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Solving inverse problems for variational equations using "generalized collage methods," with applications to boundary value problems

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ABSTRACT

A number of inverse problems involving deterministic and random differential equations may be viewed in terms of the problem of approximating a target element x of a complete metric space (X, d) by the fixed point $\bar{x}$ of a contraction mapping $T : X \to X$. Most practical methods rely on a reformulation of this problem due to the "Collage Theorem," a simple consequence of Banach's Fixed Point Theorem: They search for a contraction mapping that minimizes the "collage distance" $d(x, Tx)$. One may consider the collage method as a kind of regularization procedure for the inverse problem. In this paper, after recalling some applications of the Collage Theorem to the solution of inverse problems for fixed point equations and applications of it to initial value problems, with the help of the Lax–Milgram representation, we develop some generalizations of the collage method in order to solve inverse problems for variational equations. We consider both deterministic and stochastic problems. We then show some applications to inverse boundary value problems.

1. Introduction

A number of inverse problems in applied mathematics may be viewed in terms of the approximation of a target element x in a complete metric space (X, d) by the fixed point $\bar{x}$ of a contraction mapping $T : X \to X$. In practice, from a family of contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, one wishes to find the parameter $\bar{\lambda}$ for which the approximation error $d(x, \bar{x}_\lambda)$ is as small as possible. A simplification of this problem—at the expense of suboptimality—is afforded by the "Collage Theorem", a simple consequence of Banach's Fixed Point Theorem.

Theorem 1 (*"Collage Theorem" [1,2]*). *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$,*

$$d(x, \bar{x}) \le \frac{1}{1-c} d(x, Tx), \tag{1}$$

where $\bar{x}$ is the fixed point of T.

One now seeks a contraction mapping T_λ (with contraction factor bounded away from one) that minimizes the so-called *collage error* $d(x, T_\lambda x)$—in other words, a mapping that sends the target x as close as possible to itself. This is the essence of the method of *collage coding* which has been the basis of most, if not all, fractal image coding and compression methods and

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their applications to inverse problems and related fields [3–22]. The following examples demonstrates how the method can be applied to an inverse problem for a delay integral equation and for a stochastic differential equation, respectively.

Example 1. Consider the following integral equation with delay

$$x'(t) = \phi(t, x(\lambda t)), \quad x(0) = x_0. \tag{2}$$

Under certain conditions this equation is equivalent to

$$x(t) = (Tx)(t) = \int_0^t \phi(s, x(\lambda s))\mathrm{d}s + x_0, \tag{3}$$

where x_0 is an initial condition, $0 < \lambda < 1$ and $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $C_\phi \geq 0$. Consider now for simplicity the case where the kernel assumes the form

$$\phi(u, v) = a_0 + a_1 u + a_2 v. \tag{4}$$

This means that

$$|\phi(\alpha_1, \beta_1) - \phi(\alpha_2, \beta_2)| \leq \sqrt{a_1^2 + a_2^2}\,(|\alpha_1 - \alpha_2| + \beta_1 - \beta_2|)\,,$$

and so the condition for contractivity of the integral operator becomes

$$a_1^2 + a_2^2 < 2\lambda. \tag{5}$$

Also assume that the target function $x(t)$ can be expressed as a power series, i.e.,

$$x(t) = \sum_k x_k t^k. \tag{6}$$

By straightforward calculations, we compute the $\mathcal{L}^2$ collage distance to be

$$\|x - Tx\|_2^2 = \int_0^1 \left| \sum_k x_k t^k - a_0 t - a_1 \frac{t^2}{2} - a_2 \sum_k x_k \frac{\lambda^k t^{k+1}}{k+1} - x_0 \right|^2 \mathrm{d}t. \tag{7}$$

The inverse problem can be reduced to the following minimization problem:

$$\min_A \|x - Tx\|_2^2, \tag{8}$$

where $A = \{(a_1, a_2) : a_1^2 + a_2^2 \leq 2\lambda\}$. As a numerical example, consider the delay differential equation

$$x'(t) = 1 + \frac{1}{4}t + \frac{1}{4}x\left(\frac{t}{8}\right), \quad x(0) = 0,\ 0 \leq t \leq 1. \tag{9}$$

The integral operator (3) associated with (9) is

$$(Tx)(t) = \int_0^t \left(1 + \frac{1}{4}s + \frac{1}{4}x\left(\frac{s}{8}\right)\right) \mathrm{d}s + x_0. \tag{10}$$

The solution $x(t)$ to (9) satisfies the fixed point equation $x = Tx$ with $x(0) = x_0 = 0$. We can generate successive approximations x_i of this fixed point by introducing the relation $x_{i+1} = Tx_i$, $i = 0, 1, 2, \ldots$. In fact, one can calculate the power series solution to (9) and verify that, for $i \geq 1$, $x_i(t)$ agrees with the solution up to the t^i term. We construct $x_{20}(t)$ and fit to it a sixth-degree polynomial that becomes our target function. For our inverse problem, suppose that we seek a delay differential equation of the form

$$x'(t) = p(t) + bx\,(\lambda t)\,, \quad x(0) = x_0,\ 0 \leq t \leq 1, \tag{11}$$

where $p(t) = \sum_{i=0}^N a_i t^i$ is a polynomial in t of degree N and $0 < \lambda < 1$, that admits our target function as an approximate solution. The parameters in (11) are λ, x_0, b, and a_i, $i = 0, \ldots, N$. We minimize the $\mathcal{L}^2$ collage distance. For each N, observe that the family of polynomial differential equations given by (11) with $\lambda = 0$ is

$$x'(t) = (a_0 + bx_0) + \sum_{i=1}^N a_i t^i, \quad x(0) = x_0.$$

Based on the work in [10], when N is large enough, we expect that a member of the corresponding family with $\lambda = 0$ will admit our target function as an approximate solution with very small error. As a result, one must be particularly careful to steer away from the local minima of the collage distance with $\lambda = 0$. As a systematic approach, we choose a positive integer M and consider $\lambda = \frac{i}{M}$, $1 \leq i \leq M - 1$. When λ is fixed to each such value, the squared $\mathcal{L}^2$ collage distance is a

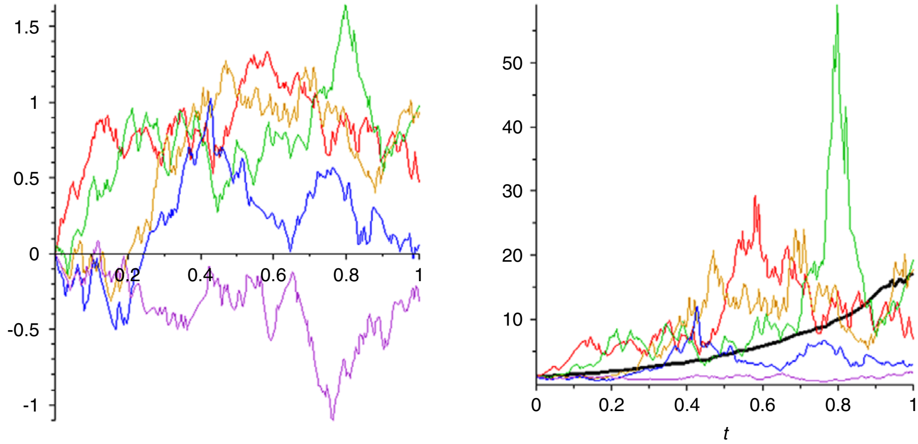

Fig. 1. The Brownian motion (left) and the resulting paths S_t^i (right) along with the mean path S_t^* (thicker), with $M = 1000$ and $N = 300$.

Table 1
Minimal collage distance parameters for different N and M, to six decimal places.

N	M	a	b
100	300	2.721127	1.728953
100	600	3.231600	1.837984
100	1000	3.029700	1.869433
300	300	2.396741	1.776105
300	600	2.583953	1.794659
300	1000	2.858033	1.779098

quadratic function of the remaining variables and, hence, easily minimizable via differentiation. We use the values of λ and the parameters corresponding to the overall minimum collage distance as starting values for further minimization. When $1 \leq N \leq 4$, we find the correct parameter values to five decimal places. For higher N values, we require a better-fitted target function.

Example 2. Suppose that the stochastic process S_t is believed to follow a geometric Brownian motion; then it satisfies the stochastic differential equation

$$\mathrm{d}S_t = aS_t\,\mathrm{d}t + bS_t\,\mathrm{d}W_t, \tag{12}$$

where W_t is a Wiener process and the constants a and b are the percentage drift and the percentage volatility, respectively. We consider the inverse problem: given realizations/paths S_t^i, $1 \leq i \leq N$, estimate the values of a and b. Taking the expectation in (12), we see that $\mathbb{E}(S_t)$ satisfies the simple fixed point equation

$$\mathbb{E}(S_t) = T(\mathbb{E}(S_t)) = S_0 + \int_0^t a\mathbb{E}(S_r)\,\mathrm{d}r.$$

Hence, to solve the inverse problem, we construct the mean of the realizations

$$S_t^* = \frac{1}{N}\sum_{i=1}^{N} S_t^i$$

and use collage coding to determine the value of a that minimizes the collage distance $d_2(S_t^*, TS_t^*)$. We can then estimate the value of b by using the known formula $\mathrm{var}(S_t) = \mathrm{e}^{2at}S_0^2(\mathrm{e}^{b^2t} - 1)$: approximate $\mathrm{var}(S_t)$ from the realizations and minimize $d_2\left(\ln\left(\mathrm{e}^{-2at}\frac{\mathrm{var}(S_t)}{S_0^2} + 1\right), b^2t\right)$ with respect to b.

As an example, we set $a = 3$, $b = 2$, and $S_0 = 1$, and then generate N paths on $[0, 1]$, dividing the interval into M subintervals in order to simulate the Brownian motion on $[0, 1]$. Beginning with these paths, we seek estimates of a and b using collage coding. Fig. 1 shows five paths for the Brownian motion and the process S_t, as well as the mean path S_t^*.

Table 1 presents the numerical results of the example.

Section 2 will be devoted to the analysis of inverse problems for variational problems. Most, if not all, practical methods of solving such problems begin with a weak or variational formulation which is obtained by integrating both sides with respect to elements of a suitable set of test functions. Using the Lax–Milgram representation theorem, we formulate the inverse problem in terms of the minimization of a functional. We view this procedure as a "generalized collage method". In Section 3, the method described in Section 2 will be developed in a stochastic environment for the purpose of solving inverse problems for random ODE/PDE boundary value problems.

2. Inverse problems for variational equations

An important class of inverse problems for boundary value problems which has understandably received a great deal of attention over the years is the problem of parameter estimation in distributed systems: the determination of unknown parameters in a mathematical model of a physical process from experimental data. Most, if not all, practical methods of solving inverse problems for boundary value problems begin with its weak or variational form which is obtained by integrating both sides with respect to elements of a suitable set of test functions, for example, the finite element "hat" functions. This produces a linear system of equations involving the known values u_k and the associated unknown values κ_k over a set of grid points. The "solution" of this system is often accomplished by minimizing an appropriate least squares functional possibly with the inclusion of additional penalty functions for the purpose of regularization. This procedure is described nicely in the more recent monograph by Vogel [23].

Here we step back a bit from these traditional practical approaches and view the inverse problem for linear ODE/PDE boundary value problems in terms of the Lax–Milgram representation theorem. Very briefly, we recall that in the forward problem, it is a standard procedure to multiply both sides with an element $v \in H$, where H is a suitable Hilbert space of functions, i.e., $H_0^1(\Omega)$, and integrate over Ω to obtain the variational equation,

$$a(u, v) = \phi(v), \quad v \in H. \tag{13}$$

Here $\phi(v)$ and $a(u, v)$ are linear and bilinear maps, respectively. Let us denote by $\langle\cdot\rangle$ the inner product in H, $\|u\|^2 = \langle u, u\rangle$ and $d(u, v) = \|u - v\|$, for all $u, v \in H$. The inverse problem may now be viewed as follows. Suppose that we have an observed solution u and a given (restricted) family of bilinear functionals $a_\lambda(u, v)$, $\lambda \in \mathbb{R}^n$. We now seek to find "optimal" values of λ, for example, those that minimize the function

$$F(\lambda) = \sup_{v\in H, \|v\|=1} |a_\lambda(u, v) - \phi(v)|. \tag{14}$$

Now suppose that

$$\lambda^* = \arg\min F(\lambda). \tag{15}$$

Then by the Lax–Milgram theorem (reviewed below), there exists a unique function u_λ^* such that

$$a_{\lambda^*}(u_{\lambda^*}, v) = \phi(v), \quad \text{for all } v \in H. \tag{16}$$

As we shall show below, the error in the approximation of our given solution u with u_λ^* is given by

$$\|u - u_{\lambda^*}\| \leq \frac{1}{m_{\lambda^*}} F(\lambda^*), \tag{17}$$

where the constant m_λ characterizes the expansivity of the bilinear form a_λ. The choice of λ according to (17) for minimizing the residual is, in general, not stabilizing (see [24]). However, as next sections show, under the condition $\inf_{\lambda\in\mathcal{F}} m_\lambda \geq m > 0$ our approach is stable. Following our earlier studies on the inverse problems using fixed points of contraction mappings, we shall refer to the minimization of the functional $F(\lambda)$ as a "generalized collage method". We begin by developing a general collage coding framework for inverse problems for variational equations using the Lax–Milgram representation theorem. We emphasize that this framework which, to the best of our knowledge, has not appeared in the literature, provides a rigorous basis for most practical methods of solving inverse problems in boundary value problems.

Theorem 2 (*Riesz Representation*). *Let H be a Hilbert space and ϕ be a bounded linear functional; then there is a unique vector $x \in H$ such that $\phi(y) = \langle x, y\rangle$ for all $y \in H$, where $\langle\cdot,\cdot\rangle$ denotes an inner product in H.*

Theorem 3 (*Lax–Milgram Representation*). *Let H be a Hilbert space and ϕ be a bounded linear nonzero functional. Suppose that $a(u, v)$ is a bilinear form on $H \times H$ which satisfies the following:*

1. *There exists a constant $M > 0$ such that $|a(u, v)| \leq M\|u\|\,\|v\|$ for all $u, v \in H$.*
2. *There exists a constant $m > 0$ such that $|a(u, u)| \geq m\|u\|^2$ for all $u \in H$.*

Then there is a unique vector $u^* \in H$ such that $\phi(v) = a(u^*, v)$ for all $v \in H$.

Suppose that we have a given Hilbert space H, a "target" element $u \in H$ and a family of bilinear functionals a_λ. Then by the Lax–Milgram theorem, there exists a unique vector u_λ such that $\phi(v) = a_\lambda(u_\lambda, v)$ for all $v \in H$. We would like to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|$ is small enough. The following theorem will be useful for the solution of this problem.

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Theorem 4 (*Generalized Collage Theorem [12]*). *Suppose that $a_\lambda(u, v) : \mathcal{F} \times H \times H \to \mathbb{R}$ is a family of bilinear forms for all $\lambda \in \mathcal{F}$ and $\phi : H \to \mathbb{R}$ is a given linear functional. Let u_λ denote the solution of the equation $a_\lambda(u, v) = \phi(v)$ for all $v \in H$ as guaranteed by the Lax–Milgram theorem. Given a target element $u \in H$ then*

$$\|u - u_\lambda\| \le \frac{1}{m_\lambda} F(\lambda), \tag{18}$$

where

$$F(\lambda) = \sup_{v\in H,\ \|v\|=1} |a_\lambda(u, v) - \phi(v)|. \tag{19}$$

In order to ensure that the approximation u_λ is close to a target element $u \in H$, we can, by the Generalized Collage Theorem, try to make the term $F(\lambda)/m_\lambda$ in Eq. (18) as close to zero as possible. The appearance of the m_λ factor complicates the procedure as does the factor $1/(1 - c)$ in standard collage coding, i.e., Eq. (1). As such, we shall follow the usual practice and ignore the m_λ factor, assuming, of course, that all allowable values are bounded away from zero. So if $\inf_{\lambda\in\mathcal{F}} m_\lambda \ge m > 0$ then the inverse problem can be reduced to the minimization of the function $F(\lambda)$ on the space $\mathcal{F}$, that is,

$$\min_{\lambda\in\mathcal{F}} F(\lambda). \tag{20}$$

Let $\langle e_i\rangle \subset H$ be a basis of the Hilbert space H, not necessarily orthogonal, so that each element $v \in H$ can be written as $v = \sum_i \alpha_i e_i$. Computing, we have for all $v \in H$,

$$\begin{aligned} |a_\lambda(u, v) - \phi(v)|^2 &= \left| a_\lambda\left(u, \sum_i \alpha_i e_i\right) - \phi\left(\sum_i \alpha_i e_i\right)\right|^2 \\ &\le \left(\sum_i \alpha_i |a_\lambda(u, e_i) - \phi(e_i)|\right)^2 && (21) \\ &\le \left[\sum_i \alpha_i^2\right]\left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]. && (22) \end{aligned}$$

So we get

$$\begin{aligned} \inf_{\lambda\in\mathcal{F}} \|u - u_\lambda\| &\le \inf_{\lambda\in\mathcal{F}} \left(\frac{1}{m_\lambda}\right)\left(\sup_{v\in H,\|v\|=1}\left[\sum_i \alpha_i^2\right]\right)^{\frac{1}{2}} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{m} \sup_{v\in H,\|v\|=1}\left[\sum_i \alpha_i^2\right]^{\frac{1}{2}} \inf_{\lambda\in\mathcal{F}} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]^{\frac{1}{2}}. \end{aligned}$$

Let $V_n = \langle e_1, e_2, \ldots, e_n\rangle$ be the finite-dimensional vector space generated by e_i, $V_n \subset H$. Given a target $u \in H$, let $\Pi_{V_n} u$ be the projection of u on the space V_n and consider the following problem: find $u_\lambda \in V_n$ such that $\|\Pi_{V_n} u - u_\lambda\|$ is as small as possible. Following the same analysis of Theorem 4 we have

$$\begin{aligned} \|\Pi_{V_n} u - u_\lambda\| &\le \frac{1}{m} \sup_{v\in V_n,\|v\|=1} |a_\lambda(\Pi_{V_n} u, v) - \phi(v)| \\ &\le \frac{1}{m} \max_{v=\sum_{i=1}^n \alpha_i e_i \in V_n,\|v\|=1} \left[\sum_{i=1}^n \alpha_i^2\right]^{\frac{1}{2}} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]^{\frac{1}{2}} \\ &= \frac{M}{m}\left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]^{\frac{1}{2}} && (23) \end{aligned}$$

where $M = \max_{v=\sum_{i=1}^n \alpha_i e_i\in V_n,\|v\|=1} \left[\sum_{i=1}^n \alpha_i^2\right]^{\frac{1}{2}}$, so we have reduced the problem to the minimization of the function

$$\inf_{\lambda\in\mathcal{F}} \|\Pi_{V_n} u - u_\lambda\| \le \frac{M}{m} \inf_{\lambda\in\mathcal{F}} \left[\sum_{i=1}^n |a_\lambda(u, e_i) - \phi(e_i)|^2\right]^{\frac{1}{2}} = \frac{M}{m} \inf_{\lambda\in\mathcal{F}} F_n(\lambda). \tag{24}$$

Notice that this result is similar to the famous Cea's lemma. This framework for solving inverse problems for variational equations can be applied to boundary value problems.

Example 3. We consider the following one-dimensional steady-state diffusion equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right)=f(x), \quad 0<x<1, \tag{25}$$

$$u(0)=0, \tag{26}$$

$$u(1)=0, \tag{27}$$

where the diffusivity $\kappa(x)$ varies in x. The inverse problem of interest is: given $u(x)$, possibly in the form of an interpolation of data points, and $f(x)$ on $[0, 1]$, determine an approximation of $\kappa(x)$. In [23] this problem is studied and solved via a regularized least squares minimization problem. The collage coding approach allows us to perform a different minimization to solve the inverse problem. A natural goal is to recover $\kappa(x)$ from observations of the response $u(x)$ to a point source $f(x)=\delta(x-x_s)$, a Dirac delta function at $x_s\in(0,1)$. We multiply Eq. (25) by a test function $\xi_i(x)\in H_0^1([0,1])$ and integrate by parts to obtain $a(u,\xi_i)=\phi(\xi_i)$, where

$$a(u,\xi_i) = \int_0^1 \kappa(x)u'(x)\xi_i'(x)\,\mathrm{d}x - \xi_i(x)\kappa(x)u'(x)\Big|_0^1 \tag{28}$$

$$= \int_0^1 \kappa(x)u'(x)\xi_i'(x)\,\mathrm{d}x, \quad \text{and} \tag{29}$$

$$\phi(\xi_i) = \int_0^1 f(x)\xi_i(x)\,\mathrm{d}x. \tag{30}$$

For a fixed choice of n, introduce the following partition of $[0, 1]$,

$$x_i=\frac{i}{n+1}, \quad i=0,\dots,n+1,$$

with n interior points, and define for $j=0,1,2,\dots$

$$V_n^r=\{v\in C[0,1]: v(0)=v(1)=0 \text{ and } v \text{ is piecewise degree } r \text{ on the partition}\}.$$

Denote a basis for V_n^r by $\{\xi_1,\dots,\xi_n\}$. When $r=1$, our basis consists of the hat functions

$$\xi_i(x)=\begin{cases}(n+1)(x-x_{i-1}), & x_{i-1}\le x\le x_i\\ -(n+1)(x-x_{i+1}), & x_i\le x\le x_{i+1}, \quad i=1,\dots,n,\\ 0, & \text{otherwise,}\end{cases}$$

and when $r=2$, our hats are replaced by parabolae, with

$$\xi_i(x)=\begin{cases}(n+1)^2(x-x_{i-1})(x-x_{i+1}), & x_{i-1}\le x\le x_{i+1},\\ 0, & \text{otherwise,}\end{cases} \quad i=1,\dots,n.$$

Suppose that $\kappa(x)>0$ for all $x\in[0,1]$. Then, for b a positive constant independent of u, we have

$$\begin{aligned} a(u,u) &= \int_0^1 \kappa(x)u'(x)u'(x)\,\mathrm{d}x \\ &\ge \inf_{x\in[0,1]}\kappa(x)\int_0^1 \left(u'(x)\right)^2\,\mathrm{d}x \\ &\ge b\inf_{x\in[0,1]}\kappa(x)\int_0^1 \left((u(x))^2+(u'(x))^2\right)\,\mathrm{d}x \\ &= m_\kappa\|u\|^2_{H_0^1}. \end{aligned}$$

As a result, because we divide by m_κ, we expect our results will be good when $\kappa(x)$ is bounded away from 0 on $[0, 1]$.

We consider a continuous framework scenario. Assume that we are given data points u_i measured at various x-values having no relation to our partition points x_i. These data points are interpolated to produce a continuous target function $u(x)$, a polynomial, say. Let us now assume a polynomial representation of the diffusivity, i.e.,

$$\kappa(x)=\sum_{j=0}^N \lambda_j x^j. \tag{31}$$

In essence, this introduces a regularization into our method of solving the inverse problem. Working on V_n^r, we have

Table 2
Collage coding results when $f(x) = -2+40x^2$, $\kappa_{\text{true}}(x) = 1+2x^2$, data points = 10, number of test functions = 30, and degree of κ_{collage} is 2. We work on V_{30}^1 and note that, in each case, $F_{30}(\lambda)$ is equal to 0 to 10 decimal places.

Noise ε	$d_2(u_{\text{true}}, u_{\text{target}})$	κ_{collage}	$d_2(\kappa_{\text{collage}}, \kappa_{\text{true}})$
0.000	0.0000	$1.0000 + 0.0000x + 2.0000x^2$	0.0000
0.001	0.0000	$0.9390 + 0.0568x + 2.0027x^2$	0.0360
0.01	0.0010	$0.5932 + 0.2126x + 2.1798x^2$	0.2663
0.05	0.0242	$0.2456 - 0.6951x + 3.3737x^2$	0.6810

$$a_\lambda(u, \xi_i) = \sum_{j=0}^{N} \lambda_j A_{ij}, \quad \text{with } A_{ij} = \int_{x_{i-1}}^{x_{i+1}} x^j u'(x)\xi_i'(x)\,\mathrm{d}x. \tag{32}$$

Letting

$$b_i = \int_0^1 f(x)\xi_i(x)\,\mathrm{d}x = \int_{x_{i-1}}^{x_{i+1}} f(x)\xi_i(x)\,\mathrm{d}x, \quad i = 1, \ldots, n, \tag{33}$$

we now minimize

$$(F_n(\lambda))^2 = \sum_{i=1}^{n} \left[\sum_{j=0}^{N} \lambda_j A_{ij} - b_i\right]^2. \tag{34}$$

Various minimization techniques can be used; in this work we used the quadratic program solving package in Maplesoft's Maple 10. As a specific experiment, consider $f(x) = -2 + 40x^4$ and $\kappa_{\text{true}}(x) = 1 + 2x^2$. We shall sample the true solution at 10 data points, add Gaussian noise of small amplitude ε to these values and then fit to the data points a polynomial of degree 4, to be denoted $u_{\text{target}}(x)$. Given $u_{\text{target}}(x)$ and $f(x)$, we seek a degree 2 polynomial $\kappa(x)$ with coefficients λ_i so that the steady-state diffusion equation admits $u_{\text{target}}(x)$ as an approximate solution. We now construct $(F_{30}(\lambda))^2$ and minimize it with respect to the λ_i. Table 2 presents the results. d_2 denotes the standard $\mathcal{L}^2$ metric on [0, 1].

3. Inverse problems for random variational equations

We now analyze the case in which the linear operator ϕ and the bilinear form a also depend on $\omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a probability space. It is well known how one can define random operators, which are natural extensions of deterministic operators, on a metric space (see [25–27]) and use them to prove random fixed point equations (see [11]). In a similar way, given a Hilbert space H a bilinear mapping $a : \Omega \times H \times H \to \mathbb{R}$ is called a *random bilinear form* if for any $x, y \in H$ the function $a(\cdot, x, y)$ is measurable and for a.e. $\omega \in \Omega$, $a(\omega, \cdot, \cdot)$ is a bilinear form on H. We then consider the following problem: find $u \in H$ such that the equation

$$a(\omega, u, v) = \phi(\omega, v) \tag{35}$$

is satisfied for all $v \in H$ and a.e. $\omega \in \Omega$. We also suppose that for a.e. $\omega \in \Omega$ the following hypotheses are satisfied:

1. there exists a constant $M > 0$ such that $|a(\omega, u, v)| \le M\|u\|\,\|v\|$ for all $u, v \in H$ and a.e. $\omega \in \Omega$,
2. there exists a constant $m > 0$ such that $|a(\omega, u, u)| \ge m\|u\|^2$ for all $u \in H$ and a.e. $\omega \in \Omega$.

Then, for a.e. $\omega \in \Omega$, there is a unique vector $u^*(\omega) \in H$ such that $\phi(\omega, v) = a(\omega, u^*(\omega), v)$ for all $v \in H$. Of course using this approach we cannot conclude anything about the measurability of u^*. For this purpose we consider the following generalized approach where we turn the pointwise problem into an averaged problem for which we know (by construction) that u^* depends measurably on ω. Of course it is not necessarily the case that $a(\omega, u^*(\omega), v) = \phi(\omega, v)$ for all v and for a.e. $\omega \in \Omega$ since just the averaged equation works. On the other hand, if u^* is measurable and $\mathcal{L}^2$ integrable, then u^* will be a solution of the averaged equation.

Consider the space H^* which is built as

$$H^* = \left\{u : \Omega \to H,\ \text{measurable}, \int_\Omega \|u(\omega)\|^2 \mathrm{d}P(\omega) < +\infty\right\} \tag{36}$$

and consider on it the following function

$$\langle u, v\rangle_* = \int_\Omega \langle u(\omega), v(\omega)\rangle \mathrm{d}P(\omega) = \mathbb{E}(\langle u, v\rangle). \tag{37}$$

It is easy to prove that $\langle\cdot,\cdot\rangle_*$ is an inner product on H^* and the following d_* and $\|\cdot\|_*$ are a distance and a norm on H^*, respectively:

$$\|u\|_* = \left(\int_\Omega \|u(\omega)\|^2 \mathrm{d}p(\omega)\right)^{\frac{1}{2}} \quad (38)$$

$$d_*(u, v) = \|u - v\|_*. \quad (39)$$

We have the following result.

Theorem 5. *H^* is an Hilbert space.*

Proof. The only property to prove is completeness with respect to the metric d_*. We follow the proof of Theorem 1.2 in [28]. Let u_n be a Cauchy sequence in H^*. So for all $\epsilon > 0$ there exists n_0 such that for all $n, m \geq n_0$ we have $d_*(u_n, u_m) < \epsilon$. Let $\epsilon = 3^{-k}$ so you can choose an increasing sequence n_k such that $d_*(u_n, u_{n_k}) < 3^{-k}$ for all $n \geq n_k$. So choosing $n = n_{k+1}$ we have $d_*(u_{n_{k+1}}, u_{n_k}) < 3^{-k}$. Let

$$A_k = \left\{\omega \in \Omega : d(u_{n_{k+1}}(\omega), u_{n_k}(\omega)) > 2^{-k}\right\}. \quad (40)$$

Then

$$\mu(A_k)2^{-k} \leq \int_{A_k} d(u_{n_{k+1}}(\omega), u_{n_k}(\omega))\mathrm{d}w \leq 3^{-k}, \quad (41)$$

and so $\mu(A_k) \leq \left(\frac{2}{3}\right)^k$. Let $A = \bigcap_{m=1}^\infty \bigcup_{k\geq m} A_k$. We observe that

$$\mu\left(\bigcup_{k\geq m} A_k\right) \leq \sum_{k\geq m} \mu(A_k) = \frac{\left(\frac{2}{3}\right)^m}{1-\left(\frac{2}{3}\right)}, \quad (42)$$

and so $\mu(A) \leq 3\left(\frac{2}{3}\right)^m$ for all m, and this implies $\mu(A) = 0$. Now for all $\omega \notin \Omega \setminus A$ there exists $m_0(\omega)$ such that for all $m \geq m_0$ we have $\omega \notin A_m$ and so $d(u_{n_{m+1}}(w), u_{n_m}(\omega)) < 2^{-m}$. This implies that $u_{n_m}(\omega)$ is Cauchy for all $\omega \notin \Omega \setminus A$ and so $u_{n_m}(\omega) \to u(\omega)$ using the completeness of H. This also implies that $u : \Omega \to H$ is measurable, that is, $u \in H^*$. To prove $u_n \to u$ in d_* we have that

$$\begin{aligned} d_*(u_{n_k}, u) &= \int_\Omega d(u_{n_k}(w), u(\omega))\mathrm{d}P(\omega) \\ &= \int_\Omega \lim_{i\to+\infty} d(u_{n_k}(\omega), u_{n_i}(\omega))\mathrm{d}P(\omega) \\ &\leq \liminf_{i\to+\infty} \int_\Omega d(u_{n_k}(\omega), u_{n_i}(\omega))\mathrm{d}P(\omega) \\ &= \liminf_{i\to+\infty} d(u_{n_k}, u_{n_i}) \leq 3^{-k} \end{aligned} \quad (43)$$

for all k. So $\lim_{k\to+\infty} d_*(u_{n_k}, u) = 0$. Now we have

$$d_*(u_n, u) \leq d_*(u_n, u_{n_k}) + d_*(u_{n_k}, u) \to 0 \quad (44)$$

when $k \to +\infty$. □

Consider now the following operators a^* and ϕ^* on H^*

$$a^*(u, v) = \int_\Omega a(\omega, u(\omega), v(\omega))\mathrm{d}P(\omega) = \mathbb{E}(a(\cdot, u(\cdot), v(\cdot))) \quad (45)$$

$$\phi^*(u) = \int_\Omega \phi(\omega, u(\omega))\mathrm{d}P(\omega) = \mathbb{E}(\phi(\cdot, u(\cdot))). \quad (46)$$

One can easily prove that a^* and ϕ^* are a bilinear and a linear form on H^*, respectively. Furthermore, the following properties are satisfied

$$|a^*(u, v)| \leq M\|u\|_*\|v\|_* \quad (47)$$

$$a^*(u, u) \geq m\|u\|_*^2. \quad (48)$$

These inequalities imply the existence of a unique $u^* \in H^*$ which is the solution of the averaged equation,

$$\mathbb{E}(a(\cdot, u(\cdot), v(\cdot))) = \mathbb{E}(\phi(\cdot, u(\cdot))), \quad \text{for all } v \in H^*.$$

Using the approach developed in the previous section, the inverse problem for a random variational equation can be reduced to an inverse problem for a deterministic variational equation in the Hilbert space H^*. The following example shows an application of this framework to a random PDE with boundary conditions.

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Table 3
Results for the random steady-state diffusion equation of Example 4. Results for the random variables are given as (mean, variance).

Case	N	Minimal collage values	
		λ_0	λ_1
1	10	(1.0051, 0.0144)	(1.9878, 0.0068)
1	30	(0.9785, 0.0297)	(1.9821, 0.0314)
1	100	(1.0318, 0.0168)	(2.0064, 0.0281)
2	10	(1.1650, 2.4400)	(1.2095, 4.1010)
2	30	(0.6946, 0.7788)	(2.3938, 7.6380)
2	100	(0.8480, 1.0350)	(2.0368, 6.1920)

Example 4. We consider the following one-dimensional random steady-state diffusion equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(\omega,x)\frac{\mathrm{d}u}{\mathrm{d}x}\right)=f(\omega,x),\quad 0<x<1, \tag{49}$$

$$u(0)=0, \tag{50}$$

$$u(1)=0. \tag{51}$$

The inverse problem of interest is: given $u(\omega, x)$ and $f(\omega, x)$ on $[0, 1]$, determine an approximation of $\kappa(\omega, x)$. Suppose that

$$\kappa(\omega,x)=\sum_{j=0}^{N}\lambda_j(\omega)x^j, \tag{52}$$

where $\lambda_j(\omega)$ are random variables with unknown means μ_j and variances σ_j^2. We are given data from realizations $u(\omega_i, x)$, $i = 1, \dots, M$, of the random variable $u(\omega, x)$ corresponding to a set of realizations $f(\omega_i, x)$, $i = 1, \dots, M$ of the random variable $f(\omega, x)$. Each realization $u(\omega_i, x)$ is the solution of the following problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{j=0}^{N}\lambda_j(\omega)x^j\frac{\mathrm{d}u}{\mathrm{d}x}\right)=f(\omega_i,x),\quad 0<x<1, \tag{53}$$

$$u(0)=0, \tag{54}$$

$$u(1)=0. \tag{55}$$

Upon treating all realizations, we will have determined a set of parameter realizations of $\lambda_j(\omega_i)$, $i = 1, \dots, M$ and $j = 1, \dots, N$. We then construct the approximations

$$\mu_j \approx \frac{1}{M}\sum_{i=1}^{M}\lambda_j(\omega_i), \tag{56}$$

and

$$\sigma_j^2 \approx \frac{1}{M-1}\sum_{i=1}^{M}(\lambda_j(\omega_i)-\mu_j)^2. \tag{57}$$

As numerical experiment, we replace the constant coefficients of Example 3 by random variables. In order to generate realizations, we assume that

$$f_{\mathrm{true}}(\omega,x)=a_0(\omega)+a_1(\omega)x^4 \quad \text{and} \quad \kappa_{\mathrm{true}}(\omega,x)=\lambda_0(\omega)+\lambda_1(\omega)x^2,$$

where the coefficients $a_0(\omega)$, $a_1(\omega)$, $\lambda_0(\omega)$, and $\lambda_1(\omega)$ are random variables selected from a chosen distribution. For Case 1, the distributions are $\mathcal{N}(-2, 0.05)$, $\mathcal{N}(40, 0.1)$, $\mathcal{N}(1, 0.02)$, and $\mathcal{N}(2, 0.03)$, respectively, while in Case 2, the distributions are $\mathcal{N}(-2, 0.05)$, $\mathcal{N}(40, 0.1)$, $\chi(1)$, and $\chi(2)$, respectively. We generate N realizations, and fit a polynomial target to uniformly sampled data points. Next, we collage code each target and finally calculate the mean parameter values and corresponding mean operator's fixed point. Results are presented in Table 3.

Example 5. We consider the steady-state diffusion equation in two spatial dimensions, setting

$$\kappa_{\mathrm{true}}(\omega,x,y)=1+\lambda_0(\omega)x^2+\lambda_1(\omega)xy,$$

where λ_0 and λ_1 are selected from the distributions $\mathcal{N}(1, 0.05)$, and $\mathcal{N}(2, 0.04)$, respectively. We set the true solution $u(x, y) = \sin(\pi x)\sin(2\pi y)$, and generate N realizations, treating each one as a target. Working in V_3^1, upon collage coding each target, we calculate the mean parameter values. The results are presented in Table 4.

Table 4
Results for the random steady-state diffusion equation of Example 5. Results for the random variables are given as (mean, variance).

N	Minimal collage values	
	$\lambda_0(\omega)$	$\lambda_1(\omega)$
10	(0.9965, 0.0822)	(1.9267, 0.0307)
30	(1.0045, 0.0477)	(1.9233, 0.0351)
100	(0.9871, 0.0562)	(1.9379, 0.0351)

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