Theory of Generalized Fractal Transforms

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Chapter 1

Introduction

The aim of this paper is to present a unified treatment of the various *fractal transform* methods for the representation and compression of computer images which have been based, in some way, on the method of Iterated Function Systems (IFS). These methods, which include "traditional" IFS and IFS with probabilities (IFSP), Iterated Fuzzy Set Systems (IFZS), Iterated Function Systems with Maps (IFSM) and variations, have been designed following a common pattern. Let (X, d) denote a complete metric space, the "base space" which may represent the computer screen, e.g. $[0, 1], [0, 1]^2$ with Euclidean metric. The IFS component, consisting of N contraction maps, $w_i : X \to X$, will be written as $\mathbf{w}$. An image or target is then represented as a point in an appropriate complete metric space (Y, d_Y). The metric spaces used in the various IFS-type methods are listed below:

IFS [15, 2, 1]: $\mathcal{H}(X)$, the set of nonempty compact subsets of X.

IFZS [5]: $\mathcal{F}^*(X)$, the set of all functions $u : X \to [0, 1]$ which are 1) upper semicontinuous on (X, d) and 2) normalized, i.e. for each $u \in \mathcal{F}^*(X)$ there exists an $x_0 \in X$ for which $u(x_0) = 1$.

IFSM [12, 13, 10]: $\mathcal{L}^p(X, \mu)$, the space of p-integrable functions with respect to a measure μ, $1 \leq p \leq \infty$. Fractal Transforms [4, 8, 9] are a special case of IFSM. The Bath Fractal Transform [19, 20] is an IFSM with place-dependent grey level maps.

IFSP [15, 2]: $\mathcal{M}(X)$, the set of probability measures on $\mathcal{B}(X)$, the σ-algebra of Borel subsets of X.

Along with the IFS maps $\mathbf{w}$ (except in the case of IFS on $\mathcal{H}(X)$) there is an associated set of functions $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$, $\phi_i : \mathbf{R} \to \mathbf{R}$, which satisfy conditions that depend on the particular metric space (Y, d_Y) being used. The pair of vectors $(\mathbf{w}, \Phi)$ then determine a *fractal transform operator* T which is designed to map Y into itself. It is desirable that T be contractive on (Y, d_Y) so that it possesses a unique and globally attracting fixed point $\bar{y} \in Y$, i.e. $T\bar{y} = \bar{y}$.

Given a $u \in Y$, its image Tu will be constructed for each point $x \in X$ (or each subset $S \in \mathcal{H}(X)$). Except in the case of IFSP (which is no longer considered for image representation), most practical as well as theoretical studies devise methods which either assume that the $w_i(X)$ do not overlap, or at least ignore any such overlap. (Indeed, if $(Tu)(x)$ is to exist for all $x \in X$ and X is closed and not discrete or finite, then some sets $w_i(X)$ *must* overlap with each other, if only at one point.) As a result, each point $x \in X$ is considered to have only one preimage $w_{i(x)}^{-1}(x) \in X$. The value of the fractal transform of u at $x \in w_i(X)$ is simply $(Tu)(x) = \phi_i(u(w_i^{-1}(x)))$.

In the spirit of our earlier work on IFS-type methods on function spaces, namely IFZS and IFSM, we consider the more general non-overlapping case when x has more than one preimage, i.e. $w_{i_k^*}^{-1}(x), 1 < k \leq n(x)$. There is then the question of how to combine the $n(x)$ *fractal components* $\phi_{i_k^*}(u(w_{i_k^*}^{-1}(x)))$, to form our *generalized fractal transform* $(Tu)(x)$. In this paper, we postulate a set of common rules for combining fractal components. Some of these rules were already considered in the development of IFZS [5]. Understandably, such a method of generalized fractal transforms may not necessarily be useful in the problem of fractal image compression since the coding of any region of an image with more than one fractal component is usually viewed as redundant and contrary to the goal of data compression. Our study, however, is based on a view of fractal transform methods as viable methods of approximating functions and measures in the same spirit as Fourier series/transforms, orthogonal function expansions and, more recently, wavelet expansions.

Previously [12], we compared briefly the various IFS-type methods before outlining a solution of the inverse problem for IFSM [13]. In this paper, we again consider all of these methods, but with the purpose of unifying them *under one common scheme*. The first step is to establish the IFS method, traditionally viewed as a method of geometrically constructing fractal-type sets in $\mathcal{H}(X)$, as a fractal transform method over an appropriate function space whose elements are *bitmap*, i.e. black and white, images. A method of representing images with varying grey levels is clearly desirable. There is a straightforward transition from this IFS approach to the method of IFZS which works with the grey level range [0,1]. In the overlapping case for IFZS, the prescription for combining several fractal components, namely the *supremum* operator, carries over from the IFS method. A modification of the IFZS method [10], motivated in part by restrictions associated with the Hausdorff metric, yields the method of IFSM on the function space $\mathcal{L}^1(X)$. It is then natural to consider IFSM on $\mathcal{L}^p(X)$.

The final step is to provide a link between the IFS, IFZS and IFSM methods on function spaces and the method of IFSP on the probability measure space $\mathcal{M}(X)$. This is accomplished by constructing an IFS-type fractal fransform on the space of distributions $\mathcal{D}'(X)$. IFSP and IFSM correspond to particular cases of this distributional IFS (IFSD). Another noteworthy result of this method is an expression for integrals of the form $\int_X f(x)\bar{u}(x)dx$, where $\bar{u}$ denotes the fixed point of an IFSM. (This is analogous to the expression for integrals of functions with respect to an IFSP invariant measure.)

Finally, we mention that the theory described in this paper may be easily extended to the "block encoding" [17] or "Local IFS" methods [4] which are currently employed in fractal image compression. In these methods, the IFS maps are assumed to map subsets of X, the *domain* or *parent* blocks, to smaller subsets, the *range* or *child* blocks. In a followup paper presented at this conference (henceforth referred to as Paper II), we consider inverse problems for generalized fractal transforms.

Chapter 2

Generalized Fractal Transforms

In this section, we formally define generalized fractal transforms of image functions and provide a set of common rules for constructing such transforms from their fractal components. The mathematical setting is provided by the following ingredients:

1. **The base space:** denoted, as above, by (X, d). The space representing the pixels; a compact subset of $\mathbf{R}^n$. Without loss of generality, $X = [0, 1]^n$ with Euclidean metric.

2. **The IFS component:** For an $N \in \mathbf{N}$, let $\mathbf{w} = \{w_1, w_2, \ldots, w_N\}$. In many cases we can relax the condition that the w_i be contraction maps on X. Note that different sets $w_i(X)$ may overlap. The principal classes of IFS functions used are:

   $Con(X) = \{w : X \to X | d(w(x), w(y)) \leq cd(x, y)$ for some $c \in [0, 1)$, $\forall x, y \in X\}$, the set of contraction maps on X.

   $Con_1(X) \subset Con(X)$: the set of one-to-one contraction maps on X.

   $Aff_1(X)$: the set of affine maps on X which are one-to-one. The representation of such maps in $X = [0, 1]^n$ is given by

   $$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}, \tag{2.1}$$

   where $\mathbf{A}$ is an $n \times n$ matrix with nonzero determinant and $\mathbf{b}$ is an n-vector. The Jacobian of this transformation will be denoted by $|J| = |\lambda_1 \lambda_2 \ldots \lambda_n|$, where the λ_i are the eigenvalues of $\mathbf{A}$.

3. **The image function space:** $\mathcal{F}(X) = \{u : X \to R_g \subseteq \mathbf{R}^+\}$, the functions which will represent our images. The **grey level range** R_g will denote the range of a particular class of image functions used in a given fractal transform method. (In practical applications, R_g is bounded.)

4. **The grey level component:** Associated with the IFS maps $\mathbf{w}$ will be a vector of N functions $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$, $\phi_i : R_g \to R_g$. We may also consider $\phi_i : R_g \times X \to R_g$, i.e. "place-dependent" grey level maps. See, for example, Ref. [20, 19, 10].

5. **The fractal components of** u will be given by $f_i : X \to R_g,\ 1 \le i \le N$, where
$$f_i(x) = \begin{cases} \phi_i(u(w_i^{-1}(x))), & x \in w_i(X), \\ 0, & x \notin w_i(X). \end{cases} \tag{2.2}$$
In other words, the fractal component $f_i(x)$ represents a modified value of the grey level of u at the ith preimage of x (if it exists).

6. **The generalized fractal transform of** $u \in \mathcal{F}(X)$: $\quad F_k : [R_g]^k \to R_g$, $1 \le k \le N$, where
$$F_k(\mathbf{t}) = F_k(t_1, t_2, \ldots, t_k), \quad t_i \in R_g,\ \ 1 \le i \le k. \tag{2.3}$$
The F_k combine the k distinct fractal components $t_i = f_i(x)$ subject to conditions described below. The transform F_N defines an operator $T : \mathcal{F}(X) \to \mathcal{F}(X)$ that associates to each image function $u \in \mathcal{F}(X)$ the image function $v = Tu$. The way in which the F_k combine the fractal components depends on the image function space used.

The grey level ranges and fractal transforms for IFZS and IFSM are given below.

IFZS: $R_g = [0,1]$. $F_N(x) = \sup_{1 \le i \le N}\{f_i(x)\}$.

IFSM: $R_g = \mathbf{R}^+$. $F_N(x) = \sum_{i=1}^N f_i(x)$.

We now outline the properties which are to be satisfied by the fractal transform operators F_k. This discussion follows the same pattern discussed in [5] with some additional comments.

1. $F_N(t_1, t_2, \ldots, t_N) = F_N(t_{i_1}, t_{i_2}, \ldots, t_{i_N})$, where $\{i_1, i_2, \ldots, i_N\}$ is any permutation of the index set $\{1, 2, \ldots, N\}$ (symmetry).

2. $F_N(t_1, t_2, \ldots, t_N) = F_2(F_{N-1}(t_1, \ldots, t_{N-1}), t_N)$ (recursivity).
Properties 1 and 2 imply that
$$F_N(t_1, t_2, \ldots, t_N) = F_2(F_2(F_2(\ldots F_2(t_1, t_2), t_3), t_4), \ldots, t_N), \tag{2.4}$$

and, in particular, that $F_2(F_2(t_1,t_2),t_3) = F_2(t_1, F_2(t_2,t_3))$. Thus, F_2 is an associative binary operation on $\mathbf{R}^+ \times \mathbf{R}^+$. We shall let S denote such a binary operation and assume that it satisfies the following set of additional properties:

3. $S : [R_g]^2 \to R_g$ is continuous.

4. $S(0,y) = y,\ \forall y \in R_g$, i.e. 0 is an *identity* element; the combination of a pixel with brightness $y > 0$ with one of zero brightness yields a pixel with brightness y.

5. S is nondecreasing, i.e. $x_a < x_b$ implies that $S(x_a,y) \leq S(x_b,y),\ \forall y \in R_g$. The brighter a pixel, the brighter its combination with another pixel.

6. $S(x,x) \geq x,\ \forall x \in R_g$; the combination of two pixels of equal brightness should not result in a darker pixel.

7. For all $y \in R_g$, $S(s,y) = s$ for $s = \sup z \in R_g$ (s may be infinite), i.e. s is the *annihilator*.

There is a representation theorem for topological semigroups on $\mathbf{R}$ which will be useful for the construction of appropriate associative operators for our fractal transforms.

Theorem 1 *[18] If $S : [\mathbf{R}^+]^2 \to \mathbf{R}^+$ satisfies Properties (1)-(7) above, then there exist:*

1. *a discrete (finite or countably infinite) index set I,*

2. *a sequence of disjoint open intervals $\{(a_i,b_i)\} \subset \mathbf{R}^+$, $i \in I$, with $0 = a_1 < b_1 \leq a_2 < b_2 \leq \ldots$,*

3. *a sequence of numbers $s_i \in \mathbf{R}^+$, $i \in I$ and*

4. *a sequence of continuous and strictly increasing functions $f_i : [a_i,b_i] \to [0,s_i]$, $i \in I$, with $f_i(a_i) = 0$ and $f_i(b_i) = s_i$, such that*

$$S(x,y) = g_i(f_i(x) + f_i(y)), \quad \forall (x,y) \in [a_i,b_i]^2, \tag{2.5}$$

where $g_i : [0,\infty] \to [0,s_i]$, the pseudoinverse *of f_i is defined as*

$$g_i(x) = \begin{cases} f_i^{-1}(x), & x \in [0,s_i], \\ b_i, & x \notin [s_i,\infty], \end{cases} \tag{2.6}$$

and finally,

$$S(x,y) = \sup(x,y) \quad if \quad (x,y) \in [\bar{\mathbf{R}}^+]^2 - \bigcup_{i\in I}[a_i,b_i]^2. \tag{2.7}$$

Examples:

1. With the condition that $S(x, x) = x$ for all $x \in R_g = [0, s] \subset \mathbf{R}^+$, we have

$$S(x, y) = \sup\{x, y\} \quad \forall (x, y) \in [0, s]^2. \tag{2.8}$$

   This represents an extreme case where the index set $I = \{1\}$ and the sequence $\{(a_i, b_i)\}$ reduces to the interval $(0, s)$. Then all $x \in [0, s]$ are idempotents of S, with 0 being the identity and s the annihilator. This was the natural choice for the IFZS case [5].

2. $S(x, x) = x$ for only $x = 0$ or $x = s$. Again, there is only one interval $(a, b) = (0, s)$. If we choose $f(x) = x$, then

$$S(x, y) = \min\{x + y, s\} \quad \forall (x, y) \in [0, s]^2. \tag{2.9}$$

3. $R_g = \mathbf{R}^+$, i.e. $s = \infty$. A possible binary operation is the summation operator, i.e.

$$S(x, y) = x + y \quad \forall (x, y) \in [0, \infty)^2. \tag{2.10}$$

   The binary operation in Eq. (2.10) is employed by the IFSM method [12, 13, 10].

Chapter 3

From IFS to IFSM Fractal Transforms

In this section, we construct a scheme to unify existing IFS-type fractal transforms on function spaces, as outlined in the Introduction. At the end we review the basic properties of IFSP on the space of probability measures $\mathcal{M}(X)$, in preparation for Section 4, where fractal transforms of functions and IFSP are related through distributions. In what follows, we define the contraction factor of a map $w \in Con(X)$ to be

$$c := \sup_{x,y \in X, x \neq y} d(w(x), w(y))/d(x, y). \tag{3.1}$$

For an N-map IFS $\mathbf{w} = (w_1, w_2, \ldots, w_N)$, the contraction factors of the w_i will be denoted by c_i. We then define $c = \max_{1 \leq i \leq N}\{c_i\}$.

3.1 IFS

Here $Y = \mathcal{H}(X)$, the set of nonempty compact subsets of X and d_Y is the Hausdorff metric h, defined as follows. Let the distance between a point $x \in X$ and a set $S \in \mathcal{H}(X)$ be given by

$$d(x, S) = \inf_{z \in S} d(x, z). \tag{3.2}$$

Then for each $S_1, S_2 \in \mathcal{H}(X)$,

$$h(S_1, S_2) = \max\{\sup_{x \in S_1} d(x, S_2), \sup_{z \in S_2} d(z, S_1)\}. \tag{3.3}$$

Now let $\mathbf{w} = \{w_1, w_2, ..., w_N\}$, $w_i \in Con(X)$. Associated with each contraction map w_i is a set-valued mapping $\hat{w}_i : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $\hat{w}_i(S) =$

$\{w_i(x)|x \in S\}$ for $S \in \mathcal{H}(X)$. Then the usual IFS operator $\hat{\mathbf{w}}$ associated with the N-map IFS $\mathbf{w}$ is defined as follows:

$$\hat{\mathbf{w}}(S) = \bigcup_{i=1}^{N} \hat{w}_i(S), \quad S \in \mathcal{H}(X). \tag{3.4}$$

The IFS operator $\hat{\mathbf{w}}$ is contractive on $(\mathcal{H}(X), h)$ [15]:

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B) \leq ch(A, B), \quad \forall A, B \in \mathcal{H}(X). \tag{3.5}$$

The completeness of $(\mathcal{H}(X), h)$ guarantees the existence of a unique fixed point $\bar{y} = A \in \mathcal{H}(X)$. The set A, also called the *attractor*, is the IFS representation of an image. From Eq. (3.4), it satisfies the following self-tiling property,

$$A = \bigcup_{i=1}^{N} \hat{w}_i(A). \tag{3.6}$$

We now formulate the IFS method over an appropriate function space. First, let $w_i \in Con_1(X)$, $1 \leq i \leq N$. Let $I_S(x)$ denote the characteristic function of a set $S \in \mathcal{H}(X)$, i.e.

$$I_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases} \tag{3.7}$$

Now let $A, B \in \mathcal{H}(X)$ and $C = A \cup B \in \mathcal{H}(X)$. It follows that

$$I_C(x) = \sup\{I_A(x), I_B(x)\}. \tag{3.8}$$

From the property that $I_{\hat{w}_i(S)}(x) = I_S(w_i^{-1}(x))$, we then have, from Eq. (3.8),

$$I_{\hat{\mathbf{w}}(S)}(x) = \sup_{1 \leq i \leq N} \{I_S(w_i^{-1}(x))\}. \tag{3.9}$$

Now consider the function space $\mathcal{F}^*_{BW}(X)$ (for "black and white") defined by

$$\mathcal{F}^*_{BW}(X) = \{u : X \to \{0, 1\} \mid supp(u) \in \mathcal{H}(X)\}. \tag{3.10}$$

In this case, the support of $u \in \mathcal{F}^*_{BW}(X)$ is given by

$$\begin{aligned} supp(u) &= \{x \in X \mid u(x) = 1\} \\ &=: [u]^1, \end{aligned} \tag{3.11}$$

where we have introduced the IFZS notation $[u]^1$ to denote the "1-level" set of u. In other words, u represents a *bitmap* (black and white) image whose white region $[u]^1$ is nonempty and closed. In fact, $\mathcal{F}^*_{BW}(X) \subset \mathcal{F}^*(X)$, the latter being the complete metric space on which IFZS is formulated. It is thus natural to consider the following metric on $\mathcal{F}^*_{BW}(X)$:

$$d_{BW}(u,v) = h([u]^1,[v]^1), \quad \forall u,v \in \mathcal{F}^*_{BW}. \tag{3.12}$$

Completeness of $(\mathcal{F}^*_{BW}(X), d_{BW})$ follows from the completeness of $(\mathcal{H}(X), h)$. From Eq. (3.9), the fractal transform operator $T : \mathcal{F}^*_{BW}(X) \to \mathcal{F}^*_{BW}(X)$ associated with the N-map IFS **w** is given by

$$(Tu)(x) = \sup_{1\leq i\leq N}\{u(w_i^{-1}(x))\}, \ \ \forall x \in X. \tag{3.13}$$

The contractivity of T on $(\mathcal{F}^*_{BW}(X), d_{BW})$ follows immediately from the contractivity of the IFS operator $\hat{\mathbf{w}}$ on $(\mathcal{H}(X), h)$. Thus there exists a unique fixed point $\bar{u} \in \mathcal{F}^*_{BW}(X)$ of the operator T, i.e. $T\bar{u} = \bar{u}$. Moreover, $[\bar{u}]^1 = A$, the attractor of the IFS **w** so that

$$[\bar{u}]^1 = \bigcup_{i=1}^{N} \hat{w}_i([\bar{u}]^1). \tag{3.14}$$

In this formulation, pixels can assume only two grey level values, namely 0 and 1, or black and white (or vice versa). As such, only the geometry of an attractor is revealed. “Real” images, however, are not only black and white. Instead, their pixels can assume a range of nonnegative grey level values, e.g. $u(x) \in [0,1]$. For this reason, it would be desirable to modify the above IFS method so that such a range of grey level values could be produced. This is easily accomplished by modifying the fractal components in Eq. (3.13) as follows:

$$(Tu)(x) = \sup_{1\leq i\leq N}\{\phi_i(u(w_i^{-1}(x)))\}, \ \ \forall x \in X. \tag{3.15}$$

where the $\phi_i : [0,1] \to [0,1]$ are *grey level maps*. Subject to some conditions on these maps (which guarantee that $T : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$) we then arrive at the method of Iterated Fuzzy Set Systems (IFZS) [5].

3.2 IFZS

Here, images are represented by functions $u \in \mathcal{F}^*(X)$ (cf. Section 1). The grey level range is $R_g = [0,1]$. The metric d_Y for the space $Y = \mathcal{F}^*(X)$ is defined as follows. We first define the α-level sets of $u \in \mathcal{F}^*(X)$ for $\alpha \in [0,1]$:

$$\begin{aligned} [u]^\alpha &:= \{x \in X : u(x) \geq \alpha\}, \ \ \alpha \in (0,1], \\ [u]^0 &:= \overline{\{x \in X : u(x) > 0\}}, \end{aligned} \tag{3.16}$$

where $\bar{S}$ denotes the closure of the set S in (X, d). Clearly, $[u]^\alpha \in \mathcal{H}(X)$ for $0 \leq \alpha \leq 1$. Then for $u, v \in \mathcal{F}^*(X)$, define

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha). \tag{3.17}$$

The metric space $(\mathcal{F}^*(X), d_\infty)$ is complete [6].

The IFZS is defined by the following:

1. **The IFS component:** $\mathbf{w} = \{w_1, \ldots, w_N\}$, $w_i \in Con(X)$, $1 \leq i \leq N$,
2. **The grey level component:** $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$, $\phi_i : [0, 1] \to [0, 1]$, such that for all $i \in \{1, 2, \ldots, N\}$:

   (i) ϕ_i is nondecreasing on [0,1],

   (ii) ϕ_i is right continuous on [0,1),

   (iii) $\phi_i(0) = 0$.

   In addition,

   (iv) $\phi_{i^*}(1) = 1$ for at least one $i^* \in \{1, 2, ..., N\}$.

The IFZS fractal transform $T : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$ is defined as

$$(Tu)(x) = \sup_{1 \leq i \leq N} \{\phi_i(\tilde{u}(w_i^{-1}(x))\}, \quad \forall x \in X, \tag{3.18}$$

where, for $B \subseteq X$, (i) $\tilde{u}(B) = \sup_{z \in B}\{u(z)\}$ if $B \neq \emptyset$ and (ii) $\tilde{u}(\emptyset) = 0$. The conditions imposed on the functions ϕ_i guarantee that T maps $(\mathcal{F}^*(X), d_\infty)$ to itself. The relation between level sets of a function u and those of its image Tu is given by

$$[Tu]^\alpha = \bigcup_{i=1}^{N} \hat{w}_i([\phi_i \circ u]^\alpha), \quad \alpha \in [0, 1]. \tag{3.19}$$

The contractivity of the IFS maps w_i implies that that T is a contraction map on $(\mathcal{F}^*(X), d_\infty)$ since [5]

$$d_\infty(Tu, Tv) \leq c d_\infty(u, v), \quad \forall u, v \in \mathcal{F}^*(X). \tag{3.20}$$

The completeness of this space guarantees the existence of a unique fixed point $\bar{u} \in \mathcal{F}^*(X)$ of the operator T. From Eq. (3.19), the α-level sets of $\bar{u}$ obey the following generalized self-tiling property:

$$[\bar{u}]^\alpha = \bigcup_{i=1}^{N} w_i([\phi_i \circ \bar{u}]^\alpha), \quad \alpha \in [0, 1], \tag{3.21}$$

which can be compared to the self tiling of $\alpha = 1$ level sets for the IFS case in Eq. (3.14).

We draw the reader's attention to the fact that the use of the *sup* operator in the IFZS operator T is a natural choice. The IFZS method is based on the properties of level sets of functions in $\mathcal{F}^*(X)$. Taking the *sup* of two functions $u(x)$ and $v(x)$ for all $x \in X$ corresponds to taking the union of their respective α-level sets.

The Hausdorff metric d_∞ is very restrictive, however, from both practical (i.e. image processing) as well as theoretical perspectives. In [10], two fundamental modifications were made to the IFZS approach:

1. For a $\mu \in \mathcal{M}(X)$ and $u, v \in \mathcal{F}^*(X)$, define $\forall \alpha \in [0, 1]$,

$$\begin{aligned} G(u, v; \alpha) &= \mu([u]^\alpha \triangle [v]^\alpha) \\ &= \int_X |I_{[u]^\alpha}(x) - I_{[v]^\alpha}(x)| d\mu(x), \end{aligned} \tag{3.22}$$

   where $\triangle$ denotes the symmetric difference operator: For $A, B \subseteq X$, $A \triangle B = (A \cup B) - (A \cap B)$.

2. Now let ν be a finite measure on $\mathcal{B}(R_g)$ and define

$$g(u, v; \nu) = \int_{R_g} G(u, v; \alpha) d\nu(\alpha). \tag{3.23}$$

   An application of Fubini's Theorem yields

$$\begin{aligned} g(u, v; \nu) &= \nu(\{0\})\mu([u]^0 \triangle [v]^0) + \int_{X_u} \nu((v(x), u(x)]) d\mu(x) \\ &+ \int_{X_v} \nu((u(x), v(x)]) d\mu(x), \end{aligned} \tag{3.24}$$

   where $X_u = \{x \in X \mid u(x) > v(x)\}$ and $X_v = \{x \in X \mid v(x) > u(x)\}$.

It can be shown that $g(u, v; \nu)$ is a pseudometric on $\mathcal{L}^1(X, \mu)$. In the particular case that $\nu = m$, the Lebesgue measure on the grey level range R_g, Eq. (34) becomes

$$g(u, v; m) = \int_X |u(x) - v(x)| d\mu(x), \tag{3.25}$$

the $\mathcal{L}^1(X, \mu)$ distance between u and v. The restrictive Hausdorff metric d_∞ over α-level sets has been replaced by a weaker pseudometric (metric on the measure algebra) involving integrations over X and R_g. The result is a fractal transform method on the function space $\mathcal{L}^1(X, \mu)$. While it appears that only the $\mathcal{L}^1$ distance can be generated by a measure ν on $\mathcal{B}(R_g)$, it is still natural to consider fractal transforms over the general function spaces $\mathcal{L}^p(X, \mu)$.

3.3 IFSM

Let μ be a measure on $\mathcal{B}(X)$ and for any integer $p \geq 1$, let $\mathcal{L}^p(X,\mu)$ denote the linear space of all real–valued functions u such that u^p is integrable on $(\mathcal{B}(X),\mu)$. We choose $Y = \mathcal{L}^p(X,\mu)$. The metric d_Y is defined by the usual $\mathcal{L}^p$–norm, i.e. $d_Y(u,v) := d_p(u,v)$, where

$$d_p(u,v) = \| u - v \|_p = \left[\int_X |u(x) - v(x)|^p d\mu(x) \right]^{1/p}. \tag{3.26}$$

The IFSM is then defined by the following:

1. **The IFS component:** $\mathbf{w} = \{w_1, \ldots, w_N\}$, $w_i \in Aff_1(X)$, $1 \leq i \leq N$,
2. **The grey level component:** $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$, $\phi_i \in Lip(\mathbf{R};\mathbf{R})$, where

$$\begin{aligned} Lip(\mathbf{R};\mathbf{R}) &= \{\phi : \mathbf{R} \to \mathbf{R} \mid |\phi(t_1) - \phi(t_2)| \leq K|t_1 - t_2|, \\ &\forall \quad t_1, t_2 \in \mathbf{R} \text{ for some } K \in [0,\infty)\}. \end{aligned} \tag{3.27}$$

Since our function space involves integrations, it is natural to define the following fractal transform operator T corresponding to an N-map IFSM $(\mathbf{w},\Phi)$:

$$(Tu)(x) := \sum_{k=1}^{N} f_k(x), \tag{3.28}$$

where the fractal components f_k are defined in Eq. (2.2). The above conditions on the w_i and ϕ_i guarantee that $T : \mathcal{L}^p(X,m) \to \mathcal{L}^p(X,m)$ for all $p \in [1,\infty]$, where m denotes Lebesgue measure on X.

Now let $X = [0,1]^D$, $\mu = m$ and $1 \leq p \leq \infty$. Also let $u, v \in \mathcal{L}^p(X,m)$ with fractal components f_k and g_k, $1 \leq k \leq N$, respectively. Then from the relation

$$\begin{aligned} \| Tu - Tv \|_p &= \| \sum_{k=1}^{N} [f_k(x) - g_k(x)] \|_p \\ &\leq \sum_{k=1}^{N} \| f_k(x) - g_k(x) \|_p, \end{aligned} \tag{3.29}$$

we obtain the result

$$d_p(Tu,Tv) \leq C_p d_p(u,v), \quad C_p = \sum_{k=1}^{N} |J_k|^{1/p} K_k, \tag{3.30}$$

where $|J_k|$ is the Jacobian associated with the affine transformation $x = w_k(y)$. In the special μ-nonoverlapping case, i.e. where the sets $w_i(X)$ overlap only on sets of zero μ-measure (the standard assumption in practical applications in the literature), we may use the relation

$$\| Tu - Tv \|_p^p \; = \; \sum_{k=1}^{N} \| f_k(x) - g_k(x) \|_p^p \tag{3.31}$$

to obtain the result

$$d_p(Tu, Tv) \le \bar{C}_p d_p(u, v), \quad \bar{C}_p = \left[\sum_{k=1}^{N} |J_k| K_k^p \right]^{1/p}. \tag{3.32}$$

Note that

$$\bar{C}_p \le C_p \le K, \quad K = \max_{1 \le k \le N} K_k. \tag{3.33}$$

In the nonoverlapping case, with $p = \infty$, we also have

$$\| Tu - Tv \|_\infty \le K \| u - v \|_\infty, \quad \forall u, v \in \mathcal{L}^\infty(X, m). \tag{3.34}$$

This is the usual bound presented in the literature on fractal transforms [4, 9].

In applications, we shall be using *affine IFSM*, i.e. $w_k \in Aff_1(X)$ and

$$\phi_k(t) = \alpha_k t + \beta_k, \quad t \in \mathbf{R}, \quad 1 \le k \le N. \tag{3.35}$$

If the associated operator T is contractive on $\mathcal{L}^p(X, \mu)$, then its fixed point $\bar{u}$ satisfies the equation

$$\bar{u}(x) = \sum_{k=1}^{N} [\alpha_k \psi_k(x) + \beta_k \chi_k(x)], \tag{3.36}$$

where $\psi_i(x) = \bar{u}(w_k^{-1}(x))$ and $\chi_k(x) = I_{w_k(X)}$. In other words, $\bar{u}$ may be written as a linear combination of both functions $\psi_k(x)$ and piecewise constant functions $\chi_k(x)$ which are obtained by dilatations and translations of $\bar{u}$ and $I_X(x)$, respectively. This is somewhat reminiscent of the role of scaling functions in wavelet theory.

The following result guarantees that the use of affine IFSM is sufficient from a theoretical perspective.

Theorem 2 *Let $X = [0,1]^D$ and $\mu \in \mathcal{M}(X)$. For a $p \ge 1$ define $\mathcal{L}_A^p(X, \mu) \subset \mathcal{L}^p(X, \mu)$ to be the set of fixed points $\bar{u}$ of all contractive N-map affine IFSM $(\mathbf{w}, \Phi)$ for $N \ge 1$. Then $\mathcal{L}_A^p(X, \mu)$ is dense in $(\mathcal{L}^p(X, \mu), d_p)$.*

The proof of this theorem is based on the property that the set of all step functions in X is dense in $(\mathcal{L}^p(X, \mu), d_p)$.

3.3.1 “Place-Dependent” IFSM

The above IFSM method may be easily generalized to “place-dependent” IFSM (PDIFSM), that is, IFSM with grey level maps having the form $\phi_k : \mathbf{R} \times X \to \mathbf{R}$, $1 \leq k \leq N$. In other words, the ϕ_i are dependent both on the grey-level value at a preimage as well as the location of the preimage itself. (This is analogous to IFS with place-dependent probabilities [3].) Much of the theory developed above for IFSM easily extends to place-dependent IFSM as we outline below. This is the basis of the “Bath Fractal Transform” and its effectiveness in coding images has been discussed in the literature [19, 20, 24].

The fractal components $f_k(x)$ of a function $u \in \mathcal{L}^p(X, \mu)$ will be given by

$$f_k(x) = \begin{cases} \phi_k(u(w_k^{-1}(x)), w_k^{-1}(x)), & x \in w_k(X), \\ 0, & x \notin w_k(X). \end{cases} \tag{3.37}$$

The operator T associated with an N-map PDIFSM $(\mathbf{w}, \Phi)$ will have the form

$$(Tu)(x) = \sum_{k=1}^{N}{}' \phi_k(u(w_k^{-1}(x)), w_k^{-1}(x)), \tag{3.38}$$

We first define the following set of uniformly Lipschitz functions,

$$\begin{aligned} Lip(\mathbf{R}, X; \mathbf{R}) \quad &= \quad \{\phi : \mathbf{R} \times X \to \mathbf{R} : |\phi(t_1, s) - \phi(t_2, s)| \leq K|t_1 - t_2|, \\ &\qquad \forall t_1, t_2 \in \mathbf{R},\ \forall s \in X \text{ for some } K \in [0, \infty)\}. \end{aligned} \tag{3.39}$$

If $w_i \in Aff_1(X)$ and $\phi_i \in Lip(\mathbf{R}, X; \mathbf{R})$ for $1 \leq i \leq N$ then $T : \mathcal{L}^p(X, m) \to \mathcal{L}^p(X, m)$ for $1 \leq p < \infty$. Furthermore, if $X \subset \mathbf{R}^D$, $D \in \{1, 2, ...\}$, and $\mu = m$ then the relation in Eq. (3.30) holds.

Some possible forms for the place-dependent grey level maps ϕ are as follows:

1. $\phi(t, s) = \sum_{i=0}^{n} a_i(s) t^i$, where the $a_i : X \to \mathbf{R}$, bounded on X,
2. $\phi(t, s) = f(t) + g(s)$ (“separable”) with suitable conditions on f and g, e.g. $f \in Lip(\mathbf{R}; \mathbf{R})$ and $g : X \to \mathbf{R}$ bounded on X.

It is convenient to work with ϕ maps which are only first degree in the grey-level variable t, i.e.

$$\phi(t, s) \quad = \quad \alpha t + \beta + g(s), \quad g : X \to \mathbf{R}, \text{ bounded on } X, \tag{3.40}$$

$$\phi(t, s) \quad = \quad \alpha(s) t + \beta(s), \quad \alpha, \beta : X \to \mathbf{R}, \text{ bounded on } X. \tag{3.41}$$

The action of the first set of maps can be considered as a “place-dependent” shift in grey-level value. The second set of maps produce a more direct interaction between position and grey-level value. In Figure 3.1 are presented histogram approximations of fixed points $\bar{u}$ of two rather simple affine PDIFSM in order to show the effects of place-dependence.

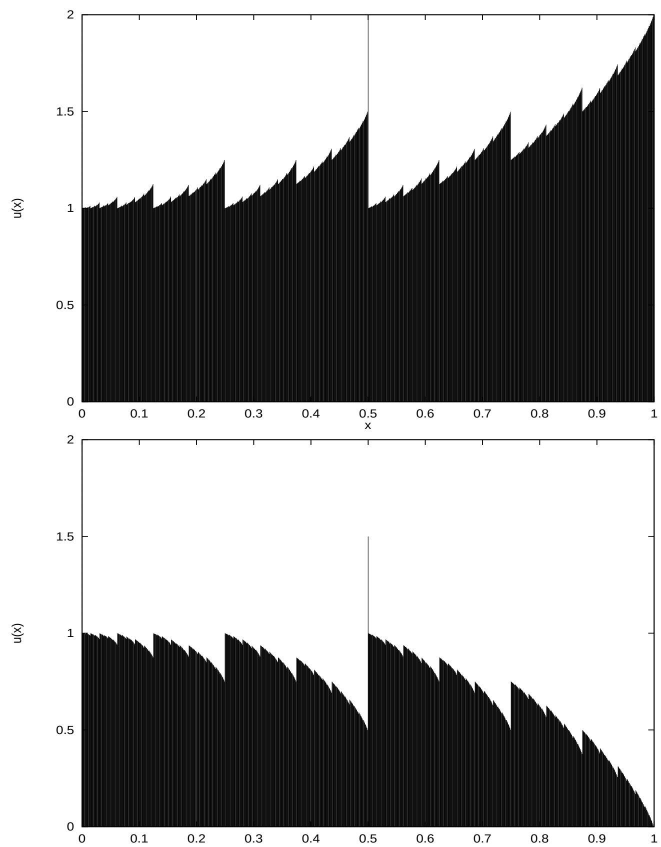

Figure 3.1: Fixed-point attractor functions $\bar{u}(x)$ of the following 2-map PDIFSM: $w_1(x) = \frac{1}{2}x$, $w_2(x) = \frac{1}{2}x + \frac{1}{2}$, $\phi_1(t, s) = \frac{1}{2}t + \frac{1}{2}$, $\phi_2(t, s) = \frac{1}{2}t + \frac{1}{2} + \gamma s$. When $\gamma = 0$, $\bar{u}(x) = 1$ (a.e.). (a) $\gamma = \frac{1}{2}$. (b) $\gamma = -\frac{1}{2}$.

3.4 IFSP

Associated with an N-map IFS $\mathbf{w}$, $w_i \in Con(X)$, is a set of probabilities $\mathbf{p} = \{p_1, p_2, ..., p_N\}$, $p_i \geq 0$, with $\sum_{i=1}^N p_i = 1$. Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X generated by all the elements of $\mathcal{H}(X)$. Then $Y = \mathcal{M}(X)$, the set of all probability measures on $\mathcal{B}(X)$. Here, the (Markov) operator associated with the IFSP $(\mathbf{w}, \mathbf{p})$ is defined as follows: For a $\mu \in \mathcal{M}(X)$ and each $S \in \mathcal{H}(X)$,

$$(T\mu)(S) = (M\mu)(S) = \sum_{i=1}^N p_i \mu(w_i^{-1}(S)). \tag{3.42}$$

The metric d_Y on $Y = \mathcal{M}(X)$ is the so-called Hutchinson metric $d_H(\mu, \nu)$:

$$d_H(\mu, \nu) = \sup_{f \in Lip_1(X;\mathbf{R})} \left[\int_X f d\mu - \int_X f d\nu \right], \quad \forall \mu, \nu \in \mathcal{M}(X), \tag{3.43}$$

where

$$\begin{aligned} Lip_1(X;\mathbf{R}) \quad = \quad & \{f : X \to \mathbf{R} \mid |f(x_1) - f(x_2)| \leq d(x_1, x_2), \\ & \forall x_1, x_2 \in X\}. \end{aligned} \tag{3.44}$$

The contractivity of the IFS maps w_i implies the contractivity of M on $(\mathcal{M}(X), d_H)$ [15]:

$$d_H(M\mu, M\nu) \leq c d_H(\mu, \nu), \quad \forall \mu, \nu \in \mathcal{M}(X). \tag{3.45}$$

There exists a unique $\bar{\mu} \in \mathcal{H}(X)$ such that (1) $M\bar{\mu} = \bar{\mu}$ and (2) $d_H(M^n\mu, \bar{\mu}) \to 0$ as $n \to \infty$. Moreover, $supp(\bar{\mu}) \subseteq A$, with the equality when all $p_i > 0$. From Eq. (3.42) it follows that

$$\bar{\mu}(S) = \sum_{i=1}^N p_i \bar{\mu}(w_i^{-1}(S)), \quad \forall S \in \mathcal{H}(X), \tag{3.46}$$

which leads to the following "change of variables" result for integration: For $f \in C(X)$ (or simple functions),

$$\int_X f(x) d\bar{\mu}(x) = \sum_{i=1}^N p_i \int_X (f \circ w_i)(x) d\bar{\mu}(x). \tag{3.47}$$

It is well known that by setting $f(x) = x^n$, $n = 1, 2, \ldots$, one can obtain recursion relations for the moments $\bar{g}_n = \int_X x^n d\bar{\mu}$ of the invariant measure.

If we use the notation

$$< f, \mu >:= \int_X f(x) d\mu(x), \quad f \in C(X), \ \ \mu \in \mathcal{M}(X), \tag{3.48}$$

then

$$< f, T\mu > = < T^{\dagger} f, \mu >, \tag{3.49}$$

where the adjoint operator $T^{\dagger} : C(X) \to C(X)$ (referred to as T in [2]) is given by

$$(T^{\dagger} f)(x) = \sum_{i=1}^{N} p_i (f \circ w_i)(x). \tag{3.50}$$

We may iterate this procedure to obtain, for $n = 1, 2, \ldots$,

$$\begin{aligned} < f, T^n \mu > &= < (T^{\dagger})^n f, \mu > \\ &= \sum_{i_1, \ldots, i_n}^{N} p_{i_1} \ldots p_{i_n} \int_X (f \circ w_{i_1} \circ \ldots \circ w_{i_n})(x) d\mu. \end{aligned} \tag{3.51}$$

For an $x_0 \in X$, let $\mu = \delta_{x_0}$, a Dirac unit mass at x_0. Since $d_H(T^n \mu, \bar{\mu}) \to 0$, one obtains

$$\int_X f(x) d\bar{\mu}(x) = \lim_{n \to \infty} \sum_{i_1, \ldots, i_n}^{N} p_{i_1} \ldots p_{i_n} (f \circ w_{i_1} \circ \ldots \circ w_{i_n})(x_0). \tag{3.52}$$

This formula has been used to provide estimates for integrals involving $\bar{\mu}$ which cannot be solved recursively. The computation of the multiple sums involve the enumeration of an N-tree to n generations.

By setting $f(x) = I_S(x)$, where $S \subseteq X$, the above relation becomes

$$\bar{\mu}(S) = \lim_{n \to \infty} \sum_{i_1, \ldots, i_n}^{N} p_{i_1} \ldots p_{i_n} I_S(w_{i_1} \circ \ldots \circ w_{i_n}(x_0)). \tag{3.53}$$

The term involving I_S indicates whether or not the point $w_{i_1} \circ \ldots \circ w_{i_n}(x_0)$ lies in S. The quantity $p_{i_1} p_{i_2} \ldots p_{i_n}$ represents the probability of choosing the finite sequence $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_n}\}$. Therefore for each $n > 0$, the sum is equal to the probability that the point x_n lies in S.

There is a connection between Eq. (3.53) and the Random Iteration Algorithm or "Chaos Game" [1], defined as follows: Pick an $x_0 \in X$ and define the iteration sequence

$$x_{n+1} = w_{\sigma_n}(x_n), \quad n = 0, 1, 2, \ldots, \tag{3.54}$$

where the σ_n are chosen randomly and independently from the set $\{1, 2, \ldots, N\}$ with probabilities $P(\sigma_n = i) = p_i$. A straightforward coding argument shows that for almost every code sequence $\sigma = \{\sigma_1, \sigma_2, \ldots)$ the orbit x_n is dense on the attractor A of the IFS $\mathbf{w}$. As such, the Chaos Game can be used to generate computer approximations of A. However, it also provides approximations to the invariant measure $\bar{\mu}$ as a consequence of the following ergodic theorem for IFS [7]: For almost all code sequences $\sigma = (\sigma_1, \sigma_2, \ldots)$,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x_k) = \int_X f(x)\bar{\mu}(x) \tag{3.55}$$

for all continuous (and simple) functions $f : X \to \mathbf{R}$. By setting $f(x) = I_S(x)$ in Eq. (3.55) for an $S \subseteq X$, we obtain

$$\bar{\mu}(S) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} I_S(x_k). \tag{3.56}$$

In other words, $\bar{\mu}(S)$ is the limit of the relative visitation frequency of S during the chaos game.

Chapter 4

IFS on the Space of Distributions $\mathcal{D}'(X)$

In what follows $X = [0,1]$ although the extension to $[0,1]^n$ is straightforward. Distributions [21, 22, 23] are defined as linear functionals over a suitable space of "test functions", to be denoted (following the standard notation in the literature) as $\mathcal{D}(X)$. In this paper, $\mathcal{D}(X) = C^\infty(X)$, the space of infinitely differentiable real-valued functions on X. (Note: In the literature, $\mathcal{D}(X)$ is normally taken to be $C_0^\infty(X)$, the set of $C^\infty(X)$ functions with compact support on X. With this choice, the expressions for distributional derivatives simplify due to the vanishing of boundary terms.) The space of distributions on X, to be denoted as $\mathcal{D}'(X)$, is the set of all bounded linear functionals on $\mathcal{D}(X)$, that is, $F : \mathcal{D}(X) \to \mathbf{R}$, such that

1. $|F(\psi)| < \infty$ for all $\psi \in \mathcal{D}(X)$,
2. $F(c_1\psi_1 + c_2\psi_2) = c_1F(\psi_1) + c_2F(\psi_2)$, $c_1, c_2 \in \mathbf{R}$, $\psi_1, \psi_2 \in \mathcal{D}(X)$.

The space $\mathcal{D}'(X)$ will include the following as special cases:

1. Functions $f \in \mathcal{L}^p(X, m)$, $1 \leq p \leq \infty$, for which the corresponding distributions are given by

$$F(\psi) = \int_X f(x)\psi(x)dx, \quad \forall\psi \in \mathcal{D}(X), \tag{4.1}$$

2. Probability measures $\mu \in \mathcal{M}(X)$, for which the corresponding distributions are given by

$$F(\psi) = \int_X \psi(x)d\mu(x), \quad \forall\psi \in \mathcal{D}(X), \tag{4.2}$$

3. The “Dirac delta function”, $\delta(x-a)$, which may be defined in the distributional sense as follows: For a point $a \in X$, $F(\psi) = \psi(a)$ for all $\psi \in \mathcal{D}(X)$. This is often written symbolically as

$$F(\psi) = \int_X \psi(x)\delta(x-a)dx. \tag{4.3}$$

Our goal is to construct an IFS-type fractal transform operator $T : \mathcal{D}'(X) \to \mathcal{D}'(X)$ which, under suitable conditions, will be contractive with respect to a given metric on $\mathcal{D}'(X)$. In the spirit of Section 2, the fractal components of a distribution $u \in \mathcal{D}'(X)$ would be defined (symbolically) as $f_i(x) = (\phi_i \circ u \circ w_i^{-1})(x)$ and then combined to form T. Such a transform would serve to join the IFSM method over function spaces and the IFSP method over measure spaces under one common scheme.

The following property is very important in establishing a representation theory for distributions in $\mathcal{D}'(X)$.

Theorem 3 *[23] For any distribution/linear functional $F \in \mathcal{D}'(X)$, there exists a sequence of test functions $f_n \in \mathcal{D}(X)$, $n = 1, 2, \ldots$, such that for all $\psi \in \mathcal{D}(X)$,*

$$\begin{aligned} \lim_{n\to\infty} F_n(\psi) &= \lim_{n\to\infty} \int_X f_n(x)\psi(x)dx \\ &=: F(\psi). \end{aligned} \tag{4.4}$$

By recourse to this result, it will be convenient to express the linear functional $F \in \mathcal{D}'(X)$ symbolically as

$$\begin{aligned} F(\psi) &= \int_X f(x)\psi(x)dx \\ &= <\psi, f>, \end{aligned} \tag{4.5}$$

even though there may not exist a pointwise function $f(x)$ which defines the distribution F (e.g. Dirac distribution). The sequence of test functions f_n in the above theorem will then be said to converge to the distribution f “in the sense of distributions”. For notational convenience we shall write that “$f \in \mathcal{D}'(X)$”.

Lemma 1 *Let $w \in Aff_1(X)$ with Jacobian $|J| \neq 0$ and $u \in \mathcal{D}'(X)$ with associated linear functional*

$$F(\psi) = \int_X u(x)\psi(x)dx, \quad \psi \in \mathcal{D}(X). \tag{4.6}$$

Then the distribution $v = u \circ w^{-1} \in \mathcal{D}'(X)$ *may be defined (symbolically) as*

$$\begin{aligned} G(\psi) &= \int_X u(w^{-1}(x))\psi(x)dx \\ &= |J| \int_X u(x)(\psi \circ w)(x)dx, \quad \psi \in \mathcal{D}(X). \end{aligned} \tag{4.7}$$

Proof: Since $u \in \mathcal{D}'(X)$ there exists a sequence $u_n \in \mathcal{D}(X)$ which converges to u in the sense of distributions, i.e.

$$\lim_{n\to\infty} \int_X u_n(x)\psi(x)dx = \int_X u(x)\psi(x)dx, \quad \forall\ \psi \in \mathcal{D}(X). \tag{4.8}$$

For $n \geq 1$ define $v_n(x)$ as

$$v_n(x) = \begin{cases} u_n(w^{-1}(x)), & x \in \hat{w}(X), \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

By the change of variable $x = w(y)$ (with Jacobian $|J| \neq 0$),

$$\int_X v_n(y)\psi(x)dx = |J| \int_X u_n(y)(\psi \circ w)(y)dy. \tag{4.10}$$

Since w is affine, $\psi \circ w \in \mathcal{D}(X)$ for any $\psi \in \mathcal{D}(X)$. Therefore for each $n \geq 1$,

$$\begin{aligned} \lim_{n\to\infty} \int_X v_n(x)\psi(x)dx &= |J| \lim_{n\to\infty} \int_X u_n(y)(\psi \circ w)(y)dy \\ &= |J| \int_X u(y)(\psi \circ w)(y)dy, \end{aligned} \tag{4.11}$$

and the theorem is proved.

Example: Let $w(x) = \frac{1}{2}x$ and $u(x) = \delta(x)$, the "Dirac delta function" at $a = 0$. Then $u(w^{-1}(x)) = \delta(2x) = \frac{1}{2}\delta(x)$.

Definition 1 *Let* $f \in \mathcal{D}'(X)$ *and* $\{f_n\}$ *any sequence in* $\mathcal{D}(X)$ *such that* $f_n \to f$ *in the sense of distributions. Now let* $g : \mathbf{R} \to \mathbf{R}$ *be such that*

$$L(\psi) := \lim_{n\to\infty} \int_X g(f_n(x))\psi(x)dx$$

exists for all $\psi \in \mathcal{D}(X)$ *independently of the sequence* $\{f_n\}$. *Then we define the distribution* $g \circ f \in \mathcal{D}'(X)$ *in terms of the above limits, i.e.*

$$\begin{aligned} < \psi, g \circ f > &= \lim_{n\to\infty} \int_X g(f_n(x))\psi(x)dx, \quad \forall\ \psi \in \mathcal{D}(X) \\ &=: (G \circ F)(\psi). \end{aligned} \tag{4.12}$$

If g is affine on $\mathbf{R}$, i.e. $g(x) = ax + b$, where $a, b \in \mathbf{R}$, then trivially the distribution $g \circ f$ exists for all $f \in \mathcal{D}(X)$. Note, however, that if g is Lipschitz on $\mathbf{R}$, then the distribution $g \circ f$ need not exist. For example, $\sin(\delta(x))$ is not defined.

Definition 2 *A function $g : \mathbf{R} \to \mathbf{R}$ will be said to satisfy a weak Lipschitz condition on $\mathcal{D}'(X)$ if there exists a $K \geq 0$ such that for all $\psi \in \mathcal{D}(X)$,*

$$\begin{aligned} \left| \int_X [(g \circ f_1)(x) - (g \circ f_2)(x)]\psi(x)dx \right| \leq & \quad K \quad \left| \int_X [f_1(x) - f_2(x)]\psi(x)dx \right| \\ & \forall f_1, f_2 \in \mathcal{D}'(X). \end{aligned} \tag{4.13}$$

If g is affine on $\mathbf{R}$, then it satisfies a weak Lipschitz condition on $\mathcal{D}'(X)$.

Lemma 2 *Let $g : \mathbf{R} \to \mathbf{R}$ satisfy a weak Lipschitz condition on $\mathcal{D}'(X)$. Then for any $f \in \mathcal{D}'(X)$, $g \circ f \in \mathcal{D}'(X)$ exists.*

Proof: Let $f \in \mathcal{D}'(X)$ and, from Theorem 1, $f_n \in \mathcal{D}(X)$, $n = 1, 2, \ldots$ such that $f_n \to f$ as $n \to \infty$ in distribution. This implies that for any $\psi \in \mathcal{D}(X)$, given an $\epsilon > 0$, there exists an $M_\psi > 0$ such that

$$\left| \int_X [f_n(x) - f_m(x)]\psi(x)dx \right| < \epsilon, \quad \forall m, n \geq M_\psi. \tag{4.14}$$

Since

$$\begin{aligned} \left| \int_X [(g \circ f_n)(x) - (g \circ f_m)(x)]\psi(x)dx \right| & \leq K \left| \int_X [f_n(x) - f_m(x)]\psi(x)dx \right| \\ & \leq K\epsilon \quad \forall m, n \geq M_\psi, \end{aligned} \tag{4.15}$$

we may define, for each $\psi \in \mathcal{D}(X)$,

$$\int_X (g \circ f)(x)\psi(x)dx = \lim_{n \to \infty} \int_X (g \circ f_n)(x)\psi(x)dx. \quad \blacksquare \tag{4.16}$$

We now define an N-map IFS on Distributions (IFSD) $(\mathbf{w}, \Phi)$ as follows:

1. **The IFS component:** $\mathbf{w} = \{w_1, w_2, \ldots, w_N\}$, $w_i \in Aff_1(X)$, with Jacobian $|J_i| \neq 0$,

2. **The grey level component:** $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$, $\phi_i : \mathbf{R} \to \mathbf{R}$ satisfies a weak Lipschitz condition on $\mathcal{D}'(X)$ (with Lipschitz constant K_i).

An operator $T : \mathcal{D}'(X) \to \mathcal{D}'(X)$ will now be associated with this N-map IFSD. For any $f \in \mathcal{D}'(X)$, the distribution $g = Tf$ will be defined by the linear functional

$$\begin{aligned} G(\psi) &= \int_X g(x)\psi(x)dx \\ &= \int_X (Tf)(x)\psi(x)dx \\ &= \sum_{i=1}^{N} \int_X (\phi_i \circ f \circ w_i^{-1})(x)\psi(x)dx. \end{aligned} \tag{4.17}$$

From Lemmas 1 and 2, it follows that T maps $\mathcal{D}'(X)$ into itself. We now define the following metric on $\mathcal{D}'(X)$:

$$d_{D'}(f,g) = \sup_{\psi \in \mathcal{D}_1(X)} \left| \int_X f(x)\psi(x)dx - \int_X g(x)\psi(x)dx \right|, \quad \forall f, g \in \mathcal{D}'(X). \tag{4.18}$$

where $\mathcal{D}_1(X) = \{\psi \in C^\infty(X) \mid \ \| \psi \|_\infty \leq 1\}$.

Theorem 4 *The metric space $(\mathcal{D}'(X), d_{D'})$ is complete.*

Proof: Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(\mathcal{D}'(X), d_{D'})$, that is, for any $\epsilon > 0$, there exists an $\bar{N}(\epsilon)$ such that $d_{D'}(f_m, f_n) < \epsilon$ for all $m, n \geq \bar{N}(\epsilon)$. From the definition of $d_{D'}$ in Eq. (4.18), it follows that for any fixed $\psi \in \mathcal{D}_1(X)$, the sequence of real numbers $\{t_n(\psi)\}_{n=1}^\infty$, where

$$t_n(\psi) = \int_X f_n(x)\psi(x)dx, \tag{4.19}$$

is a Cauchy sequence on **R**. Let $\bar{t}(\psi)$ denote the limit of this sequence. By setting $F(\psi) = \bar{t}(\psi)$ we define a bounded linear functional F on $\mathcal{D}_1(X)$. This procedure can easily be extended to all test functions $\psi \in C^\infty(X)$ by noting that $M^{-1}\psi \in \mathcal{D}_1(X)$ where $M = \| \psi \|_\infty$. Therefore $(\mathcal{D}'(X), d_{D'})$ is complete.

Remark: By restricting the test functions employed in the $d_{D'}$ metric to $\mathcal{D}_1(X)$ (as opposed to the entire space $C^\infty(X)$), we ensure that the set of Cauchy sequences in $\mathcal{D}'(X)$ is nonempty.

Theorem 5 *Let $(\mathbf{w}, \Phi)$ be an N-map IFSD, $w_i \in Aff_1(X)$. Then for any $f, g \in \mathcal{D}'(X)$,*

$$d_{D'}(Tf, Tg) \leq C_D d_{D'}(f,g), \qquad C_D = \sum_{i=1}^{N} |J_i| K_i. \tag{4.20}$$

Proof: Let $\psi \in \mathcal{D}_1(X)$. Then

$$\begin{aligned} \left|\int_X [(Tf)(x) \right. &- \left. (Tg)(x)]\psi(x)dx\right| \\ &= \left|\sum_{i=1}^N \int_{X_i} [\phi_i(f(w_i^{-1}(x))) - \phi_i(g(w_i^{-1}(x)))]\psi(x)dx\right| \\ &= \left|\sum_{i=1}^N |J_i| \int_X [\phi_i(f(y)) - \phi_i(g(y))](\psi \circ w_i)(y)dy\right| \\ &\leq \sum_{i=1}^N |J_i| \left|\int_X [\phi_i(f(y)) - \phi_i(g(y))](\psi \circ w_i)(y)dy\right| \\ &\leq \sum_{i=1}^N |J_i| K_i \left|\int_X [f(y) - g(y)](\psi \circ w_i)(y)dy\right|. \end{aligned} \tag{4.21}$$

The desired result follows from the fact that $\psi \circ w_i \in \mathcal{D}_1(X)$. ∎

Corollary 1 *Let $(\mathbf{w}, \Phi)$ be an N-map IFSD, $w_i \in Aff_1(X)$, such that*

$$C_D = \sum_{i=1}^N |J_i| K_i < 1. \tag{4.22}$$

Then there exists a unique distribution $\bar{u} \in \mathcal{D}'(X)$ such that $T\bar{u} = \bar{u}$, where T is the operator asssociated with the IFSD. Furthermore, $d_{D'}(T^n u, \bar{u}) \to 0$ as $n \to \infty$ for all $u \in \mathcal{D}'(X)$.

4.1 Affine IFSD and the Connection with IFSP and IFSM

For affine IFSD $(\mathbf{w}, \Phi)$ on $X = [0, 1]$, the IFS and grey level maps have the forms

$$w_i(x) = s_i x + a_i, \quad \phi_i(t) = \alpha_i t + \beta_i, \quad 1 \leq i \leq N, \tag{4.23}$$

with $c_i = |J_i| = |s_i|$. Given an operator $T : \mathcal{D}'(X) \to \mathcal{D}'(X)$ associated with an affine IFSD, an adjoint operator $T^\dagger : \mathcal{D}(X) \to \mathcal{D}(X)$ may be defined as follows: For all $\psi \in \mathcal{D}(X)$, $< \psi, Tf > = < T^\dagger \psi, f >$, where

$$\begin{aligned} < T^\dagger \psi, f > &= \sum_{i=1}^N \alpha_i c_i \int_X f(x)(\psi \circ w_i)(x)dx \\ &+ \sum_{i=1}^N \beta_i c_i \int_X (\psi \circ w_i)(x)dx. \end{aligned} \tag{4.24}$$

In the special case that $\alpha_i > 0$ and $\beta_i = 0$, $1 \leq i \leq N$ and $\sum_{i=1}^{N} c_i \alpha_i = 1$, then $T^\dagger$ becomes the adjoint operator on $C(X)$ associated with the N-map IFSP $(\mathbf{w}, \mathbf{p})$ with probabilities $p_i = c_i \alpha_i$, cf. Eq. (3.50). The associated IFSD operator T coincides with the Markov operator M on $\mathcal{M}(X)$. However, T is not necessarily contractive in the complete metric space $(\mathcal{D}'(X), d_{D'})$ since we may have $C_D = 1$. (See Example 3 below.) However, in the subset $\mathcal{M}(X) \subset \mathcal{D}'(X)$, T is contractive with respect to the Hutchinson metric d_H. (Note that the test functions used for the d_H metric are Lip_1 functions.) By construction T maps each "shell" of measures $\mathcal{M}_K(X) = \{\mu | \mu(X) = K > 0\}$ to itself and is contractive on that shell with respect to the d_H metric. Therefore, there exists a ray of fixed point measures of T which belong to $\mathcal{D}'(X)$.

Now let the fixed point distribution $\bar{u}$ of an N-map affine IFSD be considered as a density function for a measure $\bar{\mu}$ on $\mathcal{B}(X)$, i.e.

$$\bar{\mu}(S) = \int_X I_S(x) \bar{u}(x) dx = \int_S \bar{u}(x) dx. \tag{4.25}$$

(Note that $\bar{\mu}$ is not necessarily a probability measure although we may suitably rescale $\bar{u}$ to make it so.) From Eq. (3.36),

$$\bar{\mu}(S) = \sum_{k=1}^{N} \alpha_k c_k \bar{\mu}(w_k^{-1}(S)) + \sum_{k=1}^{N} \beta_k c_k m(w_k^{-1}(S)), \tag{4.26}$$

where m denotes Lebesgue measure. Again in the special case that $\alpha_i > 0$ and $\beta_i = 0$, $1 \leq i \leq N$ and $\sum_{i=1}^{N} \alpha_i c_i = 1$, Eq. (4.26) is identical to Eq. (3.46) for invariant measures of IFSP.

Examples: In all cases $X = [0,1]$ and $N = 2$ with $w_1(x) = \frac{1}{2}x$ and $w_2(x) = \frac{1}{2}x + \frac{1}{2}$.

1. $\phi_1(t) = \phi_2(t) = \frac{1}{2}t + \frac{1}{2}$. The associated fractal transform operator T is contractive in $(\mathcal{D}'(X), d_{D'})$ with contraction factor $C_D = \frac{1}{2}$. The fixed point $\bar{u}(x) = 1$ is an element of $\mathcal{D}'(X)$ as well as of $\mathcal{L}^p(X, m)$ for all $p \in [0, \infty]$, since T is also contractive in $(\mathcal{L}^p, d_p)$.

2. $\phi_1(t) = \phi_2(t) = \frac{1}{2}t$. T is again contractive in $(\mathcal{D}'(X), d_{D'})$ with $C_D = \frac{1}{2}$. Here, $\bar{u}(x) = 0$.

3. $\phi_1(t) = \phi_2(t) = t$. T is not contractive in $(\mathcal{D}'(X), d_{D'})$. Here, $C_D = 1$ and the equality in Eq. (4.20) holds. The functions $u(x) = c$ (a.e.), $c \in \mathbf{R}$, are fixed points of T. A fixed point attractor $\bar{u} \in \mathcal{D}'(X)$ is the Lebesgue measure on [0,1]. In order to see this, let us take $u_0(x) = \delta(x)$, the Dirac delta function at $x = 0$ and form the sequence $u_{n+1} = Tu_n$, for $n \geq 0$.

From Eq. (4.17), $u_1(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x - \frac{1}{2})$ and

$$u_n(x) = \frac{1}{2^n}\sum_{k=0}^{2^n-1}\delta(x - \frac{k}{2^n}). \tag{4.27}$$

Let F_n denote the linear functionals associated with the u_n, i.e. $F_n(\psi) = \int_X u_n(x)\psi(x)dx$. Then $\lim_{n\to\infty} F_n(\psi) = \int_X \psi(x)dx$.

If $u_0(x) = K\delta(x)$, $K > 0$, then the sequence $u_{n+1} = Tu_n$ converges to the uniform Lebesgue measure m_K where $m_K([0,1]) = K$.

One final remark concerning this example: It is also a 2-map IFZS on [0,1] (cf. Section 3.2). The associated IFZS operator T is contractive on $(\mathcal{F}^*(X), d_\infty)$ and the attractor is $\bar{u}(x) = 1$.

4.2 Integrals Involving Affine IFSD

Let $(\mathbf{w}, \Phi)$ be an N-map affine IFSD on $X = [0,1]$ with associated operator $T : \mathcal{D}'(X) \to \mathcal{D}'(X)$. If $f \in \mathcal{D}'(X)$ is defined by $F(\psi) = <\psi, f>$, the distribution $g = Tf$ will be given by $G(\psi)$, where

$$G(\psi) = \sum_{i=1}^{N}\alpha_i c_i \int_X f(y)(\psi\circ w_i)(y)dy + \sum_{i=1}^{N}\beta_i c_i \int_X (\psi\circ w_i)(y)dy \tag{4.28}$$

By iterating this procedure, we obtain, for $n = 1, 2, \ldots$,

$$\begin{aligned} <\psi, T^n f> &= <(T^\dagger)^n\psi, f> \\ &= \sum_{i_1,\ldots,i_n}^{N} p_{i_1}\ldots p_{i_n}\int_X f(y)(\psi\circ w_{i_1}\circ\ldots\circ w_{i_n})(y)dy \\ &+ \sum_{k=1}^{n}\sum_{i_1,\ldots,i_k}^{N} p_{i_1}\ldots p_{i_{k-1}}q_{i_k}\int_X(\psi\circ w_{i_1}\circ\ldots\circ w_{i_k})(y)dy, \end{aligned} \tag{4.29}$$

where $p_i = \alpha_i c_i$ and $q_i = \beta_i c_i$, $1 \le i \le N$. (This result may be compared with the IFSP case in Eq. (3.51).)

If T is contractive then it possesses a fixed point $\bar{u} \in \mathcal{D}'(X)$. Moreover, $T^n f \to \bar{u}$ as $n \to \infty$ in distribution for any $f \in \mathcal{D}'(X)$. Setting $f = \bar{u}$ and $n = 1$ in Eq. (4.29), we obtain

$$\int_X \bar{u}(x)\psi(x)dx = \sum_{i=1}^{N}\alpha_i c_i \int_X \bar{u}(y)(\psi\circ w_i)(y)dy$$

$$+ \quad \sum_{i=1}^{N} \beta_i c_i \int_X (\psi \circ w_i)(y) dy. \tag{4.30}$$

For example, in the case $\psi(x) = x^n$, we obtain a set of equations which permit the recursive computation of the moments $g_n = \int_X x^n \bar{u}(x) dx$. These equations necessarily coincide with those obtained from the IFSM method [10]. In the special case $\beta_i = 0$, $1 \le i \le N$, and $\sum_{i=1}^{N} p_i = 1$, we obtain the recursion equations for moments g_n of an invariant measure of the affine IFSP $(\mathbf{w}, \mathbf{p})$ [10].

In general, however, integrals involving the fixed point $\bar{u}$ can not be solved recursively or in closed form, e.g. $\int_X x^{1/2} \bar{u}(x) dx$. We may, however, use the fact that

$$\int_X \bar{u}(x) \psi(x) dx = \lim_{n \to \infty} \int_X (T^n f)(x) \psi(x) dx, \quad \forall\, \psi \in \mathcal{D}'(X). \tag{4.31}$$

For an $x_0 \in X$, set $f(x) = \delta(x - x_0)$, the Dirac delta function at x_0, to obtain

$$\begin{aligned} \int_X (T^n f)(x) \psi(x) dx = & \sum_{i_1, \ldots, i_n}^{N} p_{i_1} \ldots p_{i_n} (\psi \circ w_{i_1} \circ \ldots \circ w_{i_n})(x_0) \\ & + \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k}^{N} \alpha_{i_1} \ldots \alpha_{i_{k-1}} \beta_{i_k} \int_{X_{i_1, i_2, \ldots, i_k}} \psi(x) dx, \end{aligned} \tag{4.32}$$

where $X_{i_1, i_2, \ldots, i_k} = \hat{w}_{i_1} \circ \hat{w}_{i_2} \circ \ldots \circ \hat{w}_{i_k}(X)$. This expression is somewhat more complicated than Eq. (3.51), its counterpart for measures. However, the integrals are generally easy to compute. As with Eq. (3.51), the evaluation of this expression involves the enumeration of N-trees to n generations.

Example: We consider the following 3-map affine IFSM:

$$\begin{aligned} w_1(x) &= 0.4x, & \phi_1(t) &= t, \\ w_2(x) &= 0.4x + 0.3, & \phi_2(t) &= 0.25t + 0.25, \\ w_3(x) &= 0.4x + 0.6, & \phi_3(t) &= t. \end{aligned} \tag{4.33}$$

The associated operator T is contractive in $(\mathcal{L}^p(X, m), d_p)$ for $p = 1$. A histogram approximation of the attractor $\bar{u}$ of this IFSM is shown in Figure 4.1. The power moments g_n of $\bar{u}$,

$$g_n = \int_X x^n \bar{u}(x) dx, \quad n = 0, 1, 2, \ldots, \tag{4.34}$$

may be computed in closed form via Eq. (4.30) [10]. The first four moments are:

$$g_0 = 1, \quad g_1 = \frac{1}{2}, \quad g_2 = \frac{431}{1284}, \quad g_3 = \frac{217}{856}. \tag{4.35}$$

In Table 4.1 are shown approximations to these moments as well as to the integral $g_{1/2} = \int_X x^{1/2}\bar{u}(x)dx$, as computed from Eq. (4.32). (There is no closed form expression for $g_{1/2}$.) The convergence of the approximations with increasing n is evident. (The results for g_1 are not shown since all approximations were in agreement to at least seven digits of accuracy.) Also note that the Hausdorff inequalities must be satisfied by the moments. In this case, we observe that $g_0 > g_{1/2} > g_1 > g_2 > g_3$.

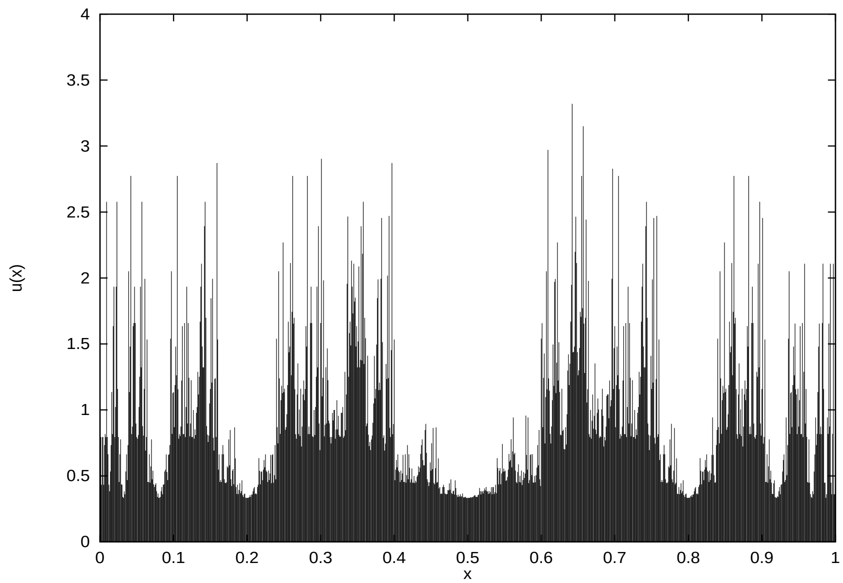

Figure 4.1: The fixed point attractor $\overline{u}(x)$ of the IFSM given in Eq. (4.33) of the text.

n	$g_{1/2}$	g_2	g_3
1	0.677591201	0.323333333	0.235
2	0.668876203	0.333893333	0.25084
3	0.666482494	0.335413973	0.25312096
4	0.665849446	0.335632945	0.253449418
5	0.665685332	0.335664477	0.253496716
6	0.665643243	0.335669018	0.253503527
7	0.665632513	0.335669671	0.253504508
8	0.665629787	0.335669766	0.253504649
9	0.665629095	0.335669780	0.253504669
10	0.665628920	0.335669782	0.253504672
Exact		0.335669782	0.253504673

Table 4.1: Approximations to integrals $g_\alpha = \int_X x^\alpha \bar{u}(x)dx$ as computed from Eq. (4.32). Here $\bar{u}$ is the fixed point of the IFSM given in Eq. (4.33).

Chapter 5

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