# Iterated function systems on functions of bounded variation

D. La Torre*, F. Mendivil† and E.R. Vrscay‡

February 4, 2016

**Abstract**

We show that under certain hypotheses, an Iterated Function System on Mappings (IFSM) is a contraction on the complete space of functions of bounded variation (BV). It then possesses a unique attractor of bounded variation. Some BV-based inverse problems based on the Collage Theorem for contraction maps are considered.

## 1 Introduction

Iterated Function Systems on Mappings (IFSM) extend the classical notion of Iterated Function Systems (IFS) to the case of space of functions [6] and can be used to generate integrable "fractal" functions. The purpose of this short note is to show that, under some hypotheses, an IFSM operator is a contraction with respect to the usual norm introduced into the space of functions of bounded variation.

**Definition 1.1** The *total variation* of a function $f : [a, b] \to \mathbb{R}$ is defined as

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{j=0}^{n} |f(x_{j+1}) - f(x_j)| , \tag{1.1}$$

where the supremum is taken over the set of all partitions of $[a, b]$,

$$\mathcal{P} = \{P = \{x_0, \ldots, x_n\} | P \text{ is a partition of } [a, b], x_0 = a, x_n = b\} .$$

*Department of Economics, Management, and Quantitative Methods, University of Milan, Italy, and Department of Applied Mathematics and Science, Khalifa University, Abu Dhabi, UAE. Email: davide.latorre@unimi.it

†Department of Mathematics and Statistics, Acadia University, Wolfville, Nova Scotia, Canada. Email: franklin.mendivil@acadiau.ca

‡Department of Applied Mathematics, University of Waterloo, Ontario, Canada. Email: ervrscay@uwaterloo.ca

If $f : [a,b] \to \mathbb{R}$ is differentiable and its derivative is Riemann-integrable, its total variation is given by

$$V_a^b(f) = \int_a^b |f'(x)| \, \mathrm{d}x \,. \tag{1.2}$$

**Definition 1.2** A real-valued function $f : [a,b] \to \mathbb{R}$ is said to be of *bounded variation* (or a "BV function") on $[a,b]$ if its total variation is finite, i.e. $V_a^b(f) < +\infty$. Let us denote by $BV([a,b])$ the space of functions of bounded variation on $[a,b]$.

Here are a few properties of BV functions:

- If $f \in BV([a,b])$, then $f$ is continuous at all $x \in [a,b]$ except at most on a countable set.
- If $f \in BV([a,b])$, then $f'(x)$ exists at a.e. $x \in [a,b]$.
- If $f$ is differentiable and has bounded derivative on $[a,b]$, then $f \in BV([a,b])$.
- A function with unbounded derivative could also be in $BV([a,b])$: For example $f(x) = \sqrt{x} \in BV([0,1])$ because it is monotone.
- More generally, a bounded function with finitely many maxima and minima on $[a,b]$ is in $BV([a,b])$.

**Theorem 1.3** [7] *The functional $f \to |f(a)| + V_a^b f$ is a norm over $BV([a,b])$. We shall denote this norm as $\|f\|_{BV}$. The normed space $(BV([a,b]), \|f\|_{BV})$ is complete.*

An $N$-map *iterated function system* (IFS), $\mathbf{w} = \{w_1, \cdots, w_N\}$, is a finite collection of contraction mappings $w_i : [a,b] \to [a,b]$, $i = 1, \cdots, N$. (See[5;1;6].) Associated with an $N$-map IFS is the following set-valued mapping $\hat{\mathbf{w}}$ on the space $\mathcal{H}([a,b])$ of nonempty compact subsets of $[a,b]$:

$$\hat{\mathbf{w}}(S) := \bigcup_{i=1}^{N} w_i(S) \,, \quad S \in \mathcal{H}([a,b]) \,. \tag{1.3}$$

**Theorem 1.4** [5] *For $A, B \in \mathcal{H}([a,b])$,*

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \leq cH(A,B) \quad \text{where} \;\; c = \max_{1 \leq i \leq N} c_i < 1 \tag{1.4}$$

*and $h$ denotes the Hausdorff metric on $\mathcal{H}([a,b])$.*

**Corollary 1.5** *There exists a unique set $A \in \mathcal{H}([a,b])$, the* attractor *of the IFS $\mathbf{w}$, such that*

$$A = \hat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} w_i(A). \tag{1.5}$$

*Moreover, for any $B \in \mathcal{H}([a,b])$, $h(A, \hat{\mathbf{w}}^n B) \to 0$ as $n \to \infty$.*

An *iterated function system with mappings* (IFSM) is an $N$-map IFS $\mathbf{w} = \{w_i\}$ on $[a,b]$ with an associated set of Lipschitz functions $\Phi = \{\phi_i\}$, where $\phi_i : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $K_i \geq 0$, $i = 1, \cdots, N$. In addition, the following covering condition on the $w_i$ is assumed:

$$[a,b] = \bigcup_{i=1}^{N} w_i([a,b])\,. \tag{1.6}$$

Associated with the IFSM $(\mathbf{w}, \Phi)$ is a so-called *IFSM operator* or *fractal transform* on the space of $L^p$ integrable functions on $[a,b]$ via the action

$$Tf(x) = \sum_{i=1}^{N} \phi_i(f(w_i^{-1}(x)))\,, \tag{1.7}$$

where the sum operates on all those terms for which $w_i^{-1}(x)$ is defined. (The covering condition in (1.6) guarantees that for each $x \in [a,b]$, there exists at least one term in the summation.)

**Theorem 1.6** [3;6] *For $p \geq 1$ and $u, v \in L^p[a,b]$,*

$$\|Tu - Tv\|_p \leq \left[\sum_{i=1}^{N} c_i^{1/p} K_i\right] \|u - v\|_p\,. \tag{1.8}$$

*In the special case that the following* open set condition *is satisfied,*

$$w_i((a,b)) \cap w_j((a,b)) = \emptyset \quad \text{if} \;\; i \neq j\,, \tag{1.9}$$

*the above estimate can be improved:*

$$\|Tu - Tv\|_p \leq \left[\sum_{i=1}^{N} c_i K_i^p\right]^{1/p} \|u - v\|_p\,. \tag{1.10}$$

**Corollary 1.7** *If*

$$C_p = \sum_{i=1}^{N} c_i^{1/p} K_i < 1\,, \tag{1.11}$$

*then the IFSM operator $T$ is contractive in $L^p[a,b]$ which, from Banach's Fixed Point Theorem*[7]*, implies the existence of a unique $\bar{u} \in L^p[a,b]$ such that $T\bar{u} = \bar{u}$. If the open set condition in (1.9) is satisfied, then $T$ is contractive in $L^p[a,b]$, etc., if*

$$\sum_{i=1}^{N} c_i K_i^p < 1\,. \tag{1.12}$$

The above results can be extended to the case $p = \infty$, i.e., for $u, v \in L^\infty([a,b])$,

$$\|Tu - Tv\|_\infty \leq \left[\sum_{i=1}^{N} K_i\right] \|u - v\|_\infty . \tag{1.13}$$

If the open set condition in (1.9) is satisfied, then

$$\|Tu - Tv\|_\infty \leq \left[\max_{1\leq i\leq N} K_i\right] \|u - v\|_\infty . \tag{1.14}$$

Note that the contractivity factors $c_i$ of the IFS maps do not appear in the Lipschitz constants of the IFSM operator $T$ in the case $p = \infty$.

# 2 IFSM on $BV([a,b])$

For the remainder of this paper, we assume that the IFS maps $w_i : [a,b] \to [a,b]$ and associated maps $\phi_i : \mathbb{R} \to \mathbb{R}$ are affine, i.e.,

$$w_i(x) = s_i x + a_i \quad \text{and} \quad \phi_i(t) = \alpha_i t + \beta_i \quad \text{so that} \quad K_i = |\alpha_i| . \tag{2.15}$$

**Theorem 2.1** *The operator $T$ defined in Eq. (1.7) maps $BV([a,b])$ into itself.*

**Proof.** From the definition of $\|\cdot\|_{BV}$,

$$\begin{aligned}
\|Tf\|_{BV} &= |Tf(a)| + V_a^b(Tf) \\
&= \left|\sum_{i=1}^{N} \phi_i(f(w_i^{-1}(a)))\right| + \sup_{P\in\mathcal{P}} \sum_{j=0}^{n} \left|\sum_{i=1}^{N} \phi_i(f(w_i^{-1}(x_{j+1}))) - \sum_{i=1}^{N} \phi_i(f(w_i^{-1}(x_j)))\right| \\
&\leq \sum_{i=1}^{N} K_i \left|f(w_i^{-1}(a))\right| + |\beta_i| + \sup_{P\in\mathcal{P}} \sum_{j=0}^{n} \sum_{i=1}^{N} \left|\alpha_i f(w_i^{-1}(x_{j+1})) - \alpha_i f(w_i^{-1}(x_j))\right| \\
&\leq \sum_{i=1}^{N} K_i \left|f(w_i^{-1}(a)) - f(a)\right| + K_i|f(a)| + |\beta_i| \\
&+ \sup_{P\in\mathcal{P}} \sum_{i=1}^{N} K_i \sum_{j=0}^{n} \left|f(w_i^{-1}(x_{j+1})) - f(w_i^{-1}(x_j))\right| \\
&\leq \sum_{i=1}^{N} K_i|f(a)| + \sum_{i=1}^{N} |\beta_i| + \sum_{i=1}^{N} K_i V_a^b(f) \\
&= \sum_{i=1}^{N} |\beta_i| + \|f\|_{BV} \sum_{i=1}^{N} K_i .
\end{aligned}$$

■

**Theorem 2.2** *For $f, g \in BV([a,b])$,*

$$\|Tf - Tg\|_{BV} \le K\|f-g\|_{BV} \quad \textit{where} \quad K = \sum_{i=1}^{N} K_i\,. \tag{2.16}$$

**Proof.**

$$\begin{aligned}
\|Tf - Tg\|_{BV} &= |Tf(a) - Tg(a)| + V_a^b(Tf - Tg)\\
&= \left|\sum_{i=1}^{N} \phi_i(f(w_i^{-1}(a))) - \phi_i(g(w_i^{-1}(a)))\right|\\
&+ \sup_{P\in\mathcal{P}} \sum_{j=0}^{n} \left|\sum_{i=1}^{N} \phi_i(f(w_i^{-1}(x_{j+1}))) - \phi_i(g(w_i^{-1}(x_{j+1}))) - \phi_i(f(w_i^{-1}(x_j))) + \phi_i(g(w_i^{-1}(x_j)))\right|\\
&\le \sum_{i=1}^{N} K_i \left|f(w_i^{-1}(a)) - g(w_i^{-1}(a))\right|\\
&+ \sup_{P\in\mathcal{P}} \sum_{i=1}^{N}\sum_{j=0}^{n} \left|\alpha_i f(w_i^{-1}(x_{j+1})) - \alpha_i g(w_i^{-1}(x_{j+1})) - \alpha_i f(w_i^{-1}(x_j)) + \alpha_i g(w_i^{-1}(x_j))\right|\\
&\le \sum_{i=1}^{N} K_i\, |f(a) - g(a)| + \sum_{i=1}^{N} K_i V_a^b(f-g)\\
&= \left[\sum_{i=1}^{N} K_i\right] \|f-g\|_{BV}\,.
\end{aligned}$$

■

**Theorem 2.3** *If $K$ in Eq. (2.16) satisfies $K < 1$, then the IFSM operator $T$ possesses a unique fixed point $\bar{f} \in BV([a,b])$. Moreover, for all $f_0 \in BV([a,b])$, the sequence $T^n f_0$ converges to $\bar{f}$ when $n \to +\infty$. Finally, the following estimate holds*

$$\|f\|_{BV} \le \frac{\sum_{i=1}^{N} |\beta_i|}{1-K}\,. \tag{2.17}$$

**Proof.** The first part of the theorem follows from the contractivity of $T$ over the complete metric space $BV([a,b])$ and Banach's Fixed Point Theorem. The estimate can be easily proved by replacing $f$ with the fixed point $\bar{f}$ in the inequality derived in the proof of Theorem 2.1,

$$\|\bar{f}\|_{BV} \le \sum_{i=1}^{N} |\beta_i| + \|\bar{f}\|_{BV} \sum_{i=1}^{N} K_i\,.$$

A rearrangement yields the desired result. ■

Note that, from Eq. (1.13), the condition $K < 1$ implies that $\bar{f} \in L^\infty([a,b])$, as expected.

**Theorem 2.4** *If (i) the maps $w_i$ are non-overlapping, i.e., the open set condition in (1.9) is satisfied, (ii) $K = \sum_{i=1}^{N} K_i > 1$, and (iii) $\sum_{i=1}^{N} c_i^{1/p} K_i < 1$, then $T$ possesses a unique fixed point $\bar{f} \in L^p[a,b]$. If, in addition, $\bar{f}$ is not constant then $\bar{f} \notin BV([a,b])$.*

**Proof.** The existence of a unique fixed point $\bar{f}$ of $T$ in $L^p[a,b]$ follows from Theorem 1.6. Then we have that

$$\begin{aligned}
V_a^b(\bar{f}) &= V_a^b(T\bar{f}) \\
&= \sup_{P\in\mathcal{P}} \sum_{j=0}^{n} \left| \sum_{i=1}^{N} \phi_i(\bar{f}(w_i^{-1}(x_{j+1}))) - \sum_{i=1}^{N} \phi_i(\bar{f}(w_i^{-1}(x_j))) \right| \\
&= \sup_{P\in\mathcal{P}} \sum_{i=1}^{N} \sum_{P\cap w_i([a,b])} \left| \sum_{i=1}^{N} K_i \bar{f}(w_i^{-1}(x_{j+1}))) - K_i \bar{f}(w_i^{-1}(x_j)) \right| \\
&= \sup_{P\in\mathcal{P}} \sum_{i=1}^{N} K_i \sum_{P\cap w_i([a,b])} \left| \bar{f}(w_i^{-1}(x_{j+1}))) - \bar{f}(w_i^{-1}(x_j)) \right| \\
&\geq \left[ \sum_{i=1}^{N} K_i \right] V_a^b(\bar{f}).
\end{aligned}$$

Thus either $V_a^b(\bar{f}) = 0$ or $V_a^b(\bar{f}) = \infty$. ■

**Example 2.5** Consider the interval $[a,b] = [0,1]$ and the IFSM operator $T_\alpha$ defined by

$$\begin{aligned}
w_1(x) &= \frac{1}{2}x, \quad \phi_1(t) = \alpha t - 1, \\
w_2(x) &= \frac{1}{2}x + \frac{1}{2}, \quad \phi_2(t) = \alpha t + 1 \,.
\end{aligned} \tag{2.18}$$

Note that the open set condition in (1.9) is satisfied. From Eqs. (1.12) and (1.14), $T_\alpha$ is contractive on $L^p[0,1]$, $1 \leq p \leq \infty$ for all $\alpha \in (-1,1)$. From Theorem 2.3, however, $T_\alpha$ is guaranteed to be contractive in $BV[0,1]$ only if $|\alpha| < 1/2$. Because of the dyadic nature of the IFS maps, the IFS "address" [1] of a point $x \in [0,1]$ (the attractor $A$ of the IFS) coincides with its binary representation. From this, and the action of $T_\alpha$ through the $w_i$ and $\phi_i$ maps, the fixed point, $\bar{f}_\alpha$, of $T_\alpha$ may be expressed in closed form as follows,

$$\bar{f}(x) = (-1)^{d_1+1} + \sum_{i=2}^{\infty} (-1)^{d_i+1} \alpha^{i-1}, \tag{2.19}$$

where the $d_i$ comprise the binary representation of $x \in [0,1]$, i.e.,

$$x = \sum_{i=1}^{\infty} d_i 2^{-i} \,. \tag{2.20}$$

For $0 \le \alpha \le 1/2$, it can be shown that $\bar{f}$ is monotone on $[0,1]$. When $\alpha = 0$,

$$\bar{f}_0(x) = -\frac{1}{2}\chi_{[0,1/2]}(x) + \frac{1}{2}\chi_{[1/2,1]}(x), \quad x \in [0,1]\,, \tag{2.21}$$

where $\chi_S(x)$ denotes the characteristic function of a set $S \subset [0,1]$. A plot of $\bar{f}$ for the case $\alpha = 1/3$ is shown in Figure 1. In the particular case $\alpha = 1/2$, it can be shown that $\bar{f}(x) = 4x - 2$. For $1/2 < \alpha < 1$, $\bar{f}$ is no longer monotone. It is this oscillation which results in $\bar{f} \notin BV[0,1]$. A plot of $\bar{f}$ for $\alpha = 3/4$ is shown in Figure 2.

**Example 2.6** Once again, the interval $[a,b] = [0,1]$ and IFS maps, $w_1$ and $w_2$, as in Example 2.5, but now with the following associated mappings,

$$\phi_1(t) = -t/2 - 1, \quad \phi_2(t) = t/2 + 1\,. \tag{2.22}$$

In this case, the fixed point $\bar{f} \notin BV[0,1]$. A plot of $\bar{f}$ is shown in Figure 3.

Figure 1: Fixed point attractor function $\bar{f} \in BV[0,1]$ for the IFSM operator $T_\alpha$ of Example 2.5 with $\alpha = 1/3$.

Figure 2: Fixed point attractor function $\bar{f} \notin BV[0,1]$ for the IFSM operator $T_\alpha$ of Example 2.5 with $\alpha = 3/4$.

# 3 Inverse problems for IFSM on $BV([a,b])$

We first outline the main ideas behind inverse problems using contraction mappings over a complete metric space. For more details, the reader is referred to [6] and references within.

Let $(X,d)$ be a complete metric space and $Con(X)$ an appropriate set of contraction maps $T : X \to X$. Given a "target" element $x \in X$, we look for a contraction map $T \in Con(X)$ with fixed point $\bar{x} \in X$ that approximates $x$ to some desired accuracy, i.e.,

$$d(x, \bar{x}) < \epsilon\,, \tag{3.23}$$

for some $\epsilon > 0$. (A discussion of whether such a fixed point $\bar{x}$ exists for any $\epsilon > 0$ is beyond the scope of this paper.)

Given the rather complicated nature of fractal transform operators, the problem of determining such operators to achieve the approximation in (3.23) is untractable. An enormous simplification is achieved by using a rather trivial consequence of Banach's Fixed Point Theorem, known in fractal coding literature as the "Collage Theorem:"

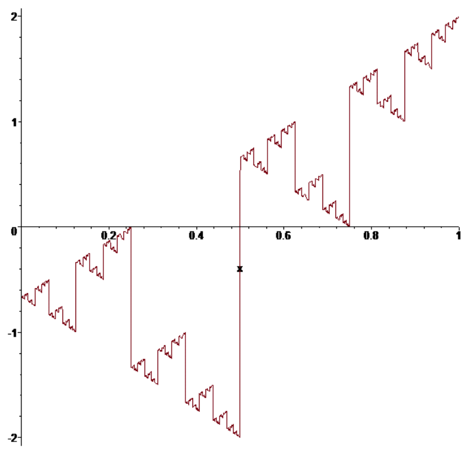


Figure 3: Fixed point attractor function $\bar{f} \notin BV[0,1]$ for the IFSM operator $T_\alpha$ of Example 2.6.

**Theorem 3.1** *(Collage Theorem[1]) Let $(X,d)$ be a complete metric space and $T \in Con(X)$ with contraction factor $c_T \in [0,1)$ and fixed point $\bar{x} \in X$. Then for any $x \in X$,*

$$d(x,\bar{x}) \leq \frac{1}{1-c_T}\, d(x,Tx)\,. \tag{3.24}$$

In an effort to produce small values of the approximation error $d(x,\bar{x})$ in (3.23), one looks for contraction maps $T$ which minimize the *collage distance* $d(x,Tx)$. This is the essence of "collage based" methods to solve the inverse problem: determining contractive operators $T$ that map a target $x$ as close as possible to itself[6].

A rather straightforward approach[4] to provide approximations of target functions in $L^p[0,1]$ (in particular $p=1,2$) by fixed points of contractive fractal operators is to consider a prescribed set of of contractive IFS maps on $[0,1]$, $\mathbf{w} = \{w_1, w_2, \cdots, w_N\}$, that satisfy the covering condition in Eq. (1.6). The $2N$ parameters $\alpha_i, \beta_i$, $1 \leq i \leq N$, which define the $\phi_i(t)$ maps associated with the fixed $w_i$ maps are considered as unknowns to be optimized. These $2N$ parameters define a fractal transform $T_N(\alpha,\beta)$.

Given a target function $f \in L^p[0,1]$, we then minimize the collage distance,

$$\Delta_{p,N}(\alpha,\beta) = \|f - T_N(\alpha,\beta)f\|_p\,, \tag{3.25}$$

considered as a function of the $\alpha_i$ and $\beta_i$, subject to inequality constraints that guarantee the contractivity of the $T_N(\alpha,\beta)$ operator in $L^p[0,1]$. By incorporating a sequence of IFS maps $w_i$ that provide increasing degrees of refinement over the interval $[0,1]$, the collage error $\Delta_{p,N}$ can be made arbitrarily small for $N$ sufficiently large.

We now consider the special case $p = 2$ which is employed in most applications. In this case the squared collage distance,

$$\Delta^2_{2,N}(\alpha,\beta) = \|f - T_N(\alpha,\beta)\|_2^2 \,, \tag{3.26}$$

is a quadratic form in the $\alpha_i$ and $\beta_i$ parameters. Minimization of the squared collage distance is a quadratic programming problem of the form,

$$\min\ \Delta^2_{2,N}(\alpha,\beta) \quad \text{subject to} \quad \sum_{i=1}^{N} c_i^{1/2}|\alpha_i| \le C_{\max} < 1 \,, \tag{3.27}$$

where $C_{\max} < 1$ is chosen to keep the $L^2$ Lipschitz constant of $T_N$ away from 1. In this way, $T_N$ is guaranteed to be contractive in $L^2$ metric with fixed point $\bar{f}_N \in L^2([0,1])$.

If we add the following constraint to the minimization problem in (3.27),

$$\|\alpha\|_1 = \sum_{k=1}^{N} |\alpha_i| \le A < 1 \,, \tag{3.28}$$

then, from Eq. (2.3), the $T_N$ operator is also contractive in $BV([0,1])$. This implies that its fixed point $\bar{f}_N \in L^2([0,1]) \cap BV([0,1])$. In other words, the fixed point approximation $\bar{f}_N$ is an $L^2$ function with bounded variation. An estimate of its total variation is given in Eq. (2.17), where $K_i = |\alpha_i|$, $1 \le i \le N$.

By decreasing the upper bound $A \to 0^+$, the total variation of the fixed point $\bar{f}_N$ can be decreased, according to Eq. (2.17). In the limiting case, $A = 0$, implying that $\alpha_i = 0$, $1 \le i \le N$, the fixed point $\bar{f}_N$ is a piecewise constant function, i.e.,

$$\bar{f}_N(x) = \sum_{k=1}^{N} \beta_i \chi_{w_i([0,1])}(x) \,, \tag{3.29}$$

where $\chi_S(x)$ denotes the characteristic function of the set $S \subseteq [0,1]$.

Another way to control the BV of collage-based $L^2$ fixed point approximations to a target function $u$ is to modify the optimization problem in (3.27) as follows: For a $\lambda > 0$,

$$\min\ \Delta^2_{2,N}(\alpha,\beta) + \lambda\|\alpha\|_1 \quad \text{subject to} \quad \sum_{i=1}^{N} c_i^{1/2}|\alpha_i| \le C_{\max} < 1 \,. \tag{3.30}$$

Here, the term $\lambda\|\alpha\|_1$ is a *regularization term*, a kind of penalty function. $L^1$ regularization is commonly employed in signal and image processing in order to enforce sparsity of solution vectors (lower number of nonzero components) [2]. By increasing the value of $\lambda$, the value of $\|\alpha\|_1$ assumed at the minimum can be decreased. It is therefore possible that there exists a critical value $\lambda^* > 0$ so that the fixed point $\bar{f}_N(\lambda) \in L^2([0,1])$ corresponding to the solution of (3.30) will behave as follows:

1. For $\lambda < \lambda^*$, $\bar{f}_N(\lambda) \notin BV([0,1])$,
2. For $\lambda > \lambda^*$, $\bar{f}_N(\lambda) \in BV([0,1])$.


### Acknowledgments

This research was partially supported by Natural Sciences and Engineering Research Council of Canada (NSERC) in the form of Discovery Grants (FM and ERV).


## References


[1] M.F. Barnsley, *Fractals Everywhere*, Academic Press, New York (1989).

[2] M. Elad, *Sparse and Redundant Representations, From Theory to Applications in Signal and Image Processing*, Springer, New York (2010).

[3] B. Forte and E.R. Vrscay, Theory of generalized fractal transforms, in *Fractal Image Coding and Analysis*, Y. Fisher, Editor, NATO ASI Series F, Volume 159, Springer-Verlag, Berlin (1998).

[4] B. Forte and E.R. Vrscay, Inverse problem methods for generalized fractal transforms, in *Fractal Image Coding and Analysis*, Y. Fisher, Editor, NATO ASI Series F, Volume 159, Springer-Verlag, Berlin (1998).

[5] J. Hutchinson, Fractals and self-similarity, Indiana Univ. J. Math., 30, 713-747 (1981).

[6] H. Kunze, D. La Torre, F. Mendivil and E.R. Vrscay, *Fractal-based Methods in Analysis*, Springer, New York (2012).

[7] W. Rudin, *Real and complex analysis*, McGraw Hill, New York (1974).