Signal Processing 101 (2014) 11–18

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Fourier transforms of measure-valued images, self-similarity and the inverse problem

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ARTICLE INFO

Article history:
Received 13 November 2013
Received in revised form
25 January 2014
Accepted 27 January 2014
Available online 5 February 2014

Keywords:
Fourier transform
Fractal transforms
Iterated function systems
Measure-valued images

ABSTRACT

After recalling the notion of a complete metric space (Y, d_Y) of measure-valued images over a base (or pixel) space X, we define a complete metric space $(\mathcal{F}, d_{\mathcal{F}})$ of Fourier transforms of elements $\mu \in Y$. We also show that a fractal transform $T: Y \to Y$ induces a mapping M on the space $\mathcal{F}$. The action of M on an element $U \in \mathcal{F}$ is to produce a linear combination of *frequency-expanded* copies of M. Furthermore, if T is contractive in Y, then M is contractive on $\mathcal{F}$: as expected, the fixed point $\overline{U}$ of M is the Fourier transform of $\mu \in Y$.

1. Introduction

In [10], a complete metric space (Y, d_Y) of measure-valued images $\mu : [0,1]^n \to \mathcal{M}(\mathbb{R}_g)$ was constructed. Here, $\mathcal{M}(\mathbb{R}_g)$ is the set of Borel probability measures supported on the *greyscale range* $\mathbb{R}_g \subset \mathbb{R}$ (or $\mathbb{R}^m$). A primary motivation for this and other constructions (see, for example, [20]) is that there are situations in image processing in which it is useful to consider the greyscale value of an image u at a point x as a random variable that can assume a range of values $\mathbb{R}_g \subset \mathbb{R}$, as opposed to a single value (or set of values, in the case of a vector-valued image, e.g., RGB color image, hyperspectral satellite image of the earth).

A measure-valued representation of images may also be useful in the analysis and implementation of various nonlocal image processing schemes, where the greyscale value of an image $u(x)$ is modified according to a number $N > 0$ of values of the image $u(y_k)$ at points y_k, $1 \le k \le N$, that are not necessarily close to x. Before computing the modified greyscale value $v(x) = T(u(x))$, where T denotes the operator associated with the nonlocal scheme, one may wish to examine the measure $\mu(x)$ that is constructed by a convex combination of point mass measures situated at the values $\phi(u(y_k))$, where $\phi : \mathbb{R}_g \to \mathbb{R}_g$ is an operator appropriate to the scheme. Such a construction represents a kind of "preprocessing step" associated with the nonlocal method.

Two such nonlocal schemes examined in [10] using measure-valued representations were (i) *nonlocal means denoising* [4] and (ii) *fractal image coding* [2,5,17]. Both of these methods may be viewed as special cases of a general model of affine image self-similarity [1,3] in which subblocks of an image are approximated by other subblocks of the image.

Measure-valued images provide a natural representation in *high angular resolution diffusion imaging* (HARDI) [21]. Here, the greyscale range is $\mathbb{R}_g = \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$. At a given $x \in X \subset \mathbb{R}^3$, $u(x;\theta,\phi)$ can represent the probability of a water molecule at X diffusing in the direction $(\theta,\phi) \in \mathbb{S}^2$. That being said, because of the

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http://dx.doi.org/10.1016/j.sigpro.2014.01.026
0165-1684

regularity of these distributions, one can employ a function-valued image mapping approach as well [20].

In [10], a fractal transform operator was also constructed over the space (Y, d_Y). For a $\mu \in Y$, the action of a fractal transform $T: Y \to Y$ is to construct N spatially contracted and range-modified copies of μ and recombine them to produce a new element $\nu = T\mu \in Y$. Under suitable conditions T is contractive in (Y, d_Y), implying the existence of a fixed point measure-valued function $\overline{\mu} = T\overline{\mu}$ (see also [11–16] for more details on this approach).

The purpose of this paper is to show how Fourier transforms of measure-valued image functions in the space (Y, d_Y) may be defined as complex-measure-valued (the measures are over the greyscale range $\mathbb{R}_g$) functions over the base space X. We also show that a fractal transform operator T on (Y, d_Y) induces a mapping M on the space $(\mathcal{F}, d_{\mathcal{F}})$ of Fourier transforms on measure-valued image functions. In the case that T is contractive on (Y, d_Y), it follows that M is contractive on $(\mathcal{F}, d_{\mathcal{F}})$. This sets up the possibility of solving inverse problems in the space $(\mathcal{F}, d_{\mathcal{F}})$: given a Fourier transform $U(\omega)$, find a contractive operator M with fixed point $\overline{U}(\omega)$ which approximates $U(\omega)$ to a prescribed accuracy.

Finally, we mention some previous studies of relationships between integral transforms and fractal transforms that are relevant to this paper. Forte and Vrscay [6] showed that the Fourier transforms of invariant measures of Iterated Function Systems (IFSs) satisfy a self-similarity relation, in that they may be expressed as a linear combination of *frequency expanded* copies of themselves. In this paper, a similar kind of self-similarity relation will be shown to exist for Fourier transforms of measure-valued functions.

Giona and Patierno [8] examined some integral transforms (Laplace, Fourier and mixed Stieltjes) of multifractal measures of affine IFS and derived appropriate self-similarity relations satisfied by the transforms. Forte et al. [7] studied the general case of an integral transform, with kernel K, of a fractally transformed function Tf, relating it to the integral transform of f. A simplification results if K satisfies a general functional equation. It is also possible that the kernel K itself can satisfy an IFS-type equation.

More recently, Mayer and Vrscay [18,19] showed that a fractal transform operator T on real-valued *functions* induces a mapping M on the space of Fourier transforms of these functions – a kind of "scalarized version" of the results presented in this paper. Moreover, they showed that if T is contractive, then M is contractive, setting up a method for solving inverse problems in the Fourier domain for standard image functions. The solution of such inverse problems is relevant to magnetic resonance imaging where either (1) image compression or (2) image superresolution may be performed entirely in the frequency domain [18].

2. Measure-valued images and their Fourier transforms

In what follows, $X = [0, 1]$ will denote the "base" or "pixel space," i.e., the support of the images. (The extension to $X = [0, 1]^n$ is straightforward.) Let $\mathbb{R}_g \subset \mathbb{R}$ denote a compact "greyscale space" of values that can be assumed by an image at any $x \in X$ and $\mathbb{B}(\mathbb{R}_g)$ the σ-algebra of Borel subsets of $\mathbb{R}_g$. Furthermore, let $\mathcal{M}(\mathbb{R}_g)$ denote the set of all Borel probability measures on $\mathbb{R}_g$ and d_H the Monge–Kantorovich metric on this set. It is well known that under the hypothesis of compactness of $\mathbb{R}_g$, the space $\mathcal{M}(\mathbb{R}_g)$ is compact and therefore complete (see [9]). We define

$$Y = \{\mu : X \to \mathcal{M}(\mathbb{R}_g), \mu \text{ is measurable}\} \tag{1}$$

and consider on this space the following metric:

$$d_Y(\mu, \nu) = \int_X d_H(\mu(x), \nu(x))\, dx. \tag{2}$$

Notice that the compactness of $\mathbb{R}_g$ implies that d_H is bounded and then d_Y is well defined. It is possible to prove that the space (Y, d_Y) is compact and, therefore, complete (see also [10]). If we relax the compactness restriction on $\mathbb{R}_g$ and consider the case $\mathbb{R}_g = \mathbb{R}$, then the space (Y, d_Y) is no longer compact but it can still be proved that it is complete if we consider only those probability measures in $\mathcal{M}(\mathbb{R}_g)$ that satisfy a finite first moment condition (see [9] for more details on this).

The connection between this measure-valued formalism and classical (image) functions is easily established as follows. For a given image function $u : X \to \mathbb{R}_g$ we can always define the measure-valued image $\mu(x) = \delta_{u(x)}$ for all $x \in X$, where δ_a denotes the point mass measure at $a \in \mathbb{R}_g$.

We are now interested in defining the Fourier transform of a measure-valued image. First of all, let $\{q_i\}_{i=0}^{\infty}$ denote a complete set of orthonormal basis functions over the Hilbert space of square integrable functions $L^2(X)$. (This could be a wavelet expansion.) For any measure-valued function $\mu : X \to \mathcal{M}(\mathbb{R}_g)$ and any measurable A subset of $\mathbb{R}_g$, define the real-valued function $g_A(x) = \mu(x)(A)$. Since $0 \le g_A(x) \le 1$, $x \in X$ and X is bounded, it follows that $g_A \in L^2(X)$, implying that it admits an expansion of the form

$$g_A(x) = \sum_{i=0}^{\infty} c_i(A) q_i(x) \tag{3}$$

(in the L^2 sense), where

$$c_i(A) = \langle g_A, q_i \rangle = \int_X \mu(x)(A) q_i(x) dx, \quad i = 0, 1, 2, \ldots. \tag{4}$$

The Fourier coefficients $c_i(A)$, $i = 0, 1, \ldots$, may be viewed as (signed) measures on $\mathbb{R}_g$. In the case that $A = \mathbb{R}_g$

$$c_i = c_i(\mathbb{R}_g) = \int_X q_i(x)\, dx, \quad i = 0, 1, 2, \ldots. \tag{5}$$

In this paper, we shall employ formulas that are consistent with the following definition of the Fourier transform $U(\omega)$ for a function $u : X \to \mathbb{R}$:

$$U(\omega) = \int_X e^{-i\omega x} u(x)\, dx, \quad \omega \in \mathbb{R}. \tag{6}$$

The associated inverse Fourier transform is given by

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega x} U(\omega)\, d\omega, \quad x \in X. \tag{7}$$

Other definitions of the Fourier transform, of course, may be used. We define the Fourier transform of any measure-valued function $\mu(x) : X \to \mathcal{M}(\mathbb{R}_g)$ as follows. For any measurable subset $A \subset \mathbb{R}_g$, define

$$U(\omega, A) = \int_X e^{-i\omega x} \mu(x)(A)\, dx$$

$$= \int_X \cos(\omega x)\mu(x)(A)\,dx - i\int_X \sin(\omega x)\mu(x)(A)\,dx \qquad (8)$$

with $\omega \in \mathbb{R}$. Notice that for any $\omega \in \mathbb{R}$, $U(\omega,\cdot)$ is a (complex) vector-valued measure with total mass that is equal to $\int_X e^{-i\omega x}\,dx$. Furthermore, for any fixed Borel set $A \in \mathbb{B}(\mathbb{R}_g)$, the map $U(\cdot,A)$ is a smooth function.

Examples.

1. Let $X \subset \mathbb{R}$ and $\mathbb{R}_g = [0,1]$. For each $x \in X$, let $\mu(x)$ be the uniform probability measure on $\mathbb{R}_g$. Then for any interval $A = [a,b] \subseteq \mathbb{R}_g$, the Fourier transform $U(\omega,[a,b])$ is easily computed

$$U(\omega,[a,b]) = \int_X e^{-i\omega x}\mu(x)([a,b])\,dx = (b-a)\int_X e^{-i\omega x}\,dx. \qquad (9)$$

In the special case that $X = [-\pi,\pi]$

$$U(\omega,[a,b]) = (b-a)\frac{2\sin(\pi\omega)}{\omega} = (b-a)2\pi\,\mathrm{sinc}(\pi\omega), \qquad (10)$$

where

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \qquad (11)$$

The appearance of the sinc function arises from the "boxcar" nature of the measure-valued function $\mu(x)$.

2. Let $X = [0,1]$ and $\mathbb{R}_g = [0,1]$. For each $x \in X = [0,1]$, define $\mu(x)$ to be the uniform probability measure on the interval $[0,x]$ so that $\mu([0,x]) = 1$. The regions of support of $\mu(x)$ for $x \in [0,1]$ comprise the light-grey triangular region shown in Fig. 1. (Note that $\mu(0) = \delta(0)$, the Dirac unit mass at $t=0$.)
Now consider the interval $A = [a,b] \subset \mathbb{R}_g$, where $0 < a < b \le 1$. With reference to Fig. 1, it follows that

$$\mu(x)([a,b]) = \begin{cases} 0, & 0 < x < a, \\ \dfrac{x-a}{x}, & a \le x \le b \le 1, \\ \dfrac{b-a}{x}, & b < x \le 1. \end{cases} \qquad (12)$$

The Fourier transform $U(\omega,[a,b])$ may then be computed as follows:

$$U(\omega,[a,b]) = \int_X e^{-i\omega x}\mu(x)([a,b])\,dx = \int_a^b e^{-i\omega x}\left(\frac{x-a}{x}\right)dx + \int_b^1 e^{-i\omega x}\left(\frac{b-a}{x}\right)dx. \qquad (13)$$

3. Let $X \subset \mathbb{R}$, $\mathbb{R}_g = \mathbb{R}$ and define

$$p(x,t) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(t-u(x))^2}{2\sigma^2}\right), \qquad (14)$$

where $\sigma > 0$ and $u : X \to \mathbb{R}_g$ is a given function. Note that the assumption that $\mathbb{R}_g$ be compact has been relaxed in this example to include the case of a normal distribution. As mentioned earlier, completeness of the space (Y, d_Y) is guaranteed if we consider only those probability measures on $\mathbb{R}$ that satisfy a finite first moment condition.
For any $x \in X$

$$u(x) = \int_{\mathbb{R}} t p(x,t)\,dt, \qquad (15)$$

the expected value of the normal distribution over $\mathbb{R}_g$ at x. Now define the measure-valued mapping $\mu(x)$ as follows: for each $x \in X$ and any measurable subset $A \subset \mathbb{R}_g$

$$\mu(x)(A) = \int_A p(x,t)\,dt. \qquad (16)$$

(In other words, $p(x,\cdot)$ is the density function associated with $\mu(x)$.)
The Fourier transform $U(\omega,A)$ may be computed by straightforward integration

$$\begin{aligned} U(\omega,A) &= \textstyle\int_X e^{-i\omega x}\mu(x)(A)\,dx \\ &= \textstyle\int_X e^{-i\omega x}\left[\int_A p(x,t)\,dt\right]dx \\ &= \textstyle\int_A \psi(\omega,t)\,dt, \end{aligned} \qquad (17)$$

where

$$\psi(\omega,t) = \int_X e^{-i\omega x}p(x,t)\,dx \qquad (18)$$

and the reversal of the order of integration is permitted by Fubini's theorem. Note that $\psi(\omega,t)$ may be viewed in two ways:
- The Fourier transform of $p(x,t)$.
- The density function of the measure $U(\omega,A)$.

We now compute the standard Fourier transform of the (real-valued) function $u(x)$ as follows:

$$\begin{aligned} F(\omega) &= \int_X e^{-i\omega x}u(x)\,dx \\ &= \int_X e^{-i\omega x}\left[\int_{\mathbb{R}} t p(x,t)\,dt\right]dx \\ &= \int_{\mathbb{R}} t\left[\int_X e^{-i\omega x}p(x,t)\,dx\right]dt \quad \text{(Fubini)} \\ &= \int_{\mathbb{R}} t\psi(\omega,t)\,dt. \end{aligned} \qquad (19)$$

In other words, the Fourier transform $F(\omega)$ of $u(x)$ is the t-expected value of $\psi(\omega,t)$, the density function associated with the measure-valued function $\mu(x)$.
Note: Of course, this result applies to any continuous density function $p(x,t)$ with t-expected value $u(x)$ at each $x \in X$.

For the remainder of this paper, unless otherwise indicated, we assume that the greyscale space $\mathbb{R}_g$ is a compact subinterval of $\mathbb{R}$.

2.1. Some notes regarding practical applications

The *boundedness* of the base space X implies that it is sufficient to work with a countably infinite set of frequencies. For example, in the case $X = [-L,L]$, the set of frequencies, $\omega_n = n\pi/L$, $n \in \mathbf{Z}$, corresponds to the complete and orthogonal set of functions $\{u_n(x) = e^{i\omega_n x}\}_{n\in\mathbf{Z}}$ on X.

(These $u_n(x)$ yield the classical Fourier series functions $\cos(\omega_n x)$ and $\sin(\omega_n x)$ on X.) It is therefore sufficient in practice to consider the subset of transform values $U(\omega_n, A)$ for any Borel set $A \in \mathbb{B}(\mathbb{R}_g)$.

Furthermore, in digital signal and image processing applications, both the base space X and greyscale range $\mathbb{R}_g$ are *discretized.* The discreteness of X makes possible the use of the standard discrete Fourier transform (DFT) along the spatial direction.

In the following brief discussion, assume that X has been discretized into N "pixels", i.e., $X = \{0, 1, 2, \ldots, N-1\}$. The discretization of the measure $\mu(n)$ at $n \in X$ in the greyscale direction may be accomplished in a straightforward manner by partitioning $\mathbb{R}_g$ into a union of L non-overlapping subintervals $I_l \subset \mathbb{R}_g$, i.e., $\mathbb{R}_g = \bigcup_{l=1}^{L} I_l$, with $I_l \cap I_m = \emptyset$ for $l \neq m$. The vector measure $\nu(n) = (\nu_1(n), \nu_2(n), \ldots, \nu_L(n))$ is then defined as

$$\nu_l(n) = \mu(n)(I_l), \quad 1 \le l \le L. \tag{20}$$

The resulting "digitization," $\nu(n)$, of the measure-valued function $\mu(x)$ may be viewed in two complementary ways:

1. As a vector-valued measure supported on the pixel space X, i.e., $\nu(n) \in \mathbb{R}^L$ for $n \in X$.
2. As a union of L real-valued "channels" supported on X, i.e., $\nu_l(X)$, $1 \le l \le L$, which correspond to the L partitions I_m of the greyscale range $\mathbb{R}_g$.

We let $U(k)$ denote the vector-valued DFT of the vector-valued measure $\nu(n)$, where $k \in K = \{0, 1, \ldots N-1\}$, the "frequency space." For a given $k \in K$, the components of $U(k)$ in greyscale space, corresponding to the L channels described above, are given by

$$U(k,l) = \sum_{n=0}^{N=1} \mu(n,l) \exp\left(-\frac{i2\pi kn}{N}\right), \quad 1 \le l \le L. \tag{21}$$

The DFT $U(k)$ may be viewed in two ways:

1. As a complex vector-valued measure defined over K: At a fixed frequency $k \in K$, $U(k,l) \in \mathbb{C}^L$.
2. As a union of L complex-valued DFTs supported over frequency space K: For a fixed channel, $l \in \{1, 2, \ldots, L\}$, $U(k,l) \in \mathbb{C}$ for $k \in K$. Each of the L DFTs, $U(\cdot, l)$, corresponds to a channel in greyscale space.

2.2. A metric space of Fourier transforms of measure-valued functions

We now let $\mathcal{F}$ denote the space of all complex-valued functions $U : \mathbb{R} \times \mathbb{B}(\mathbb{R}_g) \to \mathbb{C}$, $U(\omega, A) = U_1(\omega, A) + iU_2(\omega, A)$, $U_1, U_2 \in \mathbb{R}$, such that the following are true:

- For any fixed $A \in \mathbb{B}(\mathbb{R}_g)$, $U(\cdot, A)$ is a continuous function on $\mathbb{R}$.
- For any fixed $\omega \in \mathbb{R}$, $U(\omega, \cdot)$ is a complex vector-valued measure with total mass $U(\omega, \mathbb{R}) = \int_X e^{-i\omega x}\, dx = \int_X \cos(\omega x)\, dx - i\int_X \sin(\omega x)\, dx$.

The Monge–Kantorovich distance d_H can be easily extended to complex vector-valued measures in a quite intuitive manner which we describe here very briefly. In [9] it is shown that the Monge–Kantorovich distance can be extended to vector-valued and set-valued generalized measures having fixed total mass. Since the complex-valued measure $U(\omega, \cdot)$ is a particular case of a vector-valued measure having fixed total mass, the definition of Monge–Kantorovich distance can be easily introduced by considering the real and the imaginary parts of U. If $U = U_1 + iU_2$ and $V = V_1 + iV_2$ are two elements of $\mathcal{F}$, and $\omega \in \mathbb{R}$ is fixed, then

$$d_H(U(\omega, \cdot), V(\omega, \cdot)) := \max\{d_H(U_1(\omega, \cdot), V_1(\omega, \cdot)), d(U_2(\omega, \cdot), V_2(\omega, \cdot))\} \tag{22}$$

This is a particular case of the d_H metric presented in [9] for vector-valued measures. Using these results, we define the following metric $d_{\mathcal{F}}$ on $\mathcal{F}$: given two elements $U, V \in \mathcal{F}$

$$d_{\mathcal{F}}(U,V) = \sup_{\omega \in \mathbb{R}} d_H(U(\omega, \cdot), V(\omega, \cdot)). \tag{23}$$

Here we simply state that the right hand side is finite even though the supremum is taken over an infinite set, i.e., $\mathbb{R}$. This follows from the compactness of $\mathbb{R}_g$ and the finite mass of the measures over it which, in turn, implies the boundedness of the distances $d_H(U(\omega, \cdot), V(\omega, \cdot))$. Standard arguments can now be used to prove that the space $(\mathcal{F}, d_{\mathcal{F}})$ is complete.

In summary, the complete metric space $(\mathcal{F}, d_{\mathcal{F}})$ contains the Fourier transforms of all elements $\mu \in Y$. In fact, as might be expected, we have the following result.

Theorem 1. *Let $\mathcal{F}_{FT} \subset \mathcal{F}$ denote the set of Fourier transforms of all elements $\mu \in Y$. The set $\mathcal{F}_{FT}$ is closed.*

Proof. Let $\{U_n\}$ be a convergent sequence of Fourier transforms of μ_n in the $d_{\mathcal{F}}$ metric and U the limit of this sequence. We need to prove that U is the Fourier transform of a certain $\mu \in Y$. Under the hypothesis of compactness of X and $\mathbb{R}_g$, the space Y is compact. Therefore there exists a subsequence of measure-valued mappings $\mu_{n_k} \in Y$ which converges to a limit $\mu \in Y$. Let V be the Fourier transform of μ. It is quite straightforward to show that

$$d_{\mathcal{F}}(U_{n_k}, V) \le d_Y(\mu_{n_k}, \mu) \tag{24}$$

which implies that the sequence $\{U_{n_k}\}$ converges to V. This implies that $V=U$, which concludes the proof. □

3. Fractal transforms on (Y, d_Y) and induced operators on the Fourier transform space $(\mathcal{F}, d_{\mathcal{F}})$

The action of a generalized fractal transform $T : Y \to Y$ on an element u of a complete metric space (Y, d_Y) can be summarized in the following steps [9]. It first produces a set of N spatially contracted copies of u. It then modifies the values of these copies by means of a suitable range-mapping. Finally, it recombines these modified copies by means of an appropriate operator to produce an element $v = Tu \in Y$. Under appropriate conditions, the fractal transform T is a contraction in (Y, d_Y) which, by Banach's Fixed Point Theorem, implies the existence of a unique fixed point $\bar{u} = T\bar{u}$.

A fractal transform operator on the space (Y, d_Y) of measure-valued functions was defined in [10]. (*Note*: In

[10], this fractal transform was denoted as M. In this paper it is denoted as T.) It represents a kind of blending of IFS-based methods on measures (IFSP) and functions (IFSM) (see [9] for more details). In what follows, we list the ingredients for such a fractal transform over the base space $X=[0,1]$. (Once again, the extension to $[0,1]^n$ is straightforward.)

1. A set of N one-to-one affine contraction maps $w_i : X \to X$, $w_i(x) = s_i x + a_i$, with the condition that $\bigcup_{i=1}^N w_i(X) = X$.
2. A set of N greyscale maps $\phi_i : \mathbb{R}_g \to \mathbb{R}_g$, assumed to be Lipschitz, i.e., for each i, there exists a $\alpha_i \geq 0$ such that

$$|\phi_i(t_1) - \phi_i(t_2)| \leq \alpha_i |t_1 - t_2|, \quad \forall t_1, t_2 \in \mathbb{R}_g. \tag{25}$$

3. For each $x \in X$, a set of probabilities $p_i(x)$, $i = 1, \ldots, N$, with the following properties:
   (a) $p_i(x)$ are measurable,
   (b) $p_i(x) = 0$ if $x \notin w_i(X)$,
   (c) $\sum_{i=1}^N p_i(x) = 1$ for all $x \in X$.

The action of the fractal transform operator $T : Y \to Y$ defined by the above is as follows: For a $\mu \in Y$ and any subset $A \subset \mathbb{R}_g$

$$\nu(x)(A) = (T\mu(x))(A) = \sum_{i=1}^{N} p_i(x)\mu(w_i^{-1}(x))(\phi_i^{-1}(A)). \tag{26}$$

Theorem 2 (*La Torre et al. [10]*). *Let $p_i = \sup_{x \in X} p_i(x)$. Then for $\mu_1, \mu_2 \in Y$*

$$d_Y(T\mu_1, T\mu_2) \leq \left(\sum_{i=1}^{N} c_i p_i \alpha_i \right) d_Y(\mu_1, \mu_2), \tag{27}$$

where $c_i = |s_i|$, $1 \leq i \leq N$.

Corollary 1 (*La Torre et al. [10]*). *Let $p_i = \sup_{x \in X} p_i(x)$. Then T is a contraction on (Y, d_Y) if*

$$C_T = \sum_{i=1}^{N} c_i p_i \alpha_i < 1. \tag{28}$$

In this case there exists a unique measure-valued mapping $\overline{\mu} \in Y$, such that $\overline{\mu} = T\overline{\mu}$.

Example 4. Let $X = [0,1]$ and $\mathbb{R}_g = [0,1]$ and consider the two-IFS-map fractal transform defined by the following set of maps:

$$w_1(x) = \tfrac{1}{2}x, \quad \phi_1(t) = \tfrac{1}{2}t, \tag{29}$$

$$w_2(x) = \tfrac{1}{2}x + \tfrac{1}{2}, \quad \phi_2(t) = \tfrac{1}{2}t + \tfrac{1}{2}. \tag{30}$$

The sets $w_1(X)$ and $w_2(X)$ overlap at the single point $x = \frac{1}{2}$ so we let

$$p_1(x) = 1, \quad p_2(x) = 0, \quad x \in \left[0, \tfrac{1}{2}\right),$$
$$p_1(x) = 0, \quad p_2(x) = 1, \quad x \in \left(\tfrac{1}{2}, 1\right],$$
$$p_1\left(\tfrac{1}{2}\right) = p_2\left(\tfrac{1}{2}\right) = \tfrac{1}{2},$$

here $c_1 = c_2 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $p_1 = p_2 = 1$ so that the Lipschitz factor of T is

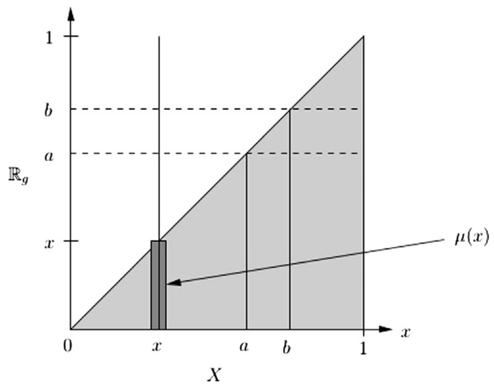

Fig. 1. Region of support of measures $\mu(x)$ from Example 2.

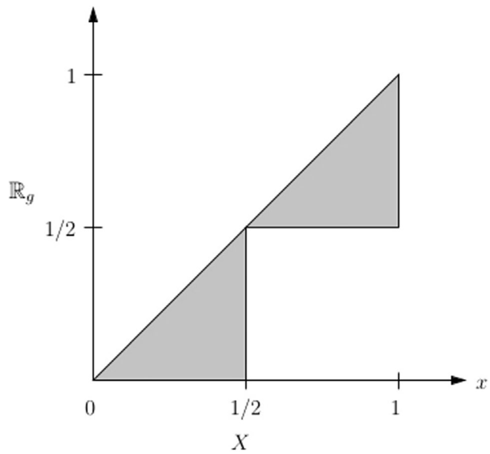

Fig. 2. Regions of support of the measure $\nu = T\mu$ of Example 4.

$$C_T = \sum_{i=1}^{2} c_i p_i \alpha_i = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \tag{31}$$

Therefore, T is contractive on (Y, d_Y).

Now consider the measure-valued function $\mu(x)$ of Example 2, the support of which is sketched in Fig. 1. The action of the operator T on μ is to produce two contracted and modified copies of it, one supported on $x \in [0, 1/2]$ and the other supported on $x \in [1/2, 1]$. The support of the measure $\nu = T\mu$ is sketched in Fig. 2. The measure-valued function $\nu(x)$ may be expressed as follows:

$$\nu(x) = \begin{cases} m_{[0,x]}, & 0 \leq x < 1/2, \\ m_{[x,1]}, & 1/2 < x \leq 1, \\ \frac{1}{2} m_{[0,x]} + \frac{1}{2}\delta_{1/2}, & x = 1/2, \end{cases} \tag{32}$$

here $m_{[a,b]}$ denotes uniform probability measure on $[a,b]$, i.e., $m_{[a,b]}([a,b]) = 1$, and δ_t denotes the Dirac unit mass measure at $t \in [0,1]$.

Now let $\nu_n = T^{\circ n}\mu$ for $n = 1, 2, \ldots$. The measures comprising $\nu_n(x)$ for $x \in [0,1]$ are supported on 2^n lower-right triangles the hypotenuses of which lie on the line $x = t$. From the contractivity of T, it follows that in the limit $n \to \infty$, ν_n must approach (in d_Y metric) the unique fixed point $\overline{\mu}$ of T which is given by

$$\overline{\mu}(x) = \delta_x, \quad x \in [0,1]. \tag{33}$$

In other words, $\mu(x)$ consists of the set of unit Dirac masses that lie on the line $t=x$.

Now let $T: Y \to Y$ denote a (not necessarily contractive) fractal transform operator with the ingredients listed above. Consider $\mu, \nu \in Y$ such that $\nu = T\mu$, as defined in Eq. (26). We now show how the Fourier transforms of μ and ν, to be denoted as $U(\omega)$ and $V(\omega)$, respectively, are related. For any subset $A \subset \mathbb{R}_g$

$$\begin{aligned} V(\omega, A) &= \int_X e^{-i\omega x} \nu_x(A)\, dx \\ &= \int_X e^{-i\omega x} (T\mu_x)(A)\, dx \\ &= \int_X e^{-i\omega x} \sum_{k=1}^N p_k(x) \mu_{w_k^{-1}(x)}(\phi_k^{-1}(A))\, dx \\ &= \sum_{k=1}^N \int_{X_k} e^{-i\omega x} p_k(x) \mu_{w_k^{-1}(x)}(\phi_k^{-1}(A))\, dx, \end{aligned} \tag{34}$$

where $X_k = w_k(X)$, $1 \le k \le N$. Now let $y = w_k^{-1}(x)$ so that $x = w_k(y) = s_k y + a_k$. Then

$$\begin{aligned} V(\omega, A) &= \sum_{k=1}^N c_k \int_X e^{-i\omega(s_k y + a_k)} p_k(s_k y + a_k) \mu_y(\phi_k^{-1}(A))\, dy \\ &= \sum_{k=1}^N c_k e^{-ia_k\omega} \int_X e^{-is_k\omega y} p_k(s_k y + a_k) \mu_y(\phi_k^{-1}(A))\, dy. \end{aligned} \tag{35}$$

3.1. The special case of constant probabilities

In the special case of constant probabilities, i.e., $p_k(x) = p_k$, Eq. (35) may be written in the form

$$V(\omega, A) = MU(\omega, A), \tag{36}$$

where

$$MU(\omega, A) = \sum_{k=1}^N c_k p_k e^{-ia_k\omega} U(s_k\omega, \phi_k^{-1}(A)). \tag{37}$$

The operator M may be viewed as a fractal transform operator on the metric space $(\mathcal{F}, d_{\mathcal{F}})$ of Fourier transforms introduced in the previous section. The action of M on a Fourier transform $U(\omega, A)$ is to produce a set of modified copies which are then multiplied by appropriate constants and complex phases and then added together.

It is noteworthy to mention that the copies $U(s_k\omega, \phi_k^{-1}(A))$ represent *expansions* of $U(\omega, A)$ in the frequency domain $\omega \in \mathbb{R}$, a consequence of the *contractions* produced by the affine IFS maps $w_k(x) = s_k x + a_k$ in the spatial domain $x \in X$ (Fourier scaling theorem). Furthermore, the multiplication by the phase factor $e^{-ia_k\omega}$ is a consequence of the translations produced by the affine IFS maps in X (Fourier shift theorem). These effects on Fourier transforms also result when fractal transforms are applied to *functions* [18,19].

In the special case, $A = \mathbb{R}_g$, Eq. (37) becomes

$$V(\omega, \mathbb{R}_g) = V(\omega) = \sum_{k=1}^N c_k p_k e^{-ia_k\omega} U(s_k\omega, \mathbb{R}_g). \tag{38}$$

Theorem 4. *For the fractal transform operator $M: \mathcal{F} \to \mathcal{F}$ defined in Eq. (37)*

$$d_{\mathcal{F}}(MU, MV) \le \left(\sum_{k=1}^N c_k p_k \alpha_k \right) d_{\mathcal{F}}(U, V). \tag{39}$$

Proof. From Eq. (37), for each $\omega \in \mathbb{R}$, we have that

$$\begin{aligned} & d_H(MU(\omega), MV(\omega)) \\ & \le d_H\left(\sum_{k=1}^N c_k p_k e^{-ia_k\omega} U(s_k\omega), \sum_{k=1}^N c_k p_k e^{-ia_k\omega} V(s_k\omega) \right) \\ & \le \left(\sum_{k=1}^N c_k p_k \alpha_k \right) d_H(U(s_k\omega), V(s_k\omega)). \end{aligned} \tag{40}$$

By taking the sup over all $\omega \in \mathbb{R}$ we obtain the desired result and the proof is complete. □

Now suppose that the N-map fractal transform T is contractive on (Y, d_Y), cf. Eq. (28), implying the existence of a unique fixed point measure-valued function $\overline{\mu} = T\overline{\mu}$. From Eq. (39), the associated operator M is contractive on $(\mathcal{F}, d_{\mathcal{F}})$. From Eqs. (37) and (38), the Fourier transform $\overline{U}(\omega)$ of $\overline{\mu}$ satisfies the following relations:

$$\overline{U}(\omega, A) = \sum_{k=1}^N c_k p_k e^{-ia_k\omega} \overline{U}(s_k\omega, \phi_k^{-1}(A)) \tag{41}$$

and

$$\overline{U}(\omega, \mathbb{R}_g) = \overline{U}(\omega) = \sum_{k=1}^N c_k p_k e^{-ia_k\omega} \overline{U}(s_k\omega). \tag{42}$$

In other words, $\overline{U}$ satisfies a self-similarity property: it may be expressed as a linear combination of modified copies of itself.

3.2. Constant probabilities, identity greyscale maps

We now consider the additional simplification in which the greyscale maps of Eq. (25) are identity maps, i.e., $\phi_i(t) = t$, $1 \le i \le N$. In this case, there is no “shuffling” of measure in the greyscale direction $\mathbb{R}_g$ since $A = \phi_i^{-1}(A)$ for all subsets $A \in \mathbb{R}_g$. Eq. (37) then becomes

$$MU(\omega, A) = \sum_{k=1}^N c_k p_k e^{-ia_k\omega} U(s_k\omega, A). \tag{43}$$

The Lipschitz factor of M is

$$K = \sum_{k=1}^N c_k p_k. \tag{44}$$

If M is contractive, i.e., $K < 1$, then from Eq. (41), the self-similarity relation satisfied by its fixed point Fourier transform $\overline{U}(\omega)$ becomes

$$\overline{U}(\omega, A) = \sum_{k=1}^N c_k p_k e^{-ia_k\omega} \overline{U}(s_k\omega, A). \tag{45}$$

3.3. More notes on practical applications

3.3.1. Discretized measures

In applications, i.e., digital images, the greyscale range is partitioned by a set of L nonoverlapping subintervals $I_l \subset \mathbb{R}_g$, as discussed in Section 2.1. The measure $\mu(x)$ and its associated Fourier transform $U(\omega)$ now become L-vectors, i.e.

$$\mu(x) = (\mu_1(x), \mu_2(x), \ldots, \mu_L(x)),$$
$$U(\omega) = (U_1(\omega), U_2(\omega), \ldots, U_L(\omega)), \tag{46}$$

and the action of the transform M is separated into channels, i.e.

$$MU_l(\omega) = \sum_{k=1}^{N} c_k p_k e^{-ia_k\omega} U_l(s_k\omega), \quad 1 \le l \le L. \tag{47}$$

Moreover, the self-similarity relation in Eq. (45) now decomposes into relations involving the components of $U(\omega)$, i.e.

$$\overline{U}_l(\omega) = \sum_{k=1}^{N} c_k p_k e^{-ia_k\omega} \overline{U}_l(s_k\omega), \quad 1 \le l \le L. \tag{48}$$

Each component $\overline{U}_k(\omega)$ may be expressed as a complex linear combination of frequency-expanded copies of that component. This decomposition simplifies the inverse problem to be discussed in the next section.

It is important to note, however, that a complete separation of the L components/channels of the Fourier transform $U(\omega)$ is not accomplished since the channels share a common set of fractal transform parameters c_k, p_k, s_k and a_k, $1 \le k \le N$.

3.3.2. Discretized "base" or "pixel space" X

As discussed briefly in Section 2.1, digital images are defined over a discretized base space, namely "pixel space," which leads to the use of the discrete Fourier transform (DFT). The DFT, $U(k)$, of the discrete measure-valued mapping, $\mu(n)$, is defined in Eq. (21). This naturally leads to the question whether a fractal transform T on the measure-valued mapping $\mu(n)$ in discrete space induces an operator on the $U(k)$ in analogy to the continuous case, i.e., the induced operator M in Eq. (37).

The answer to this question is "Yes," but with a few complications. In discrete pixel space, one normally employs a special class of contraction mappings, namely those which map an integer number n_1 of pixels to a smaller number $n_2 < n_1$ of pixels. (In one dimension, typically $n_1 = 2$ and $n_2 = 1$, corresponding to $c_k = \frac{1}{2}$ in Eq. (37).) As is well known in image processing, such mappings must be accompanied by appropriate "decimation" operations which map the n_1 greyscale values into n_2 values. The fractal transform T then modifies these n_2 greyscale values accordingly.

As a result, the expression for the discrete version of the induced operation in Eq. (37) is rather complicated in form and will not be discussed further in this paper.

3.4. Inverse problem for fractal transforms on $(\mathcal{F}, d_{\mathcal{F}})$

In this section, we outline a method of solving the inverse problem for Fourier transforms in the practical case of discretized measures, in which case the measure-valued functions become simple vector-valued mappings on the base space X.

The general problem of approximating elements of a complete metric space (Z, d_Z) by fixed points of contraction maps over Z may be summarized as follows [6]. Let $Con(Z)$ be a set of contraction maps $T : Z \to Z$. Given a target element $z \in Z$, we try to find a contractive map $T \in Con(Z)$ such that its fixed point $\overline{z}$ approximates z to a desired accuracy, i.e., $d_Z(\overline{z}, z)$ is sufficiently small.

In general, and particularly in the case of fractal transforms, such inverse problems are very difficult. An enormous simplification of the problem is possible from the following simple consequence of Banach's Fixed Point Theorem, known in the fractal coding literature as the *Collage Theorem* [2].

Theorem 5. *Let (Z, d_Z) be a complete metric space and $T : Z \to Z$ such that $d_Z(Tz_1, Tz_2) \le C_T d_Z(z_1, z_2)$ for all $z_1, z_2 \in Z$, where $C_T \in [0, 1)$. Then*

$$d_Z(z, \overline{z}) \le \frac{1}{1 - C_T} d_Z(z, Tz) \quad \text{for all } z \in Z. \tag{49}$$

In such methods of *collage coding* [9], one looks for a contraction $T \in Con(Z)$ that maps the target z as close as possible to itself, i.e., makes the *collage error* $d_Z(z, Tz)$ as small as possible, in an effort to make the approximation error $d_Z(z, \overline{z})$ small. Here, $Con(Z)$ is chosen to be a family of contractive transforms appropriate to the application. Most, if not all, fractal coding methods are based on the Collage Theorem.

In [10], the inverse problem for fractal transforms on the space of measure-valued functions (Y, d_Y) was addressed. Here, we consider the inverse problem for these functions, but in the Fourier domain, i.e., for the contractive operators M over the Fourier transform space $(\mathcal{F}, d_{\mathcal{F}})$: Given a target Fourier transform $U \in \mathcal{F}$, find an N-map fractal transform operator $M : \mathcal{F} \to \mathcal{F}$ with fixed point $\overline{U} \in \mathcal{F}$ that approximates U to an acceptable accuracy. A first step in the simplification of this problem is to adopt a collage coding strategy, i.e., to look for an operator M that minimizes the collage distance $d_{\mathcal{F}}(U, MU)$.

A few additional simplifications will also be employed in our solution of the inverse problem:

- Since we shall be working with discretized measures, hence vectors, the Monge–Kantorovich distance becomes a distance on $\mathbb{C}^L$. In practice, however, it is much easier to work with the Euclidean distance. In finite dimensions, of course, the MK and Euclidean distances are equivalent. That being said, a derivation of the relationship between Lipschitz factors (hence contractivity) in the two metrics is left for a future paper.
- We also assume that all component Fourier transforms $U(\omega, l)$, $1 \le l \le L$, are L^2 functions of the frequency ω. (This is not serious since the range of frequencies employed in practice is always finite.)
- We assume that the operators M are associated with constant probabilities and identity greyscale maps, as defined in Eq. (43).

- At each step, we work with a fixed number N of contractive maps. Increasing N will provide better approximations.
- For a given N, the affine maps w_i are also fixed. This is not a severe restriction since we can work with sets of contraction maps on X which contain maps of various degrees of refinement.

The inverse problem is then reduced to the determination of optimal probabilities p_i^* which are given by

$$\mathbf{p}^{*T} = (p_1^*, p_2^*, \ldots, p_N^*)^T = \arg\min_{\mathbf{p} \in \mathbb{R}^N} \Delta^2(\mathbf{p}), \tag{50}$$

where $\Delta^2(\mathbf{p})$ denotes the following squared collage distance:

$$\begin{aligned}\Delta^2(\mathbf{p}) &= \|U - MU\|^2_{L^2(\mathbb{R}_g)} \\ &= \int_{\mathbb{R}_g} \|U(\omega) - \sum_{k=1}^{N} c_k p_k e^{-i\alpha_k \omega} U(s_k \omega)\|^2_{\mathbb{C}^L}\, d\omega. \end{aligned}\tag{51}$$

$\Delta^2(\mathbf{p})$ is a quadratic form in the probabilities p_i, i.e.

$$\Delta^2(\mathbf{p}) = \|U\|^2 + \mathbf{p}^T \mathbf{A}\mathbf{p} + \mathbf{b}^T \mathbf{p}, \tag{52}$$

where $\mathbf{A}$ is a $N \times N$ symmetric real matrix and $\mathbf{b}$ a real N-vector. The minimization of Δ^2 may be accomplished with quadratic programming algorithms.

The practical implementation of this formalism in a discrete, i.e., digital, setting will be the subject of a future paper.

Acknowledgements

We thank the reviewer for reading our original manuscript thoroughly and for the many valuable comments and questions which, in our opinion, led to a much improved version. This work was supported in part by a Discovery Grant (ERV) from the Natural Sciences and Engineering Research Council of Canada (NSERC) which is greatly appreciated.

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