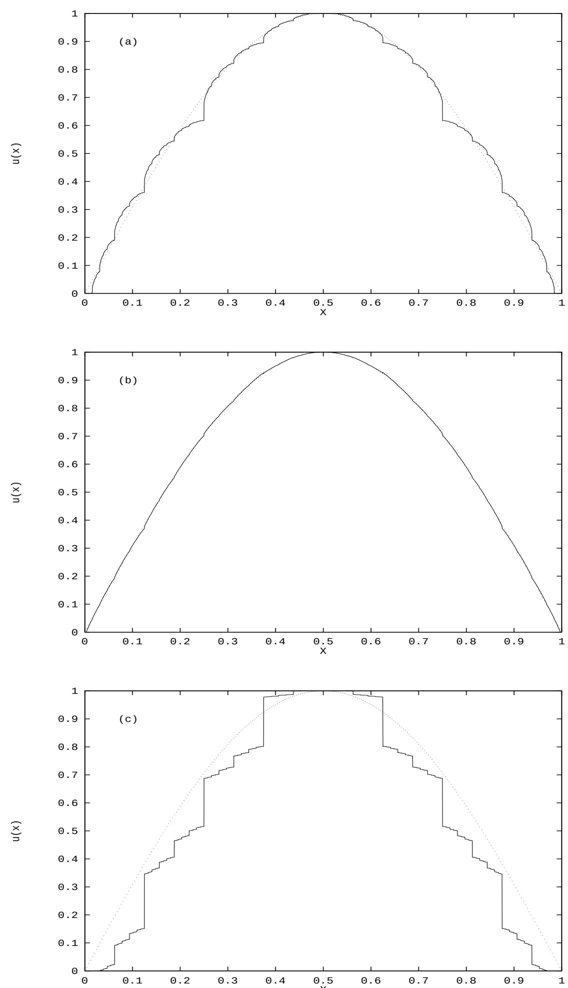

Figure 2.2: Approximations to the target set $v(x) = \sin(\pi x)$ on $X = [0,1]$ yielded by the nonoverlapping Local IFSM method of Section 2.3 using N_I parent intervals and N_J child intervals. (a) $(N_I, N_J) = (2,4)$. (b) $(N_I, N_J) = (4,8)$. (c) $(N_I, N_J) = (2,8)$.

Place-dependent grey level maps could also be used in the more general overlapping IFS case. The coefficients of the quadratic form in the a_{ij}, b_{ij} involve power moments. It is also quite straightforward to use place-dependent grey level maps in the "Local IFSM" formalism given in the previous section. In Figure 2.3 are shown some results for a PDLIFSM approach applied to the target $v(x) = \sin(\pi x)$. A comparison with Figures 2.1 and 2.2 shows that the PD method yields a better approximation even in the case where the domain block is $X = [0, 1]$.

Our numerical calculations in [11] confirmed the statements of some workers [22, 30] that there is little need for searching for optimal domain blocks when place-dependent grey level maps are used. We have found experimentally that for most domain-range block pairs, the minimum collage distances yielded by each of the eight possible affine IFS maps were equal to two, if not three, figures of accuracy. As well, we found that the minimal collage distances yielded by various domain blocks do not differ by much. Of course, the reduction of searching represents an enormous saving in computer time.

2.5 Application of IFSM Methods to Images

We summarize this section with a brief comparison of the various IFSM results outlined above, using the target image "Lena" in Figure 2.4 (512×512 pixel array with each pixel assuming one of 256 grey level values). In Figure 2.5 is shown the result of the nonoverlapping Local IFSM method of Jacquin using 16×16 pixel domain blocks and 8×8 pixel range blocks. No searching for optimal domain blocks was done - for each range block, only the domain block containing was used. All 8 possible IFS maps, however, were tested. In Figure 2.6 is shown the approximation from a place-dependent Local IFSM method - the "Bath Fractal Transform" using the same domain and range blocks as in Figure 2.5. The grey level maps were affine in both grey level value as well as spatial variable, i.e. $n_1 = 0$ and $n_2 = 1$ in Eq. (2.22):

$$\phi_i(t, s) = \alpha_{i,0} t + \beta_{i,1} s + \beta_{i,0}, \tag{2.25}$$

It is found that some improvement is made if a quadratic function for the offset term, i.e. $n_2 = 2$ in Eq. (2.22), is assumed. However, there is little, if any, improvement if higher order polynomials in the $\alpha_i(s)$ term, i.e. $n_1 > 1$ are assumed.

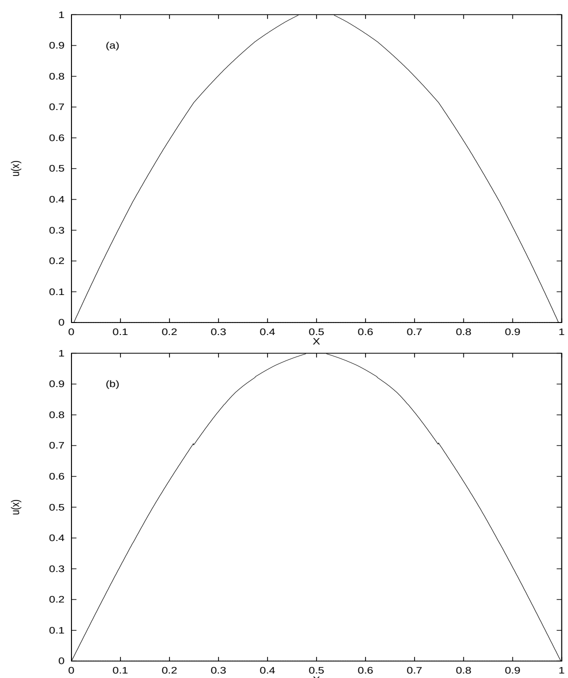

Figure 2.3: Approximations to the target set $v(x) = \sin(\pi x)$ on $X = [0, 1]$ yielded by the place-dependent nonoverlapping Local IFSM method of Section 2.4 using N_I parent intervals and N_J child intervals. (a) $(N_I, N_J) = (1, 2)$. Even with such low resolution, there is already a marked improvement in the approximation as compared to the and "local" IFSM methods of Figure 2.2. (b) $(N_I, N_J) = (2, 4)$.

Figure 2.4: The target image "Lena", a 512×512 pixel array, 8 bits (256 grey-level values) per pixel.

Figure 2.5: Approximation to target image "Lena" using nonoverlapping Local IFSM method of Jacquin with $N_I = 32^2$ domain blocks (16×16 pixel blocks) and $N_J = 64^2$ range blocks (8×8 pixel blocks). For each child block, only the parent block containing it was used. All 8 possible maps were tested. $\mathcal{L}^1$ error $\| v - \overline{u} \|_1 = 0.029$, Relative $\mathcal{L}^1$ error $= 0.068$. (Unoptimized) coding time = 34 sec..

Figure 2.6: Approximation to target image “Lena” using place-dependent Local IFSM method with $N_I = 32^2$ domain blocks (16×16 pixel blocks) and $N_J = 64^2$ range blocks (8×8 pixel blocks). No search for optimal parent blocks. For each child, only the parent containing it was used. Only the map producing no rotation or inversion was used. $\| \ v - \overline{u} \ \|_1 = \ 0.022$, Relative $\mathcal{L}^1$ error $= 0.05$. (Unoptimized) coding time = 27 sec..

Chapter 3

Approximation of Measures Using IFSP

The approximation of measures must employ indirect methods involving either moments or transforms. Here we outline the important steps behind a solution to the inverse problem for IFS approximation of measures using moments. An inverse problem may also be formulated in terms of Fourier transforms. A Collage Theorem for Fourier transforms of measures in $\mathcal{M}(X)$ is presented in the Appendix.

3.1 Approximation by "Moment Matching"

Much of the early work on the approximation of measures using IFSP was based on a knowledge of the moments of a target measure. Some form of "moment matching" was applied, in the following spirit: Given a target measure $\nu \in \mathcal{M}(X)$ ($X \subset \mathbf{R}$ for simplicity of notation), with moments $g_n = \int x^n d\nu$, $n = 0, 1, 2, \ldots$, find an IFS invariant measure $\bar{\mu}$ whose respective moments $\bar{g}_n \int x^n d\bar{\mu}$ are "close" to the g_n. In practical applications, moment matching is performed on a finite sequence of moments. In the case of an IFS with affine maps, the moments $\bar{g}_n$ of its invariant measure $\bar{\mu}$ may be computed recursively from the coefficients of the IFS maps as well as the associated probabilities. In [12], we provided a formal solution to the inverse problem for measure approximation by IFSP as well as an algorithm to approximate measures to arbitrary accuracy. Our method differs from previous efforts in two principal aspects:

1. We work with a *fixed*, infinite set of affine IFS maps which satisfy a density condition quite analogous to the μ-dense and nonoverlapping property of

IFSM. Thus, only an optimization over the probabilities p_i is required.

2. The moment matching is accomplished by means of a *Collage Theorem for Moments*. The minimization of the squared collage distance in moment space is a quadratic programming (QP) problem in the p_i with a linear constraint. This problem can be solved numerically in a finite number of steps. The advantages of QP were outlined in Section 2.1 above.

Moment matching for the approximation of measures on $[0,1]^n$ can be justified by the fact that the convergence of moments is equivalent to the weak convergence of measures. Since we are working on compact spaces, the latter convergence is equivalent to convergence in Hutchinson metric d_H. This is summarized in the following theorem.

Theorem 3 *[6] For $X = [0,1]$ let $\mu, \mu^{(n)} \in \mathcal{M}(X)$, $n = 1,2,3,\ldots$, with power moments defined by*

$$g_k = \int_X x^k d\mu, \quad g_k^{(n)} = \int_X x^k d\mu^{(n)}, \;\; k = 0,1,2,\ldots . \tag{3.1}$$

Then the following statements are equivalent:
(i) $g_k^{(n)} \to g_k$ as $n \to \infty$, $k = 0,1,2,\ldots$,
(ii) the sequence of measures $\mu^{(n)}$ converges weak* to μ, i.e. for any $f \in C(X)$, $\int f d\mu^{(n)} \to \int f d\mu$ as $n \to \infty$,
(iii) $d_H(\mu^{(n)}, \mu) \to 0$ as $n \to \infty$.

The results of this theorem apply to $[0,1]^n$.

3.2 Collage Theorem for Moments

Let $X = [0,1]$. (The extension to $[0,1]^n$ is straightforward.) As well, we consider only affine maps having the form

$$w_i(x) = s_i x + a_i, \quad c_i = |s_i| < 1, \quad 1 \le i \le N. \tag{3.2}$$

The use of affine maps leads to rather simple relations involving the moments of probability measures. That it will be sufficient to consider only affine maps is a result of the following theorem [6].

Theorem 4 *Let (X,d) denote a compact metric space and $\mathcal{M}_{AIFS}(X) \subset \mathcal{M}(X)$ the subset of invariant measures of affine IFS on X. Then $\mathcal{M}_{AIFS}$ is dense in $(\mathcal{M}(X), d_H)$.*

The above theorem is, in turn, a consequence of the following result.

Theorem 5 *[24] Let (X, d) denote a compact metric space and let $\mathcal{M}_f(X) \subset \mathcal{M}(X)$ denote the set of all measures with finite support. The $\mathcal{M}_f(X)$ is dense in $\mathcal{M}(X)$.*

We now provide the mathematical setting for the inverse problem applied to moments using IFSP. First we introduce the space $D(X)$ of all (infinite) moment vectors for probability measures in $\mathcal{M}(X)$:

$$D(X) = \{\mathbf{g} = (g_0, g_1, g_2, \ldots) \quad | \quad g_n = \int_X x^n d\mu,\ n = 0, 1, 2, \ldots, \text{ for some } \mu \in \mathcal{M}(X)\}. \tag{3.3}$$

(Note that $g_0 = 1$.) Then define the following metric on $D(X)$: For $\mathbf{u}, \mathbf{v} \in D(X)$,

$$\bar{d}_2(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{\infty} \frac{1}{k^2}(u_k - v_k)^2. \tag{3.4}$$

It was proved in [12] that $(D(X), \bar{d}_2)$ is a complete metric space.

Let $(\mathbf{w}, \mathbf{p})$ be an N-map affine IFS with Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$. Furthermore, let $\mu, \nu \in \mathcal{M}(X)$, with $\nu = M\mu$. Note that

$$\begin{aligned} \int_X f(x) d\nu(x) &= \int_X f(x) d(M\mu)(x) \\ &= \sum_{i=1}^{N} p_i \int_X (f \circ w_i)(x) d\mu(x). \end{aligned} \tag{3.5}$$

The moment vectors of μ and ν will be denoted by $\mathbf{g}, \mathbf{h} \in D(X)$, respectively. From the above equation, setting $f(x) = x^n$, $n = 1, 2, \ldots$, we have

$$h_n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left[\sum_{i=1}^{N} p_i s_i^k a_i^{n-k} \right] g_k, \quad n = 1, 2, \ldots. \tag{3.6}$$

Thus to each Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$, there corresponds a linear operator $A : D(X) \to D(X)$ so that $\mathbf{h} = A\mathbf{g}$. In the standard basis $\{\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots)\}_{i=0}^{\infty}$, the (infinite) matrix representation of A is lower triangular. The diagonal elements of this matrix are $a_{00} = 1$ and

$$a_{nn} = \sum_{i=1}^{N} p_i s_i^n, \quad n \geq 1. \tag{3.7}$$

Since $|a_{nn}| \leq c^n < 1$ for $n \geq 1$, we have the following results.

Proposition 3 *The linear operator A is contractive in $(D(X), \bar{d}_2)$.*

Corollary 1 *The operator A has a unique and attractive fixed point $\bar{\mathbf{g}} \in D(X)$.*

The components $\bar{g}_n$ of $\bar{\mathbf{g}}$ are the moments of $\bar{\mu} = M\bar{\mu}$, the invariant measure of the IFSP $(\mathbf{w}, \mathbf{p})$. (These moments may be computed recursively by a slight rearrangement of Eq. (3.6) above.)

Corollary 2 *(Collage Theorem for Moments): Assume that(X, d) is a compact metric space and $\mu \in \mathcal{M}(X)$ with moment vector $\mathbf{g} \in D(X)$. Let $(\mathbf{w}, \mathbf{p})$ be an N-map IFSP with contractivity factor $c \in [0, 1)$ such that $\bar{d}_2(\mathbf{g}, \mathbf{h}) < \epsilon$, where $\mathbf{h} = A\mathbf{g}$ is the moment vector corresponding to $\nu = M\mu$. Then*

$$\bar{d}_2(\mathbf{g}, \bar{\mathbf{g}}) < \frac{\epsilon}{1-c}, \tag{3.8}$$

where $\bar{\mathbf{g}} \in D(X)$ is the moment vector corresponding to $\bar{\mu}$, the invariant measure of the IFSP $(\mathbf{w}, \mathbf{p})$.

An inverse problem for the approximation of measures in $\mathcal{M}(X)$ may now be posed as follows: Let $\nu \in \mathcal{M}(X)$ be a target measure with moment vector $\mathbf{g} \in D(X)$. Given a $\delta > 0$, find an IFSP $(\mathbf{w}, \mathbf{p})$ with Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$ and associated linear operator $A : D(X) \to D(X)$ such that $\bar{d}_2(\mathbf{g}, A\mathbf{g}) < \delta$. As for the IFSM case, we consider N-map affine IFSP $(\mathbf{w}, \mathbf{p})$ for which the IFS maps w_i are fixed. The problem then reduces to the determination of probabilities p_i which minimize the moment collage distance $\bar{d}_2(\mathbf{g}, \mathbf{h}) < \delta$, where $\mathbf{h} = A\mathbf{g}$.

We denote the squared collage distance between moment vectors in $(D(X), \bar{d}_2)$ as

$$S(\mathbf{p}) = \sum_{n=1}^{\infty} \frac{1}{n^2}(h_n(\mathbf{p}) - g_n)^2. \tag{3.9}$$

From Eq. (3.6) the moments h_n are given by

$$h_n = \sum_{i=1}^{N} A_{ni} p_i, \quad n = 1, 2, 3, \ldots, \tag{3.10}$$

where

$$A_{ni} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) s_i^k a_i^{n-k} g_k. \tag{3.11}$$

Thus the function $S(\mathbf{p})$ may be written in the form

$$S(\mathbf{p}) = \mathbf{p}^T \mathbf{Q} \mathbf{p} + \mathbf{b}^T \mathbf{p} + s_0, \quad \mathbf{p} \in \Pi^N, \tag{3.12}$$

where

$$\Pi^N = \{\mathbf{p} = (p_1, p_2, \ldots, p_N) \mid \sum_{i=1} p_i = 1,\ p_i \geq 0\}. \tag{3.13}$$

The elements of the symmetric matrix **Q** are given by

$$q_{ij} = \sum_{n=1}^{\infty} \frac{1}{n^2} A_{ni} A_{nj}, \quad 1 \leq i, j \leq N. \tag{3.14}$$

The elements of **b** are

$$b_i = -2 \sum_{n=1}^{\infty} \frac{1}{n^2} A_{ni} g_n, \quad 1 \leq i \leq N \tag{3.15}$$

and

$$s_0 = \sum_{n=1}^{\infty} \frac{g_n^2}{n^2}. \tag{3.16}$$

The minimization of the squared moment collage distance $S(\mathbf{p})$ is a quadratic programming problem with linear constraints,

$$\text{minimize} \quad S(\mathbf{p}), \quad \sum_{i=1}^{N} p_i = 1, \quad p_i \geq 0. \tag{3.17}$$

3.3 Formal Solution to Inverse Problem

It will now be necessary to guarantee that the collage distance $S(\mathbf{p})$ can be made arbitrarily small. As in the IFSM case, we construct sequences of N-map IFSP, denoted as $(\mathbf{w}^N, \mathbf{p}^N)$, where the IFS maps in $\mathbf{w}^N$ are chosen from a fixed, infinite set $\mathcal{W}$ of contraction maps on X. A condition must be placed on this set, according to the following definition.

Definition 3.2: An infinite set of contraction maps $\mathcal{W} = \{w_1, w_2, \ldots\}$, $w_i \in Con(X)$ is said to satisfy an ϵ-contractivity condition on X if for each $x \in X$ and any $\epsilon > 0$, there exists an $i^* \in \{1, 2, \ldots\}$ such that $w_{i^*}(X) \subset N_\epsilon(x)$, where $N_\epsilon(x)\ \{y \in X \mid d(x, y) < \epsilon\}$ denotes the ϵ-neighbourhood of x.

If $\mathcal{W}$ satisfies the ϵ-contractivity condition on X, then $\inf_{1 \leq i \leq \infty} c_i = 0$. The set $\mathcal{W}$ provides N-map IFS with arbitrarily small degrees of refinement on (X, d). The "wavelet-type" basis functions of Eq. (2.13) conveniently satisfy such an ϵ-contractivity condition.

Now let $\mathcal{W} = \{w_1, w_2, \cdots\}$ be an infinite set of affine contraction maps on $X = [0, 1]$ which satisfies the ϵ-contractivity condition. Let

$$\mathbf{w}^N = \{w_1, w_2, \ldots w_N\}, \quad N = 1, 2, \ldots, \tag{3.18}$$

denote N-map truncations of $\mathcal{W}$. As well, let

$$\Pi^N = \{\mathbf{p}^N = (p_1, p_2, \cdots, p_n) \,|\, p_i \geq 0, \quad \sum_{i=1}^{N} p_i = 1\} \tag{3.19}$$

denote the set of all probability N-vectors for $\mathbf{w}^N$. Note that $\Pi^N \subset \mathbf{R}^N$ is compact in the natural topology on $\mathbf{R}^N$. Now let $\mu \in \mathcal{M}(X)$ be a target measure with moment vector $\mathbf{g} \in D(X)$. For a $\mathbf{p}^N \in \Pi^N$, let M^N be the Markov operator corresponding to the N-map IFS $(\mathbf{w}^N, \mathbf{p}^N)$. Also let $\nu_N = M^N \mu$, with associated moment vector $\mathbf{h}_N \in D(X)$. The collage distance between the moment vectors of μ and ν_N will be denoted as

$$\Delta^N(\mathbf{p}^N) \equiv \; ||\mathbf{g} - \mathbf{h}_N||_{\bar{l}^2} \; . \tag{3.20}$$

Since $\Delta^N : \Pi^N \to \mathbf{R}^+$ is continuous, it attains an absolute minimum value, to be denoted as $\Delta^N_{\min}$, on Π^N. The following result establishes that the above procedure provides a solution to the inverse problem for measure approximation.

Theorem 6 $\Delta^N_{\min} \to 0$ *as* $N \to \infty$.

Remarks:

1. Theorem 6 is a density result establishing that the set of invariant measures for all N-map IFS $(\mathbf{w}^N, \mathbf{p}^N)$ where $\mathbf{p}^N \in \Pi^N$, $N = 1, 2, \ldots$, is dense in $(\mathcal{M}(X), d_H)$. This result can be extended to $[0,1]^q, q \geq 2$.

2. Although not explicitly stated in the proof, the collage distances Δ^N and, in particular, the sequence $\Delta^N_{\min}$, are also dependent on the ordering of the w_i maps in the infinite set $\mathcal{W}$. However, at this point, we are not interested in any questions about the "optimal" ordering of the maps in $\mathcal{W}$ nor how N-map subsets $\mathbf{w}^N$ should be chosen.

3.4 Some Numerical Results

We present some results of the above approximation method to show its important features. The target measure μ considered here the Lebesgue-Stieltjes measure generated by the following distribution $F(x)$ on $[0,1]$:

$$F(x) = \begin{cases} x, & x \in [0, \frac{1}{3}), \\ \frac{1}{3}, & x \in [\frac{1}{3}, \frac{2}{3}), \\ x, & x \in [\frac{2}{3}, 1], \end{cases} \tag{3.21}$$

i.e. $\mu(a,b] = F(b) - F(a)$, $a < b$. The moments of this measure are given by

$$\begin{aligned} g_n &= \int_0^{\frac{1}{3}} x^n\,dx + \frac{1}{3}\left(\frac{2}{3}\right)^n + \int_{\frac{2}{3}}^1 x^n\,dx \\ &= \frac{1}{n+1}\left[1 + \left(\frac{1}{3}\right)^{n+1} - \left(\frac{2}{3}\right)^{n+1}\right] + \frac{1}{3}\left(\frac{2}{3}\right)^n, \quad n \geq 0. \end{aligned} \tag{3.22}$$

The wavelet-type maps of Eq. (2.13) were used here. The truncated IFS map vectors $\mathbf{w}^N$ were formed by arranging the w_{ij} maps in the same manner as in Eq. (2.16). In practical calculations, only a finite number of moments can be matched. As such, the following function,

$$S_M^N(\mathbf{p}^N) = \sum_{n=1}^{M} \frac{1}{n^2}\left[\sum_{i=1}^{N} A_{ni}p_i - g_n\right]^2, \tag{3.23}$$

was minimized, subject to the constraints on the probabilities. In calculations reported here, $M = 30$ moments were matched. The minimization of the function S_M^N was performed with a quadratic programming algorithm developed by Best and Ritter [4].

Figure 3.1 shows approximations $\bar{F}_N(x)$ to $F(x)$ yielded by the optimal IFSP for values of $N = 2, 6, 14$, respectively. The $\bar{F}_N(x)$ functions were approximated by generating discrete approximations of the invariant measures of the $(\mathbf{w}^N, \mathbf{p}^N)$ on a lattice of 2000 points on [0,1]. There are two important features of these calculations:

1. As N increases, $\bar{F}_N(x)$ is seen to converge to $F(x)$, with better approximations to the jump at $x = \frac{2}{3}$.

2. For each $N \geq 3$, the minimum of $S_M^N(\mathbf{p}^N)$ located by the QP algorithm occurred on a boundary of the feasible region Π^N, implying that there were some zero probabilities. The actual number of nonzero probabilities which existed for the cases $N = 2, 6, 14$ in Figure 3.1 were $\bar{N} = 2, 4, 8$, respectively. As such, the QP algorithm has performed a data compression, eliminating the unnecessary IFS maps. (This would not necessarily have been done if gradient methods were used to locate the minima of the objective function.)

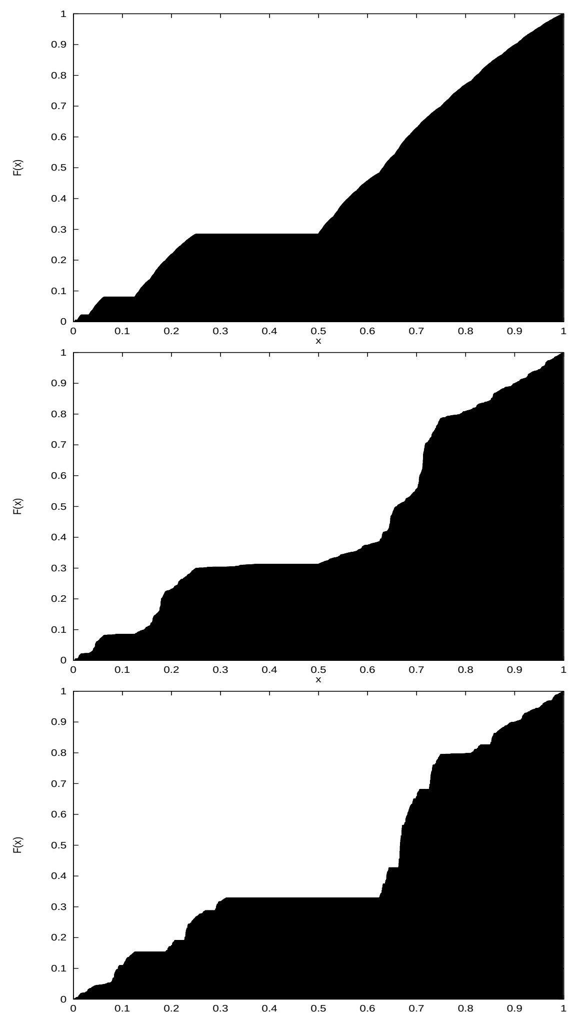

Figure 3.1: Approximations to the distribution $F(x)$ of Eq. (3.21) obtained by moment matching, using the wavelet-type basis of Eq. (2.13), with $N = 2, 6$ and 14 maps, respectively.

Chapter 4

IFSM and Fractal Wavelet Compression

Let $(\mathbf{w}, \Phi)$ be an N-map affine IFSM on $X = [0, 1]$ and suppose that its associated operator T is contractive on $(\mathcal{L}^p(X, m)$ for some (or all) $p \geq 1$. Then the fixed point $\bar{u}$ satisfies the equation (cf. Eq. (3.36) in Paper I)

$$\bar{u}(x) = \sum_{k=1}^{N} [\alpha_k \bar{u}(w_k^{-1}(x)) + \beta_k I_{X_k}(x)] \tag{4.1}$$

As noted in Paper I, $\bar{u}$ is expressed as a linear combination of dilated and translated copies of itself along with piecewise constant functions. This is somewhat reminiscent of using wavelet functions which are obtained by dilatations and translations of a "mother wavelet" function. The IFSM method may be viewed as an *adaptive* encoding since the "mother" function is the target itself. However, the copies $\psi_k(x) = \bar{u}(w_k^{-1}(x))$ are generally not orthogonal to each other. In fact, the above expansion may be considered as highly redundant - the piecewise constant functions are sufficient to create a linearly independent basis. Nevertheless, some workers have been investigating the idea of constructing orthonormal basis functions from the scaled copies $\psi_k(x)$ of the target function $v(x)$ [28].

If the affine grey level maps are now assumed to be place-dependent, having the form

$$\phi_k(t, s) = \alpha_k t + \gamma_k(s), \quad 1 \leq k \leq N, \tag{4.2}$$

(with the offset term β_k being absorbed by the function $\gamma_k(s)$), the fixed point equation for $\bar{u}$ then becomes

$$\bar{u}(x) = \sum_{k=1}^{N} [\alpha_k \bar{u}(w_k^{-1}(x)) + \gamma_k(w_k^{-1}(x))] \tag{4.3}$$

If $\alpha_k = 0$ for $1 \leq k \leq N$ (implying that the contraction factor of T is $C_p = 0$), then the above equation reduces to an expansion of $\bar{u}$ in terms of functions $\gamma_k : X \to \mathbf{R}$. In most practical applications to date (e.g. the Bath Fractal Transform [22, 23, 30]) these functions have been assumed to be polynomial. However, one could equally assume that the $\gamma_k(s)$ were linear combinations of orthonormal functions, e.g. $\{\cos(n\pi x)\}$, etc..

In the place-dependent local IFSM formalism (PDIFSM). the functions $\gamma_k(s)$ now map *domain blocks* D_k to the grey level range $\mathbf{R}$. Moreover, the compositions $(\gamma_k \circ w^{-1}_{j(k),k})(x)$ are now translations and dilatations of the $\gamma_k(s)$ functions. If the $\gamma_k(s)$ are chosen to be *wavelet* functions, then the orthogonality is guaranteed. In the special case that all grey-level scaling parameters α_k are zero, the place-dependent local IFSM method can be made to coincide with wavelet expansions. What remains is to examine wavelet expansions of functions/images from an *indirect* approach as was done for measures, i.e. through their moments.

For simplicity we restrict our attention to the one-dimensional case, i.e. $X = [0,1]$. Let $\{q_n\}_{n=0}^{\infty}$, with $q_0(x) = 1$, denote a complete set of orthonormal basis functions on X. Then for a given $u \in \mathcal{L}^2(X, m)$,

$$u(x) = \sum_{k=0}^{\infty} c_k q_k(x), \tag{4.4}$$

where

$$c_k = < u, q_k > = \int_X u(x) q_k(x) dx. \tag{4.5}$$

Now let $(\mathbf{w}, \Phi)$ denote an N-map affine IFSM on X with associated operator T and let $v = Tu$. Then

$$v(x) = \sum_{k=0}^{\infty} d_k q_k(x), \tag{4.6}$$

where

$$\begin{aligned} d_k &= < q_k, v > \\ &= < q_k, Tu > \\ &= \sum_{i=1}^{N} \alpha_i < q_k, u \circ w_i^{-1} > + \sum_{i=1}^{N} \beta_i < q_k, I_{w_i(X)} > . \end{aligned} \tag{4.7}$$

Using Eq. (4.4) to expand $u(w_i^{-1}(x))$ yields

$$d_k = \sum_{l=1}^{\infty} a_{kl} c_l \; + \; e_k, \tag{4.8}$$

where

$$a_{kl} = \sum_{i=1}^{N} \alpha_i < q_k, q_l \circ w_i^{-1} >, \quad e_k = \sum_{i=1}^{N} \beta_i < q_k, I_{w_i(X)} > . \tag{4.9}$$

This is the affine mapping $A : \mathbf{c} \to \mathbf{d}$ associated with the IFSM operator T. In this general case, there is a simple, yet important result:

Proposition 4 *Given an orthonormal basis $\{q_n\}$ on X as above. Suppose that the N-map IFSM operator T is contractive in $\mathcal{L}^2(X)$ with fixed point $\bar{u}$. Then the mapping $A : l^2(\mathbf{N}) \to l^2(\mathbf{N})$ in Eq. (4.9) is contractive in the $l^2(\mathbf{N})$ metric. The fixed point $\bar{\mathbf{c}} = A\bar{\mathbf{c}}$ is the vector of Fourier coefficients of $\bar{u}$ in the q_i basis.*

Proof: Let $u, v \in \mathcal{L}^2(X, m)$ with Fourier coefficients $\mathbf{c}, \mathbf{d} \in l^2(\mathbf{N})$, respectively, in the q_i basis. Since T is contractive, there exists a $C \in [0, 1)$ such that

$$\| Tu - Tv \|_{\mathcal{L}^2} \ \leq \ C \| u - v \|_{\mathcal{L}^2} . \tag{4.10}$$

From Parseval's relation, i.e. $\| u \|_{\mathcal{L}^2} \ = \ \| \mathbf{c} \|_{l^2}$, etc.,

$$\| Tu - Tv \|_{\mathcal{L}^2} \ = \ \| A\mathbf{c} - A\mathbf{d} \|_{l^2} \tag{4.11}$$

so that

$$\| A\mathbf{c} - A\mathbf{d} \|_{l^2} \ \leq \ C \| \mathbf{c} - \mathbf{d} \|_{l^2} . \tag{4.12}$$

Therefore A is contractive in $l^2(\mathbf{N})$. Since $l^2(\mathbf{N})$ is complete, there exists an element $\bar{\mathbf{c}} \in l^2(\mathbf{N})$ such that $A\bar{\mathbf{c}} = \bar{\mathbf{c}}$. The vector $\bar{\mathbf{c}}$ defines a unique function $\bar{u} \in \mathcal{L}^2(X)$ through the expansion in Eq. (4.4). From Eqs. (4.7)-(4.9), it follows that $\bar{\mathbf{c}} = A\bar{\mathbf{c}}$ implies $T\bar{u} = \bar{u}$. ∎

In general, the matrix elements $< q_k, q_l \circ w_i^{-1} >$ in Eq. (4.9) do not vanish, leading to a rather full (i.e. not sparse) matrix representation of A. This would occur, for example, in the case of the Discrete Cosine Transform. However, in the special case that the q_k are localized in space, e.g. *wavelets*, many of these matrix elements vanish. The relations between the d_k and the c_l simplify even further when the IFS maps w_i are *local* and map domain blocks which support wavelets of lower resolution/frequency to range blocks which support wavelets of higher resolution/frequency. This is the basis of what has been referred to in one form or another as "wavelet-based fractal compression" [7, 20, 25]. In what follows, we consider only the one-dimensional case for simplicity. The extension to functions/images in $\mathbf{R}^2$ is straightforward but more tedious.

The following standard dyadic multiresolution approximation of $\mathcal{L}^2(\mathbf{R})$ is assumed [21]:

1. A sequence of nested subspaces $V_j \in \mathcal{L}^2(\mathbf{R})$, $j \in \mathbf{Z}$, where $V_j \subset V_{j+1}$. V_j contains the set of all approximations of functions $f \in \mathcal{L}^2(\mathbf{R})$ at resolution 2^j. Moreover $\lim_{n\to\infty} V_n = \mathcal{L}^2(\mathbf{R})$.

2. The sequence of orthogonal complements $W_j - V_j$ such that $W_j \oplus V_j = V_{j+1}$, $j \in \mathbf{Z}$. This implies that for any $k \in \mathbf{Z}$ and $n > 0$,

$$V_k \oplus W_k \oplus W_{k+1} \oplus \ldots \oplus W_{k+n} = V_{k+n+1}. \tag{4.13}$$

3. A "scaling function" $\phi \in \mathcal{L}^2(\mathbf{R})$ such that the functions

$$\phi_{ij}(x) = 2^{i/2}\phi(2^i x - j), \quad j \in \mathbf{Z}, \tag{4.14}$$

   form an orthonormal basis for V_i.

4. The "orthogonal wavelet" $\psi \in \mathcal{L}^2(\mathbf{R})$ such that the functions

$$\psi_{ij}(x) = 2^{i/2}\psi(2^i x - j), \quad j \in \mathbf{Z}, \tag{4.15}$$

   form an orthonormal basis for W_i. It follows that the set $\{\psi_{ij}\}$, $i, j \in \mathbf{Z}$, forms a complete orthonormal basis for $\mathcal{L}^2(\mathbf{R})$.

In the case of the Haar wavelet system,

$$\phi_{00}(x) = 1, \;\; x \in [0,1), \quad \psi_{00}(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ -1, & x \in [\frac{1}{2}, 1). \end{cases} \tag{4.16}$$

Then $u \in \mathcal{L}^2(\mathbf{R})$ may be expanded as

$$u(x) = \sum_{j=-\infty}^{\infty} < u, \phi_{kj} > \phi_{kj}(x) + \sum_{l=0}^{\infty} \sum_{j=-\infty}^{\infty} < u, \psi_{k+l,j} > \psi_{k+l,j}(x). \tag{4.17}$$

For functions on $[0,1]$, all contributions from V_j, $j < 0$ vanish. (We assume some kind of periodic extension of the functions on [0,1] to **R**.) Then $k = 0$ in Eq. (4.17) so that

$$u(x) = b_{00}\phi_{00}(x) + c_{00}\psi_{00}(x) + \sum_{i=1}^{\infty} \sum_{j=0}^{2^i - 1} c_{ij}\psi_{ij}(x), \tag{4.18}$$

where, of course, $b_{00} =< u, \phi_{00} >$ and $c_{ij} =< u, \psi_{ij} >$. In the case of the Haar wavelets, the supports of the functions ψ_{ij} are the dyadic intervals

$$I_{ij} = \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right], \quad 0 \le j < 2^i. \tag{4.19}$$

For more generalized wavelets, the supports of the ψ_{ij} will be larger than these intervals. Nevertheless the orthogonality of the ψ_{ij} is preserved. In this case, we shall consider the wavelets ψ_{ij} to be centered on the intervals I_{ij}. The coefficients

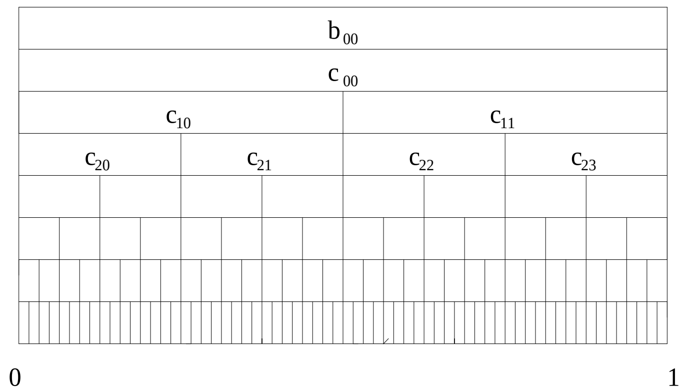

Figure 4.1: The table of wavelet coefficients b_{00} and $c_{i,j}$ associated with the expansion in Eq. (4.18). The location of each coefficient reveals the resolution as well as the spatial localization of the wavelet ψ_{ij}.

b_{00} and c_{ij} may be conveniently arranged in a table such as the one shown in Figure 4.1 which reflects the degree of resolution as well as localization in space.

For simplicity, we consider the following "nonoverlapping" local IFSM. (Our analysis may be extended to cover more general cases where "quadtree partitioning" has been used.)

1. Domain blocks given by the intervals $I_{i^*,j},\ 0 \leq j \leq 2^{i^*} - 1$,
2. Range blocks given by the intervals $I_{k^*,l},\ 0 \leq l \leq 2^{k^*} - 1$, where $k^* > i^*$.

Suppose that for each range block $I_{k^*,l}$, $l \in \{0, 1, \ldots, 2^{k^*} - 1\}$, we choose a domain block $I_{i^*,j(l)}$. The local IFSM map for this pair, denoted as $w_l : I_{i^*,j(l)} \to I_{k^*,l}$, will have the contraction factor $2^{i^*-k^*}$. There will be an associated affine grey level map $\phi_l(t) = \alpha_l t + \beta_l$ (not to be confused with the scaling function $\phi_{ij}(x)$). From Eq. (4.8) and the orthogonality property of the ψ_{ij}, we have

$$d_{k^*,l} = \alpha_l c_{i^*,j(l)} < \psi_{kl}, \psi_{ij} \circ w_l^{-1} > . \tag{4.20}$$

Since $(\psi_{ij} \circ w_l^{-1})(x) = 2^{(i-k)/2}\psi_{kl}(x)$, it follows that

$$d_{k^*,l} = \alpha_l 2^{(i^*-k^*)/2} c_{i^*,j(l)}. \tag{4.21}$$

As well, all coefficients lying below $d_{k^*,l}$ in the table will be scaled copies of the corresponding entries below $c_{i^*,j}$, that is,

$$d_{k^*+k',2^{k'}l+l'} = \alpha_l 2^{(i^*-k^*)/2} c_{i^*+k',2^{k'}j(l)+l'}, \quad k' \geq 0,\ \ 0 \leq l' \leq 2^{k'} - 1. \tag{4.22}$$

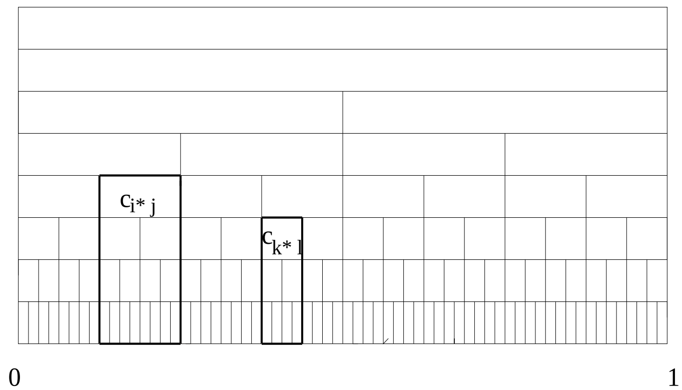

Figure 4.2: The domain and range blocks associated with the nonoverlapping IFSM in the text, cf. Eqs. (4.21) and (4.22).

This equation represents the transformation on the wavelet coefficients c_{ij} for $i \geq k^*$ induced by the local IFSM. The domain and range regions in the wavelet coefficient table are schematically illustrated in Figure 4.2. Such a transformation has been derived independently in the literature in the context of wavelet representations. Writing this transformation as an IFS-type method on wavelet coefficients has been referred to as "fractal wavelet compression". The purpose of this Section has been to show how IFSM/local IFSM induces an IFS-type transform on the wavelet coefficients.

Note that no "offset" terms involving the β_i appear in Eqs. (4.21) and (4.22). Such terms would appear only if the resolution level of the domain blocks would be $i^* = 0$, i.e. "traditional IFS", where the domain block is X itself. Again by orthogonality, the offset terms involving the β_i would contribute only to the coefficient b_{00}. (See note later in this section.)

The coefficients b_{00} and c_{ij}, $0 \leq i \leq k^* - 1$, $0 \leq j \leq 2^i - 1$ remain fixed. Therefore, once computed using Eq. (4.5), these coefficients are stored and then used to generate the higher resolution coefficients. In what follows, we provide the framework for an IFS-type inverse problem on the wavelet coefficients.

For a given function $u \in \mathcal{L}^2([0,1])$ and the above local IFSM, define the following wavelet coefficient space:

$$\begin{aligned} C_w(u,k^*) \quad &= \quad \{b_{00}, c_{kl},\ k \geq 0,\ 0 \leq l \leq 2^k - 1 \mid \sum_{k,l}^{\infty} c_{kl} < \infty, \text{ where} \\ &\qquad c_{ij} = < u, \psi_{ij} >,\ 0 \leq i \leq k^* - 1,\ 0 \leq j \leq 2^i - 1\}. \qquad (4.23) \end{aligned}$$

Let $T_w : C_w(u,k^*) \to C_w(u,k^*)$ denote the transformation induced by the

LIFSM on the wavelet coefficients. We consider the following metric on $C_w(u,k^*)$: For $\mathbf{c},\mathbf{d}\in C_w(u,k^*)$, define

$$d_w(\mathbf{c},\mathbf{d}) = \max_{0\le l\le 2^{k^*}-1} \Delta_l^2, \tag{4.24}$$

where

$$\Delta_l^2 = \sum_{k'=0}^{\infty}\sum_{l'=0}^{2^{k'}-1}(c_{k^*+k',2^{k'}l+l'} - d_{k^*+k',2^{k'}l+l'})^2. \tag{4.25}$$

Proposition 5 *The metric space $(C_w(u,k^*),d_w)$ is complete.*

Proposition 6 *For $\mathbf{c},\mathbf{d}\in C_w(u,k^*)$,*

$$d_w(T_w\mathbf{c},T_w\mathbf{d}) \le c_w d_w(\mathbf{c},\mathbf{d}), \quad c_w = \max_{0\le l\le 2^{k^*}-1} |\alpha_l| 2^{(i^*-k^*)/2}. \tag{4.26}$$

Corollary 3 *If $c_w<1$, there exists a unique $\bar{\mathbf{c}}\in C_w(u,k^*)$ such that $T_w\bar{\mathbf{c}}=\bar{\mathbf{c}}$.*

Corollary 4 *Let $\mathbf{c}\in C_w(u,k^*)$ and suppose that there exists an IFSM with associated transformation T_w such that $d_w(\mathbf{c},T_w\mathbf{c})<\epsilon$. Then*

$$d_w(\mathbf{c},\bar{\mathbf{c}}) < \frac{\epsilon}{1-c_w}, \tag{4.27}$$

where $\bar{\mathbf{c}}=T_w\bar{\mathbf{c}}$.

For the wavelet coefficient vector $\mathbf{c}$ of a target function $v\in\mathcal{L}^2([0,1])$, the squared $\mathcal{L}^2$ collage distance associated with each range block $I_{k^*,l}$ will be given by

$$\Delta_l^2 = \sum_{k'=0}^{\infty}\sum_{l'=0}^{2^{k'}-1}(c_{k^*+k',2^{k'}l+l'} - \alpha_l 2^{(i^*-k^*)/2} c_{i^*+k',2^{k'}j(l)+l'})^2. \tag{4.28}$$

The absence of terms involving β_l greatly simplifies the minimization of this distance. The optimal scaling factor is given by

$$\bar{\alpha}_l = 2^{(k^*-i^*)/2}\frac{S_{k^*,l,i^*,j}}{S_{i^*,j,i^*j}}, \tag{4.29}$$

where

$$S_{\alpha,\beta,\gamma,\delta} = \sum_{k'=0}^{\infty}\sum_{l'=0}^{2^{k'}-1} c_{\alpha+k',2^{k'}\beta+l'} c_{\gamma+k',2^{k'}\delta+l'}. \tag{4.30}$$

The minimum collage distance is

$$\Delta_{l,\min} = [S_{k^*,l,k^*,l} - \bar{\alpha}_l 2^{(i^*-k^*)/2} S_{k^*,l,i^*,j}]^{1/2}. \tag{4.31}$$

One may now proceed in a fashion similar to that of the usual LIFSM method: For a given range block $I_{k^*,l}$, find the optimal domain block $I_{k^*,j(l)}$, i.e. the block yielding the lowest minimum collage distance $\Delta_{l,\min}$. When this has been done for all range blocks, an operator T_w has then been defined. One may then generate the fixed point $\bar{\mathbf{c}}$ of T_w by the iteration process $T_w^n \mathbf{c}_0$, where $\mathbf{c}_0 \in C_w(u, k^*)$. The corresponding approximation $\bar{v}(x)$ to $v(x)$ may then be constructed by summing the resulting wavelet series in the coefficients $\bar{c}_{ij}$.

One further note regarding offset terms β_k and their role in the wavelet coefficient transformation: As stated earlier, offset terms would appear in this transformation only if $i^* = 0$, i.e. the domain blocks $D_k = X$. However, note that our local IFSM method in Paper I involves offset terms in the grey level maps. The explanation is that the wavelet expansion implicitly assumed by the LIFSM method is not Eq. (4.17) but rather the following:

$$\begin{aligned} u(x) = \sum_{j=0}^{2^{i^*}-1} b_{i^*,j}\phi_{i^*,j}(x) \quad &+ \quad \sum_{j=0}^{2^{i^*}-1} c_{i^*,j}\psi_{i^*,j}(x) \\ &+ \quad \sum_{i=1}^{\infty} \sum_{j=0}^{2^{i^*+i}-1} c_{i^*+i,j}\psi_{i^*+i,j}(x). \end{aligned} \tag{4.32}$$

In other words, the minimum resolution k in Eq. (4.17) is i^*. Each domain block D_k has been expanded separately in a wavelet series expansion. The structure of the corresponding wavelet coefficient table is sketched in Figure 4.3.

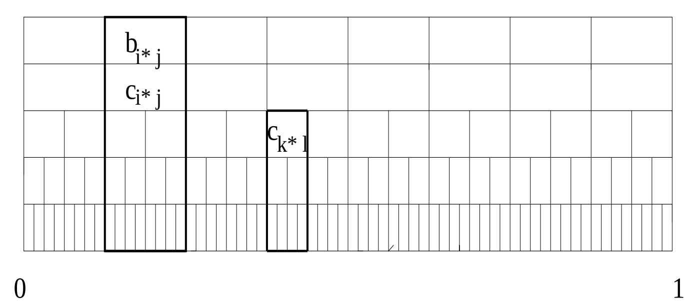

Figure 4.3: The domain and range blocks associated with the nonoverlapping IFSM in the text.

When the wavelet system used above is the Haar system, then the above fractal transform method is identical to the usual nonoverlapping LIFSM method. However, the above method also applies to more generalized wavelets ψ_{ij} which are supported on range blocks R_k which overlap with each other. Even though these blocks overlap, the wavelets remain orthogonal to each other. (Of course, the form of the local IFSM maps will have to be altered accordingly.) The overlapping of these wavelets has been shown to be beneficial as it can reduce the usual "block" effects exhibited by normal "nonoverlapping" fractal transforms [26, 27]. An example is shown in Figure 4.4. Both images use 16×16 pixel domain blocks and 8×8 pixel range blocks. However, Figure 4.4(a) was produced using the Haar wavelet basis and Figure 4.4(b) was produced using the "Coifman 12" wavelet basis [29]. Wavelet functions of the latter type which are centered on neighbouring blocks overlap with each other while remaining orthogonal to each other. The "blockiness" of the Haar expansion has been reduced somewhat by the Coifman 12 expansion. We thank Mr. A. Van de Walle for providing these images as computed from his fractal wavelet compression routine [26, 27].

Figure 4.4: Approximations to target image "Lena" using discrete wavelet fractal transform method of Section 5. Domain blocks are 16×16 pixel arrays, range blocks are 8×8 pixel arrays. (a) Haar wavelet basis. (b) "Coifman 12" wavelet basis. (Courtesy of A. Van de Walle.)

Chapter 5

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Chapter 6

Collage Theorem for Fourier Transforms

In Chapter 3, it was shown that an N-map IFSP $(\mathbf{w}, \mathbf{p})$ with associated Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$ induces a linear operator A which maps $D(X)$, the space of moment vectors for measures in $\mathcal{M}(X)$, into itself. In this section, we derive the mapping which is induced on the space of Fourier transforms of measures in $\mathcal{M}(X)$. The treatment may easily be modified to treat the discrete cosine transforms (DCT) of measures. The structure of our discussion will closely follow that of Section 3.1 on moments. As in the main text, it is assumed that $X = [0, 1]$.

Given a measure $\mu \in \mathcal{M}(X)$, we define its Fourier transform (FT) $\hat{\mu} : \mathbf{R} \to \mathbf{C}$ as

$$\hat{\mu}(\omega) = \int_X e^{-i\omega x} d\mu(x), \quad \omega \in \mathbf{R}. \tag{6.1}$$

Note that $\hat{\mu}(0) = 1$ and that $|\hat{\mu}(\omega)| \leq 1,\ \forall\, \omega \in \mathbf{R}$. Now let $FT(X)$ denote the set of FT's for all measures in $\mathcal{M}(X)$. As is well known, $\hat{\mu}(\omega) \in FT(X)$ does not necessarily imply that $\hat{\mu}(\omega) \in \mathcal{L}^2(\mathbf{R}, m)$. We thus define the following metric on $FT(X)$: For $\hat{\mu}, \hat{\nu} \in FT(X)$, let

$$d_{FT}(\hat{\mu}, \hat{\nu}) = \left[\int_{-\infty}^{\infty} |\hat{\mu}(\omega) - \hat{\nu}(\omega)|^2 \omega^{-2} d\omega \right]^{1/2}, \tag{6.2}$$

That these integrals exist is an immediate consequence of the following result: For any $\mu \in \mathcal{M}(X)$, the function $f_\mu : \mathbf{R} \to \mathbf{C}$ defined by $f_\mu(\omega) = \omega^{-1}(\hat{\mu}(\omega) - 1)$ for $\omega \neq 0$ satisfies $\int_{-\infty}^{\infty} |f_\mu(\omega)|^2 d\omega < \infty$.

Proposition 7 *$(FT(X), d_{FT})$ is a complete metric space.*

Proposition 8 *Let $\mu, \nu^{(n)} \in \mathcal{M}(X)$, $n = 1, 2, 3, \ldots$, with FT's $\hat{\mu}, \hat{\nu}^{(n)} \in FT(X)$. Then $d_T(F, F^{(n)}) \to 0$ as $n \to \infty$ iff $d_H(\mu, \nu^{(n)}) \to 0$ as $n \to \infty$.*

Now, as in Section 3.1, let $(\mathbf{w}, \mathbf{p})$ be an N-map affine IFSP with associated Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$. Let $\mu, \nu \in \mathcal{M}(X)$, with $\nu = M\mu$ and FT's $\hat{\mu}, \hat{\nu} \in FT(X)$, respectively. From Eq. (3.5) in Section 3.2, setting $f(x) = e^{-i\omega x}$, we have

$$\hat{\nu}(\omega) = \sum_{k=1}^{N} p_k e^{-i\omega a_k} \hat{\mu}(s_k \omega), \quad \omega \in \mathbf{R}. \tag{6.3}$$

Therefore, to each Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$, there corresponds a linear operator $B : FT(X) \to FT(X)$.

Proposition 9 *The linear operator B is contractive in $(FT(X), d_{FT})$.*

Proof: Let $\hat{\mu}, \hat{\nu} \in FT(X)$. Then

$$\begin{aligned} d_{FT}(B(\hat{\mu}), B(\hat{\nu})) &= \left[\int_{-\infty}^{\infty} \left| \sum_{k=1}^{N} p_k e^{-i\omega a_k} [\hat{\mu}(s_k\omega) - \hat{\nu}(s_k\omega)] \right|^2 \omega^{-2} d\omega \right]^{1/2} \\ &\leq \sum_{k=1}^{N} p_k \left[\int_{-\infty}^{\infty} |\hat{\mu}(s_k\omega) - \hat{\nu}(s_k\omega)|^2 \omega^{-2} d\omega \right]^{1/2} \\ &= \sum_{k=1}^{N} p_k |s_k|^{1/2} \left[\int_{-\infty}^{\infty} |\hat{\mu}(\tau) - \hat{\nu}(\tau)|^2 \tau^{-2} d\tau \right]^{1/2} \\ &\leq c^{1/2} d_T(\hat{\mu}, \hat{\nu}). \quad \blacksquare \end{aligned} \tag{6.4}$$

Corollary 5 *The operator B has a unique and attractive fixed point $\hat{\bar{\mu}} \in D(X)$.*

Note: $\hat{\bar{\mu}}$ is the FT of the invariant measure $\bar{\mu}$ of the affine IFSP $(\mathbf{w}, \mathbf{p})$ and satisfies the relation

$$\hat{\bar{\mu}}(\omega) = \sum_{k=1}^{N} p_k e^{-i\omega a_k} \hat{\bar{\mu}}(s_k \omega), \quad \omega \in \mathbf{R}. \tag{6.5}$$

In other words, $\hat{\bar{\mu}}(\omega)$ satisfies a "self-tiling property": its graph may be expressed as a linear combination of dilated copies of itself. However, the graphs $\hat{\bar{\mu}}(s_k\omega)$ are copies of $\hat{\bar{\mu}}(\omega)$ which are *stretched* along the ω-axis, unlike the case for IFSM functions. (This is perfectly in accord with the "duality" of space and frequency variables in Fourier transforms.)

Corollary 6 *(Collage Theorem for Fourier Transforms): Let* (X, d) *be a compact metric space and* $\mu \in \mathcal{M}(X)$ *with FT* $\hat{\mu} \in FT(X)$*. Let* $(\mathbf{w}, \mathbf{p})$ *be an* N*-map IFSP with contractivity factor* $c \in [0, 1)$ *such that* $d_{FT}(\hat{\mu}, \hat{\nu}) < \epsilon$*, where* $\hat{\nu} = B(\hat{\mu})$ *is the FT corresponding to* $\nu = M\mu$*. Then*

$$d_{FT}(\hat{\mu}, \hat{\bar{\mu}}) < \frac{\epsilon}{1 - c^{1/2}}, \tag{6.6}$$

where $\hat{\bar{\mu}} \in FT(X)$ *is the FT corresponding to* $\bar{\mu} = M\bar{\mu}$*, the invariant measure of the IFSP* $(\mathbf{w}, \mathbf{p})$*.*

The above treatment now allows us to formulate an inverse problem for Fourier transforms of measures as a minimization of the collage distance $d_{FT}(\hat{\mu}, B\hat{\mu})$.

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