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A generalized collage method based upon the Lax–Milgram functional for solving boundary value inverse problems

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ABSTRACT

Some inverse problems can be cast as approximating a given target function (perhaps the interpolation of observational data points) by the fixed points of an operator equation. The "collage method" refers to the solution strategy of bounding above the true approximation error by a more readily minimizable quantity. In this paper, we present a collage method for solving inverse problems for boundary value problems. The approach is based upon the Lax–Milgram representation theorem. For applications, we focus on one- and two-dimensional steady-state reaction–diffusion equations.

1. Introduction

Inverse problems in many different frameworks can be formulated as approximating a target element x by the fixed point $\bar{x}$ of an operator T on a complete metric space (X, d). The inverse problem requires one to minimize the approximation error $d(x, \bar{x})$. It is typically the case that the choice of T is restricted to a particular family of maps T_λ, $\lambda \in \Lambda$, where Λ is a space of parameters.

In fractal imaging, where the goal is to approximate a target image by the fixed point (image) of a contractive fractal transform, the tools are Banach's fixed point theorem and Barnsley's collage theorem [1–4]. With (X, d) a complete metric space, the conclusion of the collage theorem states that for any $x \in X$

$$d(x, \bar{x}) \leq \frac{1}{1-c} d(x, Tx), \tag{1}$$

where T has contraction factor c and fixed point $\bar{x} \in X$. With this inequality, we can shift the problem from minimizing the true approximation error to minimizing instead the so-called collage distance $d(x, Tx)$, subject to the constraint that c is bounded away from 1. In the fractal imaging framework, one typically chooses the fractal transform to be a linear function of its unknown parameters. When working with the $\mathcal{L}^2$ metric, the collage distance then becomes a quadratic function of these parameters, which is readily minimized by solving the first derivative equations.

These ideas have been extended to inverse problems in ordinary differential equations [5,6]. In this setting, the contractive Picard operator plays the role of T and the space X contains continuous and appropriately bounded functions on a closed interval of observation. Given a target function, perhaps the interpolation of observational data points, the

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collage theorem can be applied to find the Picard operator within a prescribed class that minimizes the collage distance. The machinery of this framework has also been applied to two-point boundary value problems [7,8], quasilinear partial differential equations [9], and more.

We now turn to a fundamental class of inverse problems for boundary value problems. Our model problem, which arises in steady-state heat or fluid flow, is

$$-\nabla \cdot (\kappa(x)\nabla u(x)) = f(x), \quad \text{where } x \in \Omega \subset \mathbb{R}^n, \tag{2}$$

with appropriate boundary conditions. In the heat flow setting, the function $\kappa(x)$ is the diffusivity and $f(x)$ is a source or sink term. The inverse problem seeks to determine $\kappa(x)$ (or an approximation of it) from a set of observation values of $u(x)$ [10, 11]. (Of course, we assume that conditions for existence and uniqueness of solutions to the forward problem are satisfied.) For $n > 1$, most practical methods of solving this inverse problem begin with its weak or variational form, obtained by integrating both sides with respect to elements of a suitable set of basis functions, for example, the finite element "hat" functions. This integration process produces a linear system of equations in the coefficients λ_k of κ with respect to this basis. The solution of this system is typically replaced by a minimization problem involving an appropriate least squares functional and, possibly, an additional penalty functions for the purpose of regularization. This procedure is described nicely in the more recent monograph by Vogel [12].

Our goal is to formulate the inverse problem in a manner analogous to the collage theorem, replacing the minimization of the true error by the minimization of something akin to the collage distance. In place of Banach's fixed point theorem for contraction maps on a complete metric space, we appeal to the Lax–Milgram representation theorem. Loosely, upon multiplying both sides of (2) by an element $v \in H$, where H is a suitable Hilbert space of functions, for example $H_0^1(\Omega)$, we integrate over Ω to obtain the equation

$$a(u, v) = \phi(v), \quad v \in H, \tag{3}$$

where $\phi(v) = \int_\Omega f v \, dx$ and $a(u, v)$ is a bilinear form on $H \times H$. The Lax–Milgram representation theorem states that under certain conditions on a, (3) has a unique fixed point $\bar{u} \in H$ such that (3) holds for all $v \in H$. (While we should write "solution" in place of "fixed point", we choose the latter description to stay connected to the philosophy that motivates this work.) With this result in mind, we can reformulate our inverse problem in terms of a fixed point equation: minimize the distance between a target u (generated via observation) and the fixed point $\bar{u}$ by making the best choice of the parameters which define κ. Of course, our goal is to replace the minimization of the true approximation error by the minimization of a functional with which we can work more easily. That is, we avoid solving the boundary value problem for u_λ and then minimizing the true error $d(u, u_\lambda)$ for the best λ value.

In Section 2, we offer a recap of the the mathematics surrounding the Lax–Milgram representation theorem. Then, in a new section, we formulate the framework for solving our inverse problem, as we have described briefly above. We will refer to the main theorem as a generalized collage theorem, even though the trappings of the true collage theorem (contraction maps, Banach's theorem) are not at all involved. But the spirit and philosophy of the approach is rooted in similar thinking. Finally, we develop some examples to demonstrate how the method can be used to solve boundary value inverse problems.

2. Background

We start by defining the notion of a bounded linear function on a Hilbert space H.

Definition 1. A linear functional on a real Hilbert space H is a linear map from H to $\mathbb{R}$. A linear functional ϕ is bounded, or continuous, if there exists a constant M such that

$$|\phi(x)| \leq M\|x\|$$

for all $x \in H$.

Note that by the linearity of ϕ it is trivial to prove that we may choose

$$M = \max_{x \in H, \|x\|=1} \phi(x).$$

We next state the Riesz representation theorem.

Theorem 2 (*Riesz Representation*). *Let H be a Hilbert space and ϕ be a bounded linear functional; then there is a unique $x \in H$ such that $\phi(y) = \langle x, y\rangle$ for all $y \in H$, where $\langle \cdot, \cdot \rangle$ denotes an inner product in H.*

The Lax–Milgram representation theorem, the key result used in building the work in this paper, follows. This theorem is a generalization of the Riesz representation theorem to more general quadratic forms.

Theorem 3 (*Lax–Milgram Representation*). *Let H be a Hilbert space and ϕ be a bounded linear nonzero functional. Suppose that $a(u, v)$ is a bilinear form on $H \times H$ which satisfies: there exist constants*

- *$M > 0$ such that $|a(u, v)| \leq M\|u\|\|v\|$ for all $u, v \in H$, and*
- *$m > 0$ such that $|a(u, u)| \geq m\|u\|^2$ for all $u \in H$.*

Then there exists a unique $\bar{u} \in H$ such that $\phi(v) = a(\bar{u}, v)$ for all $v \in H$.

Both the one- and two-dimensional steady-state diffusion equations can be cast into the framework of the Lax–Milgram theorem. To help understand the examples that will follow later, we present these formulations in the following example.

Example 4. We start with the two-dimensional steady-state diffusion equation on the unit square. Let $\Omega = \{0 < x, y < 1\}$, $\bar{\Omega}$ represent its closure, and $\partial\Omega$ its boundary. We consider the equation

$$-\nabla \cdot (\kappa(x, y)\nabla u(x, y)) = f(x, y), \ (x, y) \in \Omega \tag{4}$$
$$u(x, y) = 0, \ (x, y) \in \partial\Omega,$$

where the diffusivity $\kappa(x, y)$ varies in both x and y. Multiply (4) by a test function $v(x, y) \in H_0^1([0, 1]^2)$, the Hilbert space built with all L^2 functions that have a weak derivative in L^2. Next, integrate over Ω to get, suppressing the dependence upon x and y,

$$\begin{aligned}\iint_\Omega f v \,\mathrm{d}A &= -\iint_\Omega \nabla \cdot (\kappa \nabla u)\, v \,\mathrm{d}A \\ &= -\iint_\Omega (\nabla\kappa \cdot \nabla u)v, \mathrm{d}A - \iint_\Omega \kappa v \nabla^2 u, \mathrm{d}A.\end{aligned} \tag{5}$$

Upon application of Green's first identity, with $\hat{n}$ denoting the outward unit normal to $\partial\Omega$, (5) becomes

$$\iint_\Omega f v \,\mathrm{d}A = \iint_\Omega \kappa \nabla v \cdot \nabla u \,\mathrm{d}A - \int_{\partial\Omega} \kappa v (\nabla u \cdot \hat{n}) \,\mathrm{d}s, \tag{6}$$

where the boundary integral vanishes because $v \in H_0^1([0, 1]^2)$. Now, (6) can be written as $a(u, v) = \phi(v)$, with

$$a(u, v) = \iint_\Omega \kappa \nabla v \cdot \nabla u \,\mathrm{d}A, \tag{7}$$

$$\phi(v) = \iint_\Omega f v \,\mathrm{d}A. \tag{8}$$

The one-dimensional steady-state diffusion equation can be formulated similarly. For $x \in [0, 1]$, we consider

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x), \quad 0 < x < 1,$$
$$u(0) = 0,$$
$$u(1) = 0.$$

It is well known that on the Hilbert space $H_0^1([0, 1])$ this problem can be reformulated in a variational form as

$$\int_0^1 \kappa(x)u'(x)v'(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x, \tag{9}$$

for all $v \in H_0^1([0, 1])$. In terms of the notation used above, this equation can be rewritten in the form

$$a(u, v) = \phi(v).$$

In either case, by the Lax–Milgram theorem, we have the well-known result that the two problems we have presented have a unique solution in $H_0^1([0, 1]^2)$ and $H_0^1[0, 1]$, respectively.

3. The generalized collage theorem

We formulate our inverse problem. For a given Hilbert space H, suppose that we have (i) a target element $u \in H$, and (ii) a family of bilinear functionals $a_\lambda(u, v) : \Lambda \times H \times H \mapsto \mathbb{R}$ that satisfy the hypotheses of the Lax–Milgram theorem. Then, by the theorem, for each $\lambda \in \Lambda$ there exists a unique fixed point $\bar{u}_\lambda$ such that $\phi(v) = a_\lambda(\bar{u}_\lambda, v)$ for all $v \in H$. We would like to find the value of λ such that $\|u - \bar{u}_\lambda\|$ is as small as possible; that is, we wish to find

$$\min_{\lambda\in\Lambda} \|u - \bar{u}_\lambda\|.$$

As in our introductory discussion of inverse problems in other frameworks, minimizing this true approximation error is not appealing. Motivated by the collage theorem, our goal is to produce an inequality that allows us to shift our attention to a different minimization problem. We present the generalized collage theorem.

Theorem 5 (*Generalized Collage Theorem*). *Suppose that*

$$a_\lambda(u, v) : \Lambda \times H \times H \mapsto \mathbb{R}$$

is a family of bilinear forms satisfying the hypotheses of the Lax–Milgram theorem for all $\lambda \in \Lambda$, with positive constants M_λ and m_λ. Let $\phi : H \mapsto \mathbb{R}$ be a given linear functional. Let $\bar{u}_\lambda$ denote the solution of the equation $a_\lambda(u, v) = \phi(v)$ for all $v \in H$, as guaranteed by the Lax–Milgram theorem. Then for all $u \in H$, we have

$$\|u - \bar{u}_\lambda\| \le \frac{1}{m_\lambda} F(\lambda),$$

where

$$F(\lambda) = \sup_{v \in H,\ \|v\|=1} |a_\lambda(u, v) - \phi(v)| . \tag{10}$$

Proof. For each λ we know, by the Lax–Milgram theorem, there exists a unique $\bar{u}_\lambda \in H$ such that $a_\lambda(v, \bar{u}_\lambda) = \phi(v)$ for all $v \in H$. We then have

$$\begin{aligned} m_\lambda \|u - \bar{u}_\lambda\|^2 &\le |a_\lambda(u - \bar{u}_\lambda, u - \bar{u}_\lambda)| \\ &= |a_\lambda(u - \bar{u}_\lambda, u) - a_\lambda(u - \bar{u}_\lambda, \bar{u}_\lambda)| \\ &= |a_\lambda(u - \bar{u}_\lambda, u) - \phi(u - \bar{u}_\lambda)| \end{aligned}$$

so that

$$\|u - \bar{u}_\lambda\| \le \left(\frac{1}{m_\lambda}\right) \sup_{v \in H,\ \|v\|=1} |a_\lambda(u, v) - \phi(v)|, \tag{11}$$

giving the result. □

Now, if

$$m = \inf_{\lambda \in \Lambda} m_\lambda > 0, \tag{12}$$

we can replace the minimization of the true error by the minimization of $F(\lambda)$; namely, we then wish to find

$$\min_{\lambda \in \Lambda} F(\lambda).$$

That is, letting $\langle e_i \rangle \subset H$ be a basis of the Hilbert space H, not necessarily orthogonal, we have that each element $v \in H$ can be written as $v = \sum_i \alpha_i e_i$. Computing, we have for all $v \in H$,

$$\begin{aligned} |a_\lambda(u, v) - \phi(v)|^2 &= \left| a_\lambda \left(u, \sum_i \alpha_i e_i \right) - \phi \left(\sum_i \alpha_i e_i \right) \right|^2 \\ &\le \left(\sum_i \alpha_i |a_\lambda(u, e_i) - \phi(e_i)| \right)^2 \\ &\le \left[\sum_i \alpha_i^2 \right] \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2 \right]. \end{aligned}$$

Then, inequality (11) gives

$$\begin{aligned} \inf_{\lambda \in \Lambda} \|u - u_\lambda\| &\le \inf_{\lambda \in \Lambda} \left(\frac{1}{m_\lambda}\right) \left(\sup_{v \in H,\ \|v\|=1} \left[\sum_i \alpha_i^2 \right] \right) \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2 \right] \\ &\le \frac{1}{m} \sup_{v \in H,\ \|v\|=1} \left[\sum_i \alpha_i^2 \right] \inf_{\lambda \in \Lambda} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2 \right]. \end{aligned}$$

Finally, to produce a problem that we can actually solve in general, let $V_n = \langle e_1, e_2, \ldots, e_n \rangle$ be the finite-dimensional vector space generated by e_i, so that $V_n \subset H$. Given a target $u \in H$, let $\Pi_{V_n} u$ be the projection of u on the space V_n. We approximate our minimization problem for the true error by

$$\min_{u_\lambda \in V_n} \|\Pi_{V_n} u - u_\lambda\|. \tag{13}$$

Note that u_λ in (13) is in general not the same as the u_λ in the infinite-dimensional problem. But following the same analysis that led to (11), we have

$$\|\Pi_{V_n} u - u_\lambda\| \le \left(\frac{1}{m_\lambda}\right) \sup_{v \in V_n,\ \|v\|=1} |a_\lambda(\Pi_{V_n} u, v) - \phi(v)|$$

$$\leq \frac{1}{m_\lambda} \max_{v=\sum_{i=1}^{n} \alpha_i e_i \in V_n,\ \|v\|=1} \left[\sum_{i=1}^{n} \alpha_i^2\right] \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]$$

$$\leq \frac{R}{m} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right],$$

where $R = \max_{v \in V_n,\ \|v\|=1} \sum_{i=1}^{n} \alpha_i^2$. So, we have

$$\inf_{\lambda \in \Lambda} \|\Pi_{V_n} u - u_\lambda\| \leq \frac{R}{m} \inf_{\lambda \in \Lambda} \sum_{i=1}^{n} |a_\lambda(u, e_i) - \phi(e_i)|^2 = \frac{R}{m} \inf_{\lambda \in \Lambda} (F_n(\lambda))^2. \tag{14}$$

Thus, our goal is to find

$$\inf_{\lambda \in \Lambda} (F_n(\lambda))^2 . \tag{15}$$

It is worth mentioning that when using the Collage Theorem in (1), one minimizes the collage distance $d(x, Tx)$, disregarding the term involving c, even though the value of c depends upon T. Similarly, in arriving at the minimization problem in (15), we bound the term involving m_λ in order to get a simpler objective function.

4. Applications

Although the one-dimensional steady-state diffusion equation in the earlier example is a simple second-order BVP, the inverse problem which can be treated using other machinery [8], it is instructive to consider precisely this problem as a first illustrative example. As such, we consider

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x), \quad 0 < x < 1, \tag{16}$$

$$u(0) = 0, \tag{17}$$

$$u(1) = 0, \tag{18}$$

where the diffusivity $\kappa(x)$ varies in x. The inverse problem of interest is: given $u(x)$, possibly in the form of an interpolation of data points, and $f(x)$ on $[0, 1]$, determine an approximation of $\kappa(x)$. Of course, one might suggest substituting $u(x)$ and $f(x)$ into (16) to solve for $\kappa(x)$. But this suggestion ignores the approximative nature of the problem and offers little assistance in more complicated settings. In [12], a regularized least squares minimization problem is used to solve this problem by directly minimizing the distance between the given $u(x)$ and the solutions to (16). The generalized collage method allows us to perform a different minimization to solve the inverse problem. A natural goal is to recover $\kappa(x)$ from observations of the response $u(x)$ to a point source $f(x) = \delta(x - x_s)$, a Dirac delta function at $x_s \in (0, 1)$.

For a fixed choice of n, partition $[0, 1]$ at

$$x_i = \frac{i}{n+1}, \quad i = 0, \dots, n+1,$$

with n interior points. Also define

$$V_n^1 = \{v \in C[0, 1] : v \text{ is linear on } [x_{i-1}, x_i],\ i = 1, \dots, n+1 \text{ and } v(0) = v(1) = 0\}.$$

A basis for V_n^1 is the hat functions

$$\xi_i(x) = \begin{cases} (n+1)(x - x_{i-1}), & x_{i-1} \leq x \leq x_i \\ -(n+1)(x - x_{i+1}), & x_i \leq x \leq x_{i+1},\ i = 1, \dots, n. \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that continuous $\kappa(x) > 0$ for all $x \in [0, 1]$, so m in (12) can be chosen as $\min_{x\in[0,1]} \kappa(x)$.

Returning to (9) and setting $v(x) = \xi_i(x) \in H_0^1([0, 1])$, we obtain $a(u, \xi_i) = \phi(\xi_i)$ where

$$a(u, \xi_i) = \int_0^1 \kappa(x) u'(x) \xi_i'(x)\,\mathrm{d}x, \quad \text{and} \tag{19}$$

$$\phi(\xi_i) = \int_0^1 f(x) \xi_i(x)\,\mathrm{d}x. \tag{20}$$

Assume that we are given data points u_i measured at various x-values having no relation to our partition points x_i. Suppose that we fit a polynomial target function $u(x)$ to the data, and express the diffusivity as a polynomial

$$\kappa(x) = \sum_{j=0}^{N} \lambda_j x^j. \tag{21}$$

Table 1
Collage coding results when $f(x) = 5x$, $\kappa_{true}(x) = 1 + \cos(x)$, m data points, target degree P, n basis functions, degree of $\kappa_{collage} = 3$, and Gaussian noise amplitude ε.

m	P	n	ε	λ_1	λ_2	λ_3
200	20	100	0	2.00000	0.00000	−0.50000
200	20	100	0.01	2.01370	−0.30198	−9.75647
200	20	100	0.05	1.98435	−1.75487	−30.72414
100	10	50	0	2.00000	0.00002	−0.50035
100	10	50	0.01	1.72368	2.38409	−12.90815
100	10	50	0.05	2.18033	−16.29861	550.77331
20	4	10	0	1.96180	0.59654	−2.48663
20	4	10	0.01	1.88142	1.64255	−4.97099
20	4	10	0.05	1.58852	5.51423	−13.63244

In essence, this introduces a regularization into our method of solving the inverse problem. Working on V_n^1, we have

$$a_\lambda(u, \xi_i) = \sum_{j=0}^{N} \lambda_j A_{ij}, \quad \text{with } A_{ij} = \int_{x_{i-1}}^{x_{i+1}} x^j u'(x)\xi_i'(x)\,\mathrm{d}x. \tag{22}$$

Letting

$$b_i = \int_0^1 f(x)\xi_i(x)\,\mathrm{d}x = \int_{x_{i-1}}^{x_{i+1}} f(x)\xi_i(x)\,\mathrm{d}x, \quad i = 1, \ldots, n, \tag{23}$$

we now minimize

$$(F_n(\lambda))^2 = \sum_{i=1}^{n} \left[\sum_{j=0}^{N} \lambda_j A_{ij} - b_i \right]^2. \tag{24}$$

Various minimization techniques can be used.

Example 6. We set $f(x) = 5x$ and $\kappa_{true}(x) = 1 + \cos(x)$. We solve the resulting BVP numerically and sample the solution at m data points. We add Gaussian noise of small amplitude ε to these values and then fit a polynomial $u_{target}(x)$ of degree P to the data. Then, given $u_{target}(x)$ and $f(x)$, we seek to recover $\kappa(x)$ in the polynomial form (21), with coefficients λ_j, so that the steady-state diffusion equation admits $u_{target}(x)$ as an approximate solution. Finally, we construct $(F_n(\lambda))^2$ and minimize it with respect to the λ_i. Table 1 presents the results, with coefficients to 5 decimal places. In the top line of the table, the coefficients of the minimal collage κ agree to five decimal places with the corresponding terms of Taylor series of κ_{true}.

Moving to two spatial dimensions, for N_x and N_y fixed natural numbers, define $h_x = \frac{1}{N_x}$ and $h_y = \frac{1}{N_y}$, as well as the N_xN_y nodes inside $[0, 1]^2$:

$$(x_i, y_j) = (ih_x, jh_y), \quad 0 \le i \le N_x,\ 0 \le j \le N_y.$$

The corresponding finite element basis functions $\xi_{ij}(x, y)$ are pyramids with hexagonal bases, such that $\xi_{ij}(x_i, y_j) = 1$ and $\xi_{ij}(x_k, y_l) = 0$ for $k \ne i, l \ne j$. If i or j is 0, the basis function restricted to Ω is only a portion of such a pyramid.

Now, if we expand $\kappa(x, y)$ in this basis, writing

$$\kappa(x, y) = \sum_{k=0}^{N_x} \sum_{l=0}^{N_y} \lambda_{kl}\xi_{kl}(x, y),$$

then

$$a(u, \xi_{ij}) = \sum_{k=0}^{N_x} \sum_{l=0}^{N_y} \lambda_{kl} \left(\iint_\Omega \xi_{kl} \nabla\xi_{ij} \cdot \nabla u \,\mathrm{d}A - \int_{\partial\Omega} \xi_{kl}\xi_{ij}(\nabla u \cdot \hat{n})\,\mathrm{d}s \right).$$

Defining

$$A_{klij} = \iint_\Omega \xi_{kl} \nabla\xi_{ij} \cdot \nabla u \,\mathrm{d}A - \int_{\partial\Omega} \xi_{kl}\xi_{ij}(\nabla u \cdot \hat{n})\,\mathrm{d}s \quad \text{and} \quad b_{ij} = \iint_\Omega f\xi_{ij}\,\mathrm{d}A$$

means that we must minimize

$$\left(F_{N_xN_y}(\lambda)\right)^2 = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} \left[\sum_{k=0}^{N_x} \sum_{l=0}^{N_y} \lambda_{kl} A_{klij} - b_{ij} \right]^2. \tag{25}$$

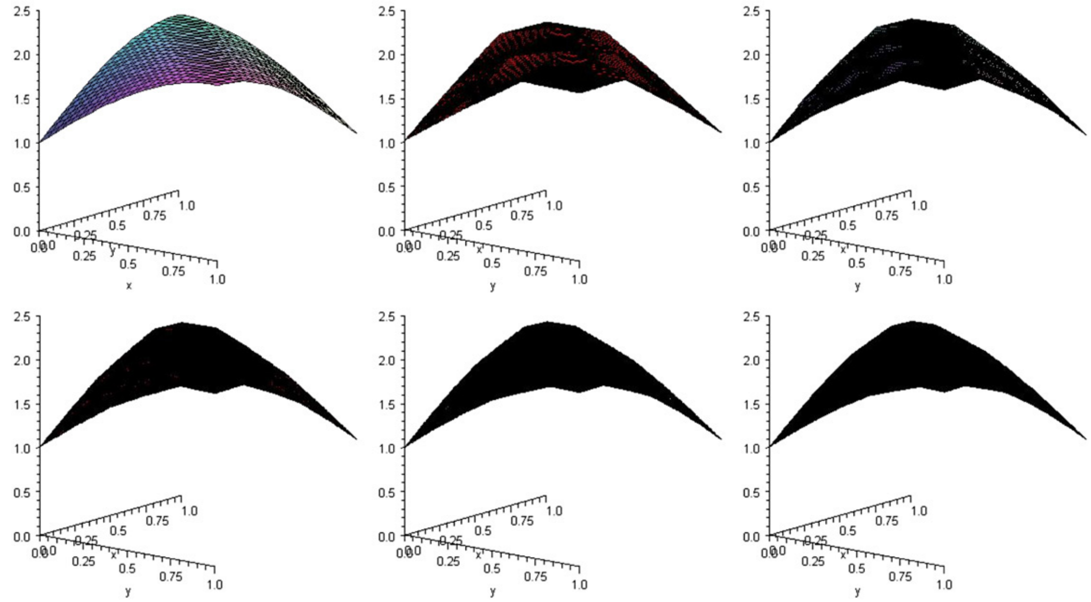

Fig. 1. For Example 7, the graphs of our actual $\kappa(x, y)$ and the collage-coded approximations of κ with $N_x = N_y = 3, 4, 5, 6, 7$ (left to right, top to bottom).

Example 7. Set $u(x, y) = x(1 - x)y(1 - x) \in H_0^1([0, 1]^2)$, and pick $\kappa(x, y) = 1 + \sin\left(\frac{\pi}{2}(x + y)\right)$. We use (4) to determine $f(x, y)$. Our inverse problem is: given $f(x, y)$ and data values of $u(x, y)$, approximate $\kappa(x, y)$. We determined the values of λ_{kl} that minimize (25) by using Maple 11's quadratic program solver. In Fig. 1, we present graphs of our actual $\kappa(x, y)$, as well as the results obtained by minimizing (25) with $N_x = N_y = 3, 4, 5, 6, 7$.

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