Existence, Uniqueness and Asymptotic Behaviour of Intensity-Based Measures Which Conform to a Generalized Weber's Model of Perception

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Abstract. In this paper, we report on further progress on our study of "Weberized" metrics for image functions presented at ICIAR 2018. These metrics allow greater deviations at higher intensity values and lower deviations at lower intensity values in accordance with Weber's model of perception. The purpose of this paper is to address some important mathematical details that were not considered in the ICIAR 2018 paper, e.g., (a) proving the existence and uniqueness of greyscale density functions $\rho_a(y)$ which conform to Weber's model, (b) complete description of the dominant asymptotic behaviour of the density functions $\rho_a(y)$ for $y \to \infty$ and $y \to 0^+$ for the cases (i) $0 < a < 1$ and (ii) $a > 1$, (c) a method of computing the asymptotic expansion of $\rho_a(y)$ for $y \to \infty$ to any number of terms for the case $0 < a < 1$.

1 Introduction

In this paper we report on some recent progress on the formulation and analysis of image function metrics designed to perform better than standard L^2-based metrics in terms of perceptual image quality. It is a continuation of work presented at ICIAR 2018 [4] and ICIAR 2014 [3], to which the interested reader is referred for more details.

Very briefly, in [3] we used the idea of "Weberizing" distance functions to show that the logarithmic L^1 distance between two image functions conforms to Weber's standard model of perception. The method employed in [3] could be viewed as rather *ad hoc*. In [4], we approached the problem from a more mathematical point of view, by employing appropriate nonuniform measures over the greyscale intensity to produce a class of alternate metrics between image functions. (The idea of such measures over greyscale space was introduced in [2].)

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These measures were defined by (nonconstant) greyscale density functions which conform (at least asymptotically, in the limit $I \to \infty$) to Weber's generalized model of perception. (The term "conform" will be defined in Sect. 2 below.)

By Weber's generalized model of perception we mean the following: Given a greyscale background intensity $I > 0$, the minimum change in intensity ΔI perceived by the human visual system (HVS) is related to I as follows,

$$\frac{\Delta I}{I^a} = C, \tag{1}$$

where $a > 0$ and C is constant, or at least roughly constant over a significant range of intensities I. The case $a = 1$ corresponds to the standard Weber model which is employed in practically all applications [7]. There are situations, however, in which other values of a, in particular, $a = 0.5$, may apply – see, for example, [5]. From Eq. (1), a Weberized distance between two functions u and v should tolerate greater/lesser differences over regions in which they assume higher/lower intensity values. The degree of toleration will be determined by the exponent a in Eq. (1).

As mentioned in [4], the well known structural similarity (SSIM) measure [8,9] is already Weberized, according to the standard model $a = 1$, since it may be rewritten in terms of intensity ratios. This may well be quite sufficient in practice, but it would be interesting to investigate how SSIM could be modified to conform to the more general case, i.e., $a \neq 1$.

In [4], we showed that for $0 \leq a \leq 1$, the density function $\rho_a(I)$ which conforms, at least asymptotically, to Weber's model of perception is $\rho_a(I) = 1/I^a$. (In the case $a = 0$, i.e., zero Weber effect, $\rho_0(y) = 1$ corresponds to uniform Lebesgue measure in I-space.) The image function metrics produced by the measures which correspond to these density functions are as follows,

$$\begin{aligned} 0 \leq a < 1: \qquad & D_a(u,v) = \int_X \left| u(x)^{-a+1} - v(x)^{-a+1} \right| \, dx \\ a = 1: \qquad & D_1(u,v) = \int_X \left| \ln u(x) - \ln v(x) \right| \, dx \,. \end{aligned} \tag{2}$$

As mentioned in [4], it is more convenient to work with the L^2-based analogues of these metrics.

The primary purpose of this paper is to address some important mathematical details that were not considered in [4], including the following:

1. The results in [4] were obtained under the assumption of the existence of a continuous density function $\rho_a(I)$ for $a > 0$. In Sect. 2, we outline the main steps in a proof of the existence of a unique density function $\rho_a(y)$ which satisfies an invariance result (Eq. (9) below) that guarantees "conformity" to Weber's generalized model of perception. It is very interesting, at least from a mathematical viewpoint, to see how the "equal-area condition" in Eq. (9) is associated with the famous Abel functional equation.
2. In Sect. 3, the dominant asymptotic behaviour of the density functions $\rho_a(y)$ for $y \to \infty$ and $y \to 0^+$ are obtained in the cases (i) $0 < a < 1$ and (ii) $a > 1$.

All of these results are obtained by means of an asymptotic analysis of an "equal-area integral," cf. Eq. (9). (In [4], the result for $y \to \infty$ in the case $0 < a < 1$ was obtained using a differential equation approach.)

3. A method of computing the asymptotic expansion of $\rho_a(y)$ for $y \to \infty$ to any number of terms in the case $0 < a < 1$ is also presented in Sect. 3.

Finally, in Sect. 2, we address and correct a couple of unfortunate typographical errors which appeared in [4], specifically in Theorem 2 of that paper.

The basic mathematical ingredients of our formalism are as follows:

1. The **base (or pixel) space** $X \subset \mathbb{R}^n$ on which our signals/images are supported. Here, without loss of generality since our discussion is purely theoretical, we simply consider the one-dimensional case $X = [a, b]$.
2. The **greyscale range:** For an $A > 0$, $\mathbb{R}_g = [A, B]$, where $B < \infty$.
3. The **signal/image function space** $\mathcal{F} = \{u : X \to \mathbb{R}_g \mid u \text{ is measurable}\}$. From our definition of the greyscale range $\mathbb{R}_g$, $u \in \mathcal{F}$ is positive and bounded almost everywhere, i.e., $0 < A \le u(x) \le B < \infty$ for almost every $x \in X$. A consequence of this boundedness is that $\mathcal{F} \subset L^p(X)$ for all $p \ge 1$, where the $L^p(X)$ function spaces are defined in the usual way. For any $p \ge 1$, define a metric d_p on $\mathcal{F}$ can be defined in the usual way, i.e.,

$$d_p(u, v) = \|u - v\|_p = \left[\int_X [u(x) - v(x)]^p \, dx\right]^{1/p}, \quad u, v \in L^2(X) . \tag{3}$$

2 Existence and Uniqueness of Greyscale Density Functions $\rho_a(y)$ for Generalized Weber's Model of Perception

In [4], we defined a class of intensity-dependent distance functions between functions in the space $\mathcal{F}$ as summarized below.

Consider two functions $u, v \in \mathcal{F}$ and define the following subsets of the base space $X = [a, b]$:

$$X_u = \{x \in X \mid u(x) \le v(x)\} \qquad X_v = \{x \in X \mid v(x) \le u(x)\} , \tag{4}$$

so that $X = X_u \cup X_v$. The sketch in Fig. 1 below should help with the visualization of our procedure.

The distance D between u and v will be defined as an integration over vertical strips of width dx and centered at $x \in [a, b]$. The contribution of each strip will **not**, in general, be determined by the usual lengths of the strips, i.e., the quantities $|u(x) - v(x)|$, but rather the **sizes** of the intervals $(u(x), v(x)] \subset \mathbb{R}_g$ and $(v(x), u(x)] \subset \mathbb{R}_g$ as assigned by a **measure** ν that is supported on the greyscale interval $\mathbb{R}_g = [A, B]$. The measures of the two intervals shown in the figure will be denoted as $\nu(u(x), v(x)]$ and $\nu(v(x), u(x)]$, respectively. The distance between u and v associated with the measure ν is now defined as follows,

$$D(u, v; \nu) = \int_{X_u} \nu(u(x), v(x)] \, dx \; + \; \int_{X_v} \nu(v(x), u(x)] \, dx , \tag{5}$$

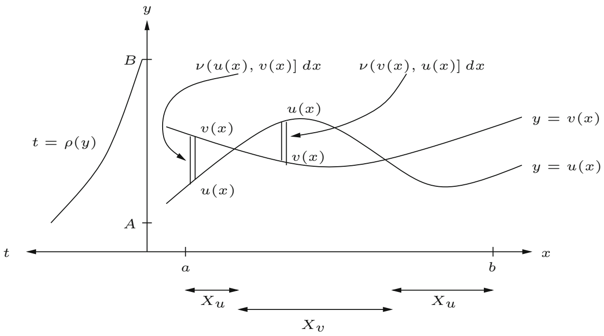

Fig. 1. Sketch of two nonnegative greyscale functions $u(x)$ and $v(x)$ with strips of width dx that will contribute to the distance $D(u, v; \nu)$. A density function $\rho(y)$ over the greyscale range $\mathbb{R}_g = [A, B]$ is sketched at the left.

It is convenient to consider measures which are defined by continuous, nonnegative density functions $\rho(y)$ for $y > 0$. (Such measures will be absolutely continuous with respect to Lebesgue measure.) Given a measure ν with density function ρ, then for any interval $(y_1, y_2] \subset \mathbb{R}_g$,

$$\nu\,(y_1, y_2] = \int_{y_1}^{y_2} \rho(y)\, dy \;= P(y_2) - P(y_1)\,, \tag{6}$$

where $P'(y) = \rho(y)$. The distance function $D(u, v; \nu)$ in Eq. (5) then becomes

$$D(u, v; \nu) = \int_X |P(u(x)) - P(v(x))|\, dx\,. \tag{7}$$

Special Case: $\nu = m_g$, uniform Lebesgue measure on $\mathbb{R}_g$, where $\rho(y) = 1$ so that $P(y) = y$ in Eq. (7). This is the measure employed in most function metrics, e.g., the L^p metrics in Eq. (3). The associated metric is

$$D(u, v, m_g) = \int_{X_u} [v(x) - u(x)]\, dx + \int_{X_v} [u(x) - v(x)]\, dx = \int_X |u(x) - v(x)|\, dx\,, \tag{8}$$

the well known L^1 distance between u and v.

As discussed in [4], the constancy of the Lebesgue density function $\rho(y) = 1$ implies that all greyscale intensity values are weighted equally in the computation of distances between image functions. However, Weber's model of perception in Eq. (1) suggests that for $a > 0$, the density function $\rho_a(y)$ should be a *decreasing* function of intensity y: As the intensity value increases, the HVS will tolerate greater differences between $u(x)$ and $v(x)$ before being perceived. We shall also require the density function $\rho_a(y)$ to conform to Weber's model in Eq. (1) according to the following criterion.

Definition: For a given $a > 0$, suppose that Weber's model of perception in Eq. (1) holds for a particular value of $C > 0$ for all values of $I \geq A$. (See Note 3 below.) We say that a measure $\nu_a(y)$ defined by the density function $\rho_y(y)$ **conforms to** or **accommodates** this Weber model if the following condition holds for all $I \geq A$,

$$\nu_a(I, I + \Delta I) = \int_I^{I+\Delta I} \rho_a(y)\, dy = K \,, \tag{9}$$

for some constant K, where $\Delta I = CI^a$ is the minimum change in perceived intensity at I according to Eq. (1).

Equation (9) may be viewed as an invariance result with respect to perception. Its graphical interpretation in terms of equal areas enclosed by the density curve $\rho_a(y)$ is shown in Fig. 2 below.

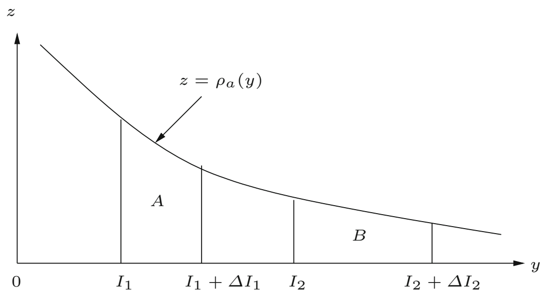

Fig. 2. Graphical interpretation of the "equal areas" invariance result in Eq. (9). Area of A = Area of B.

Notes

1. In the special case, $a = 1$, i.e., Weber's standard model, $\rho_1(y) = 1/y$ and $\nu(I, I + \Delta I) = \ln(1 + C)$ [4].
2. We may also include the special case $a = 0$, i.e., an absence of Weber's model, in the above definition. In this case $\rho_0(y) = 1$ so that $\nu = m_g$, Lebesgue measure on $\mathbb{R}_g$.
3. As mentioned earlier, Weber's model in Eq. (1) is valid only over a limited range of intensities. The requirement that Eq. (9) be true for all $I > A$ is imposed in order to establish the asymptotic behaviour of the density functions $\rho_a(y)$ for large y.

From Notes 1 and 2 above, it is natural to conjecture that for $a > 0$ in general, $\rho_a(y) = 1/y^a$ or at least approaches $1/y^a$ asymptotically as $y \to \infty$. It is easy to show that equality does not hold for $a \neq 1$. In [4], however, we proved that the asymptotic result holds for $0 < a < 1$. Unfortunately, there are two typographical errors in the presentation of this result, namely Theorem 2 in [4]:

(i) it is valid for $0 < a < 1$ and not $a > 0$ as stated, (ii) the asymptotic result $\rho_a(y) \approx 1/y^a$ is valid for $y \to \infty$ and not $y \to 0^+$ as stated. (These errors do not appear in the proof of the theorem given in the Appendix.)

We now present a significant improvement of the results obtained in [4] in terms of mathematical rigor, as summarized in Sect. 1. The following result, although seemingly trivial, will have some important consequences.

Theorem 1: For given values of $a > 0$ and $C > 0$, let $\rho_a(y)$ satisfy the invariance condition in Eq. (9). Then $\displaystyle\int_A^\infty \rho_a(y)\,dy = \infty$.

Sketch of Proof: Let $y_0 = A$, and $y_{n+1} = y_n + C y_n^a$ for $n \geq 0$. It is not difficult to show that $y_n \to \infty$ as $n \to \infty$. Furthermore, from Eq. (9),

$$\int_A^{y_n} \rho_a(y)\,dy = nK \to \infty \quad \text{as} \quad n \to \infty\,. \tag{10}$$

This result already establishes that the asymptotic property $\rho_a(y) \sim 1/y^a$ as $y \to \infty$ cannot hold for $a > 1$.

In preparation for the main result of this section of the paper, let $g(y)$ be a continuous function on $[A, \infty)$ and define $G(x) = \displaystyle\int_A^x g(y)\,dy$ for all $x \geq A$. Clearly, $G(A) = 0$. Furthermore, suppose that for fixed values of $a > 0$ and $C > 0$, $g(y)$ satisfies the invariance property in Eq. (9). It follows that $G(x)$ satisfies the following equation,

$$G(x + Cx^a) - G(x) = K\,, \quad x \geq A\,. \tag{11}$$

For convenience, we define $f(x) = x + Cx^a$ and divide both sides of Eq. (11) by K to obtain the equation,

$$H(f(x)) - H(x) = 1\,, \tag{12}$$

where $H(x) = K^{-1}G(x)$. Equation (12) is known as **Abel's equation**, a well known **functional equation** [6].

We now state the main result of this section of the paper.

Theorem 2: For given values of $a > 0$ and $C > 0$, there exists a unique, continuous function $\rho_a(y)$ defined on $[A, \infty)$ which satisfies Eq. (9).

Sketch of Proof: Consider the following linear functional equation for the function $g(x)$ for $x \geq A$,

$$g(x + Cx^a)(1 + aCx^{a-1}) - g(x) = 0\,. \tag{13}$$

(This equation may be obtained by differentiating both sides of Eq. (9) with respect to x with the assumption that $g(x)$ is continuous.) Equation (13) is a special case of the following family of linear functional equations studied by Belitskii and Lyubich in [1],

$$P(x)\psi(F(x)) + Q(x)\psi(x) = \gamma(x)\,, \quad x \in X\,, \tag{14}$$

where X is the topological space over which the equation is being considered. Here, $X = [A, \infty)$, $P(x) = 1 + aCx^{a-1}$, $F(x) = 1 + aCx^a$, $Q(x) = -1$ and $\gamma(x) = 0$. In [1], it is shown that if the Abel equation associated with Eq. (14), namely,

$$\phi(F(x)) - \phi(x) = 1\,, \tag{15}$$

has a continuous solution $\phi(x)$, then Eq. (14) is *totally solvable*, i.e., it has a continuous solution $\psi(x)$ for every continuous function $\gamma(x)$.

The existence of continuous solutions to Eq. (15) depends on the iteration dynamics of the function $F(x)$ on the space X. In this case, $F(x) = x + Cx^a$ is an increasing function on $X = [A, \infty)$, the dynamics of which is quite straightforward: $F : X \to X$ and for all $x \in X$, $F^n(x) \to \infty$ as $n \to \infty$. As such, the conditions of Corollary 1.6 of [1] are satisfied, i.e., every compact set $S \in X$ is wandering under the action of F. Therefore, Eq. (15) has a continuous solution (unique up to a constant, i.e., if $\phi(x)$ satisfies Eq. (15), then so does $\phi(x) + C$ for any $C \in \mathbb{R}$). This, in turn, implies that Eq. (14) has a unique, nonzero, continuous solution.

Additional analysis of Eqs. (13) and (9) yields the following properties of $\rho_a(y)$ for $a > 0$ which we state without formal proof because of space limitations:

1. $\rho_a(y) > 0$ for all $x > 0$. (Or at least we choose a positive solution of Eq. (13). If $g(x)$ is a solution to Eq. (13), then so is $Cg(x)$ for any $C \in \mathbb{R}$.
2. $\rho_a(y)$ is decreasing on $[0, \infty)$ and $\rho_a(y) \to 0$ as $y \to \infty$. This is to be expected: In Eq. (9), the length of the interval $[x, x + Cx^a]$ increases with x. The equal-area condition would dictate that $\rho_a(y)$ decrease with y.
3. As $y \to 0^+$, $\rho_a(y) \to \infty$. This is expected from the equal-area condition since the length of the interval $[x, x + Cx^a]$ decreases as $x \to 0^+$.

3 Asymptotic Behaviour of Density Functions $\rho_a(y)$

The determination of the asymptotic behaviour of the density functions $\rho_a(y)$ is centered on the equal-area property of Eq. (9).

Asymptotic Behaviour of $\rho_a(y)$ as $y \to \infty$ for the Case $0 < a < 1$

Theorem 3: For a given $a \in (0, 1)$ and $C > 0$, the asymptotic expansion of the Weber density function $\rho_a(y)$ satisfying Eq. (9) has the following form,

$$\rho_a(y) = \sum_{n=0}^{\infty} \frac{A_n}{y^{a+n(1-a)}} \quad \text{as} \quad y \to \infty\,. \tag{16}$$

The first three terms of this expansion are

$$\rho_a(y) = \frac{1}{y^a} + \frac{1}{2}aC\frac{1}{y} - \frac{1}{12}aC^2(2a-1)\frac{1}{y^{2-a}} + \cdots\,. \tag{17}$$

Note that the leading term of this expansion is in agreement with the asymptotic result presented in [4].

The expansion in Eq. (16) is obtained by repeated use of the following important result.

Lemma 1: For $x > 0$, $0 < a < 1$ and $b > 0$,

$$\int_x^{x+Cx^a} \frac{1}{y^b}\, dy = Cx^{a-b} - \frac{1}{2} bC^2 x^{2a-b-1} + \cdots \quad \text{as} \quad x \to \infty\,. \tag{18}$$

Sketch of Proof of Lemma: Use antiderivative of $1/y^b$ (considering the cases $b \neq 1$ and $b = 1$ separately). For $b \neq 1$, let $(x + Cx^a)^{1-b} = x^{1-b}(1 + Cx^{a-1})^{1-b}$ and expand the term in parenthesis via binomial series. Similar procedure for $b = 1$ involving the ln function and binomial series.

Sketch of Proof of Theorem 3: Here are the first few steps involved in the derivation of Eq. (16).

1. Setting $b = a$ in Eq. (18) yields

$$\int_x^{x+Cx^a} \frac{1}{y^a}\, dy = C + O(x^{a-1}) \quad \text{as} \quad x \to \infty\,, \tag{19}$$

   which implies that the equal-area condition in Eq. (9) is satisfied, to leading order, by $\rho_a(y) \sim 1/y^a$, in agreement with the asymptotic result of [4].
2. Set $b = a$ in Eq. (18) and rewrite it as follows,

$$\int_x^{x+Cx^a} \frac{1}{y^a}\, dy - C = -\frac{1}{2} aC^2 x^{a-1} + O(x^{2a-2}) \quad \text{as} \quad x \to \infty\,. \tag{20}$$

3. Up to a constant, the first term on the right side of Eq. (20) has the same behaviour, x^{a-1}, as the first term in Eq. (18) in the case $b = 1$. Now multiply both sides of Eq. (18) in the case $b = 1$ by the factor $-\dfrac{1}{aC}$ and subtract the resulting equation from Eq. (20) to obtain the result,

$$\int_x^{x+Cx^a} \left[\frac{1}{y^a} + \frac{1}{2} aC \frac{1}{y}\right] dy - C = O(x^{2a-2}) \quad \text{as} \quad x \to \infty\,. \tag{21}$$

   The term in square brackets is an improved approximation of $\rho_a(y)$ as $y \to \infty$ in terms of the equal-area condition of Eq. (9).
4. Higher order terms in Eq. (18) involve powers of the form $x^{n(a-1)}$. Matching these terms with the leading term x^{a-b} in Eq. (18) is accomplished by selecting $b = a + n(1 - a)$ for $n \geq 0$, yielding the expansion in Eq. (16).

Asymptotic Behaviour of $\rho_a(y)$ as $y \to \infty$ for the Case $a > 1$

Recall, from Theorem 1, that in the case $a > 1$, we do not expect that $\rho_a(y) \sim 1/y^a$ as $y \to \infty$. This is confirmed by the following result, which can easily be derived with some elementary Calculus:

Lemma 2: For a given $a > 0$ and $C > 0$, define the function

$$G_{a,b}(x) = \int_x^{x+Cx^a} \frac{1}{y^b}\, dy\,, \quad x > 0\,. \tag{22}$$

1. If $0 < b \leq 1$, then $G_{a,b}(x) \to \infty$ as $x \to \infty$.
2. If $b > 1$, then $G_{a,b}(x) \to 0$ as $x \to \infty$.

This leads us to consider functions which may involve $\ln y$ along with the term $1/y$. The following result, which can also be derived via Calculus, is helpful.

Lemma 3: For a given $a > 0$ and $C > 0$ define the function

$$H_{a,p}(x) = \int_x^{x+Cx^a} \frac{1}{y\,(\ln y)^p}\, dy\,, \quad x > 0\,. \tag{23}$$

1. If $0 < p < 1$, then $H_{a,p}(x) \to \infty$ as $x \to \infty$.
2. If $p = 1$, then $H_{a,1}(x) \to \ln a$ as $x \to \infty$.
3. If $p > 1$, then $H_{a,p}(x) \to 0$ as $x \to \infty$.

The above result suggests that $\rho_a(y) \sim 1/(y\, \ln y)$ as $y \to \infty$ for $a > 1$. Using some results from the asymptotic analysis involved in the proof of Lemma 3, we obtain the following two-term approximation,

$$\rho_a(y) \simeq \frac{1}{y\,\ln y} - \left(\frac{\ln C}{a}\right)\frac{1}{y\,(\ln y)^2} \quad \text{as} \quad y \to \infty\,. \tag{24}$$

This result is interesting in that the leading-order term is independent of a, unlike the situation for $a \leq 1$.

Further asymptotic analysis of this case will be quite complicated due to the possible mixed presence of powers of y and $\ln y$ in the denominators of additional terms in the expansion.

Asymptotic Behaviour of $\rho_a(y)$ as $y \to 0^+$

There is a reciprocity with regard to the integrals involved in the previous analysis of $y \to \infty$ and those involved in the case $y \to 0^+$, It is quite straightforward to show, for example, that an analysis of the asymptotic limit $x \to 0^+$ for the case $0 < a < 1$ employs the same equations as those used in the analysis of $x \to \infty$ for the case $a > 1$. For this reason, we simply state the two major results below:

$$\begin{aligned} 0 < a < 1: &\qquad \rho_a(y) \simeq \frac{1}{y\,\ln y} - \left(\frac{\ln C}{a}\right)\frac{1}{y\,(\ln y)^2} \quad \text{as} \quad y \to 0^+\,, \\ a > 1: &\qquad \rho_a(y) = \frac{1}{y^a} + \frac{1}{2}aC\frac{1}{y} - \frac{1}{12}aC^2(2a-1)\frac{1}{y^{2-a}} + \cdots \text{as} \;\; y \to 0^+\,, \end{aligned}$$

where the second result is the truncation of an asymptotic expansion of the same form, and the same constants A_n, as in Eq. (16).

Some Comments on These Asymptotic Results
The asymptotic analysis of the density functions $\rho_a(y)$ in the case $y \to 0^+$ is more of a theoretical exercise since we are working with the range space $\mathbb{R}_g = [A, B]$ with lower "cutoff" intensity level $A > 0$. Indeed, the validity of Weber's model for low intensity values is also questionable. We are more interested in the high-intensity region and expect that the behaviour of $\rho_a(y)$ is well described by the asymptotic formulas for $y \to \infty$. In fact, as was done in [4], we consider only the leading-order behaviour of the asymptotic expansions: Although it would be an interesting mathematical exercise, the inclusion of subdominant terms would most probably be "overkill" in practice, especially in light of the fact that Weber's model is, in itself, an approximation.

4 Revisiting the Distance Functions Associated with the Density Functions $\rho_a(y)$

The Case $0 < a < 1$
The metrics associated with the leading asymptotic behaviour of the density functions, $\rho_a(y) \sim 1/y^a$ for $0 \le a \le 1$, were presented in Eq. (2). As mentioned in [4], it is rather difficult – although not impossible – to work with these L^1-based metrics so we consider their L^2-based analogues,

$$\begin{aligned} 0 \le a < 1: \qquad D_{2,a}(u,v) &= \left[\int_X \left[u(x)^{-a+1} - v(x)^{-a+1}\right]^2 \, dx\right]^{1/2} \\ a = 1: \qquad D_{2,a}(u,v) &= \left[\int_X \left[\ln u(x) - \ln v(x)\right]^2 \, dx\right]^{1/2} . \end{aligned} \tag{25}$$

It is rather straightforward to show that these metrics are equivalent to the L^2 metric on our space $F(X)$, i.e., $p = 2$ in Eq. (3). First let us recall the Mean Value Theorem of elementary Calculus:

Theorem 4: Let $g : [A, B] \to \mathbb{R}$ be continuous on $[A, B]$ and differentiable on (A, B). Then for any $y_1, y_2 \in [A, B]$, there exists a c between y_1 and y_2 such that

$$g(y_2) - g(y_1) = g'(c)(y_2 - y_1) . \tag{26}$$

In what follows, we shall also exploit the fact that for $u \in \mathcal{F}(X)$, $A \le u(x) \le B$ for a.e. $x \in X = [a, b]$.

1. $0 < a < 1$. For a fixed $a \in (0, 1)$, we apply the Mean Value Theorem to the function $g(y) = y^{-a+1}$ on $[A, B]$ with $A > 0$, as follows. Then for a.e. $x \in [a, b]$,

$$u(x)^{-a+1} - v(x)^{-a+1} = \frac{1-a}{c^a}\,(u(x) - v(x)) , \tag{27}$$

where c lies between $u(x)$ and $v(x)$. Taking absolute values, and noting that $A < c < B$, we obtain the inequalities for a.e. $x \in [a, b]$,

$$\frac{1-a}{B^a}\,|u(x) - v(x)| \le \left|u(x)^{1-a} - v(x)^{1-a}\right| \le \frac{1-a}{A^a}\,|u(x) - v(x)| . \tag{28}$$

Now square all terms, integrate over $[a, b]$ and take square roots to obtain the result,

$$\frac{1-a}{B^a} d_2(u,v) \leq D_{2,a}(u,v) \leq \frac{1-a}{A^a} d_2(u,v) \,. \tag{29}$$

2. $a = 1$. The Mean Value Theorem is now applied to the function $g(y) = \ln y$ on $[A, B]$. For a.e. $x \in [a, b]$,

$$\ln u(x) - \ln v(x) = \frac{1}{c}(u(x) - v(x)) \,, \tag{30}$$

where c lies between $u(x)$ and $v(x)$. Noting once again that $A < c < B$, and proceeding as in the previous case, we arrive at the result,

$$\frac{1}{B} d_2(u,v) \leq D_{2,1}(u,v) \leq \frac{1}{A} d_2(u,v) \,. \tag{31}$$

The Case $a > 1$
This case was not considered in [4] because of a lack of knowledge of the large-y behaviour of the density function $\rho_a(y)$ for $a > 1$ at the time. The asymptotic result in Eq. (24) suggests that we consider a distance function of the following form for the case $a > 1$,

$$D_a(u,v) = \int_X |\ln(\ln u(x)) - \ln(\ln v(x))| \, dx \,, \tag{32}$$

with L^2 analogue,

$$D_{2,a}(u,v) = \left[\int_X [\ln(\ln u(x)) - \ln(\ln v(x))]^2 \, dx \right]^{1/2} \,. \tag{33}$$

In this case, it may be desirable that the lower greyscale limit $A > e$, in which case $\ln(\ln y) > 0$ for $y \in \mathbb{R}_g$.

Mathematically, the problem of approximating functions in this metric is a very interesting one, and worthy of further study. Whether or not this problem has any practical value is related to the question of whether or not Weber's model is useful or even valid in the case $a > 1$. These, of course, are open questions at this time.

5 Concluding Remarks

In this paper, we have addressed some important mathematical details involving intensity-based measures that were not considered in our earlier ICIAR 2018 paper [4]. From a practical perspective, the most important details are (a) the proof of existence and uniqueness of greyscale density functions $\rho_a(y)$ which conform, in the "equal area sense," to Weber's generalized model of perception and (b) their dominant asymptotic behaviour in the case $y \to \infty$. Of course, the actual practical value of these results remains to be explored. In our previous works, admittedly, an examination of the applications of this method was limited to a few simple examples. More work could, and should, be done here and perhaps the area of high dynamic range imaging would serve as a good testing ground.

The "equal area condition" of Eq. (9) represents a unique way of looking at Weber's generalized model of perception. Not only is it interesting from a theoretical, i.e., mathematical, perspective but it also yields a concrete result, namely, a measure ν_a defined by a density function $\rho_a(y)$ which, in turn, defines a metric which "conforms" to Weber's model. It would be most interesting to investigate whether such a condition, or suitable modification thereof, applies to other models, including those outside of perception/image processing.

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