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Random measure-valued image functions, fractal transforms and self-similarity

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ABSTRACT

We construct a complete metric space (Y, d_Y) of random measure-valued image functions. This formalism is an extension of previous work on measure-valued image functions.

1. Introduction

In this paper, we construct a complete metric space of random measure-valued functions, that is $\mu : \Omega \times X \to \mathcal{M}(\mathbb{R}_g)$, where X is the compact *base* or "pixel" space, $(\Omega, \mathcal{F}, P)$ is the underlying probability space and $\mathcal{M}(\mathbb{R}_g)$ is the set of probability measures supported on the greyscale range $\mathbb{R}_g$. This is a generalization of our formulation of *measure-valued functions*, or *place-dependent measures*, in [1]. One of the primary motivations for these constructions is signal/image processing. As we discussed in [1], there are situations in which it is useful to consider the greyscale value of an image u at a point x as a random variable that can assume a range of values $\mathbb{R}_g \subset \mathbb{R}$. For example, in diffusion MRI [2] one seeks to determine the probability of diffusion in various directions from a location/pixel $x \in X$. The measure-valued formalism is a natural venue for this problem, as we shall show elsewhere.

In many cases, large databases of images, e.g., medical, satellite, are composed of images that exhibit a significant degree of "set redundancy", [3], that is, similar pixel intensities in the same areas, comparable histograms, similar edge distributions and analogous distributions of features. In these cases, such redundancies can be exploited for the purpose of lossless compression [4]. A measure-valued formalism could also provide a natural mathematical background for such problems.

There are, however, complications that arise from greater variability, hence less redundancy, in image databases. An example is the accumulation of satellite images of similar geographical locations taken not only at different times of the day as the satellite orbits the Earth, but also on different days and at different times of the year. The different weather conditions throughout the year may cause variability that significantly reduces the similarity/redundancy of the images (e.g., existence of snow). It may be advantageous to extend the formalism of measure-valued images to random measure-valued images to address these problems. Indeed, we consider the complete metric space of random measure-valued functions constructed in this to be *a natural space in which all possible images can exist*.

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In [1], we constructed a class of affine fractal transform operators on measure-valued functions and showed how they can be useful in characterizing the affine self-similarity of an image, i.e., how well subblocks of an image are approximated by other subblocks under affine greyscale transformations. The methods of *nonlocal means denoising* [5] and *fractal image coding* [6,7] may, in fact, be considered as special cases of a more general model of affine self-similarity. In this paper, we define an appropriate class of affine fractal transform operators on random measure-valued functions. Under suitable conditions on the parameters that define these transforms, the operator T is contractive on (Y, d_Y), implying the existence of a unique fixed point random measure $\bar{\mu} = T\bar{\mu}$. This fixed point relation implies that $\bar{\mu}$ is self-similar.

2. Random measure-valued functions

In what follows, (X, d) denotes a compact metric "base space", typically $[0, 1]^n$, $(\Omega, \mathcal{F}, P)$ a probability space and $\mathcal{M}(\mathbb{R}_g)$ the set of probability measures supported on the greyscale range $\mathbb{R}_g$, a compact subset of $\mathbb{R}$. It is well known that $\mathcal{M}(\mathbb{R}_g)$ is a complete metric space with respect to the Monge–Kantorovich metric [8] defined as follows,

$$d_H(\mu, \nu) = \sup_{f \in \mathrm{Lip}_1(\mathbb{R}_g, \mathbb{R})} \left[\int_{\mathbb{R}_g} f(x) \mathrm{d}\mu - \int_{\mathbb{R}_g} f(x) \mathrm{d}\nu \right], \tag{1}$$

where

$$\mathrm{Lip}_1(\mathbb{R}_g, \mathbb{R}) = \{f : \mathbb{R}_g \to \mathbb{R} \mid |f(x_1) - f(x_2)| \le d(x_1, x_2), \forall x_1, x_2 \in \mathbb{R}_g\}. \tag{2}$$

A *random measure* is a random variable $\mu : \Omega \to \mathcal{M}(\mathbb{R}_g)$. Let us now define $\overline{\mathcal{M}}(\mathbb{R}_g)$ to be the space of all possible random measures equipped with the metric $d_{\overline{\mathcal{M}}}(\nu_1, \nu_2) = \mathbb{E}(d_H(\nu_1(\cdot), \nu_2(\cdot)))$.

Theorem 1. *$(\overline{\mathcal{M}}(\mathbb{R}_g), d_{\overline{\mathcal{M}}})$ is a complete metric space.*

Proof. It is trivial to prove that $d_{\overline{\mathcal{M}}}$ is a metric when we consider that $\mu = \nu$ if $\mu(\omega) = \nu(\omega)$ a.e. $\omega \in \Omega$. To prove the completeness (see [9] and [10] and the references therein), let μ_n be a Cauchy sequence in $\overline{\mathcal{M}}(\mathbb{R}_g)$. So for all $\epsilon > 0$ there exists n_0 such that for all $n, m \ge n_0$ we have $d_{\overline{\mathcal{M}}}(\mu_n, \mu_m) < \epsilon$. Let $\epsilon = 3^{-k}$ and select an increasing sequence n_k such that $d_{\overline{\mathcal{M}}}(\mu_n, \mu_{n_k}) < 3^{-k}$ for all $n \ge n_k$. Now let $n = n_{k+1}$ so that $d_{\overline{\mathcal{M}}}(\mu_{n_{k+1}}, \mu_{n_k}) < 3^{-k}$ and define $\Omega_k = \{\omega \in \Omega : d_H(\mu_{n_{k+1}}(\omega), \mu_{n_k}(\omega)) > 2^{-k}\}$. Then

$$P(\Omega_k)2^{-k} \le \int_{\Omega_k} d_H(\mu_{n_{k+1}}(\omega), \mu_{n_k}(\omega))\mathrm{d}P(\omega) \le 3^{-k} \tag{3}$$

so that $P(\Omega_k) \le \left(\frac{2}{3}\right)^k$. Let $\bar{\Omega} = \bigcap_{m=1}^{\infty} \bigcup_{k \ge m} \Omega_k$. We observe that $P\left(\bigcup_{k \ge m} \Omega_k\right) \le \sum_{k \ge m} P(\Omega_k) \le \sum_{k \ge m} \left(\frac{2}{3}\right)^k = \frac{\left(\frac{2}{3}\right)^m}{1 - \left(\frac{2}{3}\right)}$. Therefore $P(\bar{\Omega}) \le 3\left(\frac{2}{3}\right)^m$ for all m, which implies that $P(\bar{\Omega}) = 0$. Now for all $\omega \in \Omega \setminus \bar{\Omega}$ there exists $m_0(\omega)$ such that for all $m \ge m_0$ we have $\omega \notin \Omega_m$ and so $d_H(\mu_{n_{m+1}}(\omega), \mu_{n_m}(\omega)) < 2^{-m}$. This implies that $\mu_{n_m}(\omega)$ is Cauchy for all $\omega \in \Omega \setminus \bar{\Omega}$ and so $\mu_{n_m}(\omega) \to \mu(\omega)$ using the completeness of $\mathcal{M}(\mathbb{R}_g)$. This also implies that $\mu : \Omega \to \mathcal{M}(\mathbb{R}_g)$ is measurable, that is $\mu \in \overline{\mathcal{M}}$. To prove $\mu_n \to \mu$ in $\overline{\mathcal{M}}$ we have that

$$d_{\overline{\mathcal{M}}}(\nu_1, \nu_2)(\mu_{n_k}, \mu) = \int_{\Omega} d_H(\mu_{n_k}(\omega), \mu(\omega))\mathrm{d}P(\omega) \le \liminf_{i \to +\infty} d_{\overline{\mathcal{M}}}(\mu_{n_k}, \mu_{n_i}) \le 3^{-k} \tag{4}$$

for all k. Then $\lim_{k \to +\infty} d_{\overline{\mathcal{M}}}(\mu_{n_k}, \mu) = 0$. Now we have $d_{\overline{\mathcal{M}}}(\mu_n, \mu) \le d_{\overline{\mathcal{M}}}(\mu_n, \mu_{n_k}) + d_{\overline{\mathcal{M}}}(\mu_{n_k}, \mu) \to 0$ when $k \to +\infty$. □

We now define a *random measure-valued function* as a mapping $u : X \to \overline{\mathcal{M}}(\mathbb{R}_g)$. This represents an extension of our previous work [1] by introducing randomness as well as the correlation between pixels. The space of random measure-valued functions is to be denoted as $Y = \{\mu : X \to \overline{\mathcal{M}}(\mathbb{R}_g)\}$. On this space we consider the following metric,

$$d_Y(\mu_1, \mu_2) = \int_X d_{\overline{\mathcal{M}}}(\mu(x), \nu(x))\mathrm{d}\mu_L \tag{5}$$

where μ_L denotes the Lebesgue measure on X. We observe that d_Y is well defined; by using the same arguments we developed in the proof of Theorem 1, it is easy to prove that (Y, d_Y) is a complete metric space.

Example. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \mathbb{R}^n \times \Omega \rightrightarrows \mathbb{R}^p$ be a place-dependent closed-valued random set. Consider, for all $x \in \mathbb{R}^n$, the map $\hat{\mu}(x, \omega, K) = \mu_L(X(x, \omega) \cap K)$, where $K \subset \mathbb{R}^p$ and μ_L is the Lebesgue measure on $\mathbb{R}^p$. In the literature, $\hat{\mu}$ is sometimes referred to as the "hitting functional". Then $\hat{\mu} : \mathbb{R}^n \to \overline{\mathcal{M}}(\mathbb{R}^p)$, that is, $\hat{\mu}$ is a random measure-valued function.

3. A class of affine fractal transform operators M on (Y, d_Y)

As discussed in [11], a *generalized fractal transform* T over a complete metric space (Z, d_Z) produces several spatially contracted and range-modified copies of an element $z \in Z$ and then combines them (in a manner appropriate for the space Z) to produce a new element $Tz \in Z$. The ingredients for such a fractal transform operator over our space of random measure-valued functions are quite similar to those employed for measure-valued functions [1]. For simplicity, we assume that $X = [0, 1]$: the extension to $[0, 1]^n$ is straightforward. The ingredients for our fractal transform operator are as follows,

1. A set of N one-to-one contraction affine maps $w_i : X \to X$, $w_i(x) = s_i x + a_i$, with the condition that $\cup_{i=1}^N w_i(X) = X$,
2. A set of N greyscale maps $\phi_i : [0, 1] \to [0, 1]$, assumed to be Lipschitz, i.e., for each i, there exists an $\alpha_i \geq 0$ such that
$$|\phi_i(t_1) - \phi_i(t_2)| \leq \alpha_i |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, 1], \tag{6}$$
3. For each $x \in X$ and a.e. $\omega \in \Omega$, a set of random probabilities $p_i(x, \omega) \in [0, 1]$, $i = 1, \ldots, N$ with the following properties:
   (a) $p_i(x, \cdot)$ are measurable for all $x \in X$ and $p_i(\cdot, \omega)$ are continuous for a.e. $\omega \in \Omega$,
   (b) $p_i(x, \omega) = 0$ if $x \notin w_i(X)$ and a.e. $\omega \in \Omega$
   (c) $\sum_i^N p_i(x, \omega) = 1$ for all $x \in X$ and a.e. $\omega \in \Omega$.

The "randomization" of the probabilities represents the significant difference between this fractal transform and those constructed in [1]. The action of the fractal transform operator $M : Y \to Y$ defined by the above ingredients is defined as follows: for a $\mu \in Y$ and any subset $S \subset [0, 1]$,

$$(M\mu)(x, \omega)(S) = \nu(x, \omega)(S) = \sum_{i=1}^{N} p_i(x, \omega)\mu(w_i^{-1}(x), \omega)(\phi_i^{-1}(S)). \tag{7}$$

Theorem 2. *Let $p_i = \sup_{x \in X} \sup_{\omega \in \Omega} p_i(x, \omega)$. Then for $\mu_1, \mu_2 \in Y$,*

$$d_Y(M\mu_1, M\mu_2) \leq C d_Y(\mu_1, \mu_2), \quad \text{where } C = \sum_{i=1}^{n} |s_i| \alpha_i p_i. \tag{8}$$

Proof. From straightforward computation, we have

$$\begin{aligned} d_Y(M\mu_1, M\mu_2) &= \int_X d_{\overline{\mathcal{M}}}(M\mu_1(x), M\mu_2(x)) \mathrm{d}\mu_L(x) \\ &\leq \int_\Omega \int_X \sum_{i=1}^{n} \alpha_i p_i(x, \omega) d_H(\mu_1(w_i^{-1}(x), \omega), \mu_2(w_i^{-1}(x), \omega)) \mathrm{d}\mu_L(x) \mathrm{d}P(\omega) \\ &\leq \int_\Omega \int_X \left(\sum_{i=1}^{n} |s_i| \alpha_i p_i(x, \omega) \right) d_H(\mu_1(x, \omega), \mu_2(x, \omega)) \mathrm{d}\mu_L \mathrm{d}P(\omega) \\ &= \left(\sum_{i=1}^{n} |s_i| \alpha_i p_i \right) d_Y(\mu_1, \mu_2). \quad \square \end{aligned}$$

Corollary 1. *Let $0 \leq C < 1$ in the above theorem. Then M is a contraction on (Y, d_Y). Consequently, from Banach's Theorem, there exists a measure-valued mapping $\bar{\mu} \in Y$, such that $\bar{\mu} = M\bar{\mu}$. Moreover, for any $\mu_0 \in Y$, the sequence of random measures $\{\mu_n\}$ defined by $\mu_{n+1} = T\mu_n$ converges to $\bar{\mu}$ in metric d_Y.*

From Eq. (7), the fixed point equation $\bar{\mu} = M\bar{\mu}$ becomes

$$\bar{\mu}(x, \omega)(S) = \sum_{i=1}^{N} p_i(x, \omega)\bar{\mu}(w_i^{-1}(x), \omega)(\phi_i^{-1}(S)), \quad S \subset X. \tag{9}$$

This may be viewed as a *self-similarity* property of the random measured function $\bar{\mu}$.

Examples.

1. For purposes of comparison, we first consider the fractal transform M defined by the following two-IFS-map system on $X = [0, 1]$:
$$\begin{aligned} w_1(x) &= \frac{1}{2}x, \qquad \phi_1(t) = \frac{1}{2}t, \\ w_2(x) &= \frac{1}{2}x + \frac{1}{2}, \qquad \phi_2(t) = \frac{1}{2}t + \frac{1}{2}. \end{aligned}$$

Note that the sets $w_1(X)$ and $w_2(X)$ overlap at the single point $x = \frac{1}{2}$. We now let

$$p_1(x) = 1, \qquad p_2(x) = 0 \quad x \in \left[0, \frac{1}{2}\right),$$
$$p_1(x) = 0, \qquad p_2(x) = 1 \quad x \in \left(\frac{1}{2}, 1\right],$$
$$p_1\left(\frac{1}{2}\right) = p_2\left(\frac{1}{2}\right) = \frac{1}{2}.$$

There is essentially no "randomness" in the p_i since their associated probability distributions are Dirac measures. In this case, the fixed point $\bar{\mu}$ is the *measure-valued function* given by

$$\bar{\mu}(x) = \delta(t - x), \quad x \in [0, 1], \tag{10}$$

where $\delta(s)$ denotes the "Dirac delta function" at $s \in [0, 1]$. Clearly, its support in xt-space is the line $x = t$.

2. We now "perturb" the above fractal transform M in 1, by adding the following IFS and associated greyscale map,

$$w_3(x) = \frac{1}{2}x, \qquad \phi_3(t) = \frac{1}{2}t + 0.1.$$

The sets $w_1(X)$ and $w_3(X)$ overlap over the entire subinterval $\left[0, \frac{1}{2}\right]$ so we define

$$p_1(x) = \frac{1}{4}, \qquad p_3(x) = \frac{3}{4}, \qquad p_2(x) = 0 \quad x \in \left[0, \frac{1}{2}\right),$$
$$p_1(x) = p_3(x) = 0, \qquad p_2(x) = 1 \quad x \in \left(\frac{1}{2}, 1\right],$$
$$p_1\left(\frac{1}{2}\right) = p_2\left(\frac{1}{2}\right) = p_3\left(\frac{1}{2}\right) = \frac{1}{3}.$$

The fixed point $\bar{\mu}$ of this transform is a (nonrandom) measure-valued function. A "bird's-eye" view of this measure is sketched in Fig. 1(a), with the x-axis lying horizontally and the $\mathcal{R}_g$-axis lying vertically. The darkness of a point is proportional to the measure at that point. In xt-space, the support of this function is seen to be a Sierpinski-like gasket. A much better picture of the behavior of this measure-valued function is provided by the histogram approximation in Fig. 1(b). (100 points in each of the x and t directions were used to construct this histogram.) The height of each histogram bar is proportional to the $\bar{\mu}$-measure of the corresponding 0.01×0.01 regions in (x, t)-space on which it is located.

The fact that $w_1(X)$ and $w_3(X)$ overlap over the set $\left[0, \frac{1}{2}\right]$ is responsible for the self-similar "splitting" of the measures $\bar{\mu}(x)$ in the $\mathcal{R}_g$ (vertical) direction over this x-interval, since the map $\phi_3(t)$ produces a shift in the positive greyscale direction. And since $w_2(x)$ maps the support $[0, 1]$ of the entire measure-valued function onto $\left[\frac{1}{2}, 1\right]$, the self-similarity of the measure over $\left[0, \frac{1}{2}\right]$ is carried over (horizontally) to $\left[\frac{1}{2}, 1\right]$. And since $p_1(x) < p_3(x)$ for $x \in \left[0, \frac{2}{3}\right]$, there is a self-similar upward "slanting" of the measure in the positive $\mathcal{R}_g$ (vertical) direction.

3. We now randomize the fractal transform M in 2. as follows,

$$p_1(x) = \frac{1}{4} + \epsilon, \qquad p_3(x) = \frac{3}{4} - \epsilon, \qquad p_2(x) = 0 \quad x \in \left[0, \frac{1}{2}\right),$$
$$p_1(x) = p_3(x) = 0, \qquad p_2(x) = 1 \quad x \in \left(\frac{1}{2}, 1\right],$$
$$p_1\left(\frac{1}{2}\right) = p_2\left(\frac{1}{2}\right) = p_3\left(\frac{1}{2}\right) = \frac{1}{3}.$$

Here, ϵ is a random real-valued variable. In this case, the fixed point $\bar{\mu}$ of this transform is a random measure-valued function. Samples of $\bar{\mu}$ are sketched in Fig. 2 for the following cases:

Fig. 2(a): $\epsilon \in \left[-\frac{1}{8}, \frac{1}{8}\right]$ uniformly distributed,

Fig. 2(b): $\epsilon \in \left[-\frac{1}{4}, \frac{1}{4}\right]$ uniformly distributed.

The particular sample in Fig. 2(a) appears a little less "slanted" vertically at various x-values because there are opportunities for the probabilities $p_1(x)$ and $p_2(x)$ to come closer together at points. On the other hand, there are some vertical slits in Fig. 2(b), due to the fact that the probability $p_1(x)$ can assume values that are arbitrarily close to 0. This implies that the values of $\mu(x)$ can be small at these regions.

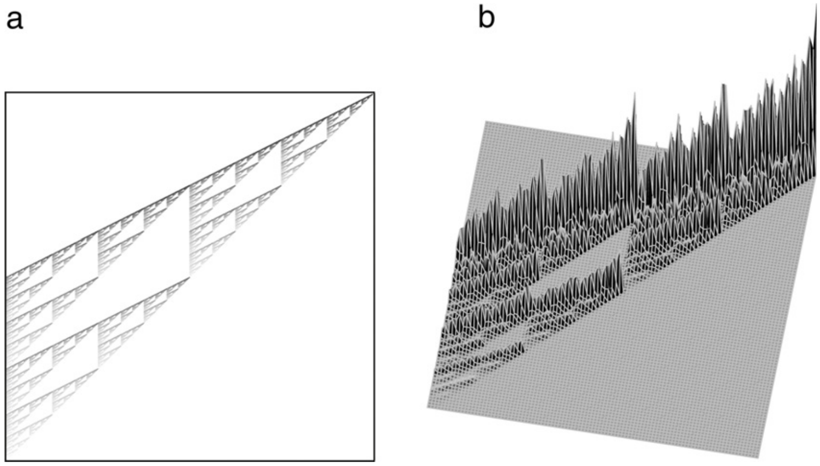

Fig. 1. (a) A top view of the measure-valued function $\bar{\mu}(x)$ of example 2 in xt-space, where $x = [0, 1]$ is horizontal axis and $t = [0, 1]$ is the vertical axis. The darkness at a point is directly proportional to the measure of the region it represents. The Sierpinski-gasket nature of the support of this measure in xt-space is quite evident. (b) A histogram approximation of $\bar{\mu}(x)$ for $x \in [0, 1]$ which provides a better picture of its behavior.

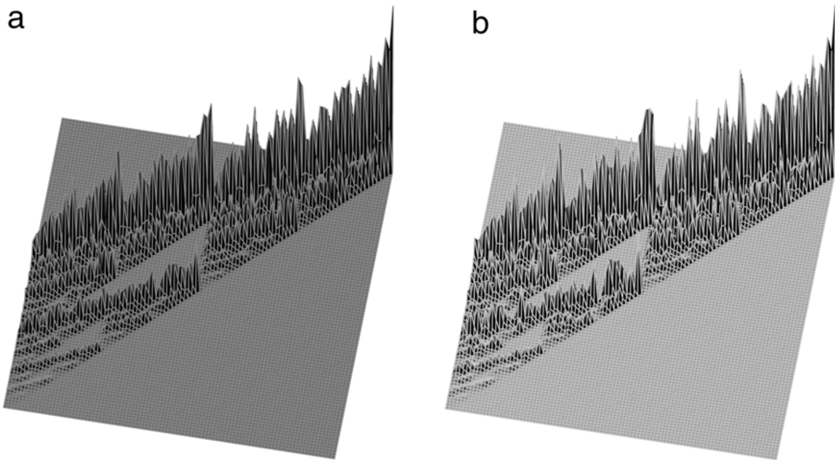

Fig. 2. Histogram approximations of particular realizations of the random measure-valued function $\bar{\mu}(x)$ of example 3, as described in the text.

4. Concluding remarks

In this paper we have constructed an appropriate complete metric space (Y, d_Y) of random measure-valued images $\mu : [0, 1]^n \to \overline{\mathcal{M}(\mathbb{R}_g)}$. A method of (random) fractal transforms was also formulated over this space. Under suitable conditions, the fractal transform operator $M : Y \to Y$ is contractive, implying the existence of a unique fixed point measure $\bar{\mu} = M\bar{\mu}$.

The fractal-based formalism makes it possible to consider the *inverse problem of approximation by random measures*: given a random measure $\nu \in Y$, find a fractal transform M with fixed point $\bar{\mu}$ such that the distance $d_Y(\nu, \bar{\mu})$ is sufficiently small. This problem is simplified enormously by a simple consequence of Banach's Theorem, known in the fractal coding literature as the *Collage Theorem* [6]. By making the *collage distance* small (and controlling c), we can make the approximation error $d_Y(y, \bar{y})$ small. This is the basis of most, if not all, methods of fractal image coding [7]. The problem of approximation by random measures is beyond the scope of this paper and will be considered elsewhere.

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