Inverse problems for DEs and PDEs using the collage theorem: a survey

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Abstract: In this paper, we present several methods based on the collage theorem and its extensions for solving inverse problems for initial value and boundary value problems. Several numerical examples show the quality of this approach and its stability. At the end we present an application to the Euler-Bernoulli beam equation with boundary measurements.

Keywords: inverse problems; collage theorem; initial value problems; boundary value problems.

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1 Inverse problems for fixed point equations

Many inverse problems or parameter identification problems may be viewed in terms of the approximation of a target element x in a complete metric space (X,d) by the fixed point $\bar{x}$ of a contraction mapping $T: X \to X$. Thanks to a simple consequence of *Banach's fixed point theorem* known as the *collage theorem*, most practical methods of solving the inverse problem for fixed point equations seek to find an operator T for which the *collage distance* $d(x,Tx)$ is as small as possible.

Theorem 1 [collage theorem (Barnsley, 1989)]: Let (X,d) be a complete metric space and $T: X \to X$ a contraction mapping with contraction factor $c \in [0,1)$. Then for any $x \in X$,

$$d(x,\bar{x}) \leq \frac{1}{1-c} d(x,Tx), \tag{1}$$

where $\bar{x}$ is the fixed point of T.

One now seeks a contraction mapping T that minimises the so-called *collage error* $d(x,Tx)$ – in other words, a mapping that sends the target x as close as possible to itself. This is the essence of the method of *collage coding* which has been the basis of most, if not all, fractal image coding and compression methods. Barnsley (1989) was the first to see the potential of using the collage theorem above for the purpose of *fractal image approximation* and *fractal image coding* (see also Barnsley et al., 1985; Forte and Vrscay, 1998). However, this method of collage coding may be applied in other situations where contractive mappings are encountered. We have shown this to be the case for inverse problems involving differential equations (see Kunze and Gomes, 2003; Kunze and Crabtree, 2005; Kunze and Vrscay, 1999; Kunze et al., 2004, 2007, 2009a, 2009b, 2010, 2012). Other applications of the collage theorem can be found in Iacus and La Torre (2005a, 2005b), La Torre and Mendivil (2008), La Torre and Vrscay (2009), and La Torre et al. (2009). In practical applications, from a family of contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, one wishes to find the parameter $\bar{\lambda}$ for which the approximation error $d(x, \bar{x}_\lambda)$ is as small as possible. In practical contexts the feasible set is defined to be $\Lambda = \{\lambda \in \mathbb{R}^n : 0 \leq c_\lambda \leq c < 1\}$ which guarantees the contractivity of T_λ for any $\lambda \in \Lambda$. This is the main difference between this approach and the one based on Tikhonov regularisation (see Tychonoff, 1963; Tychonoff and Arsenin, 1977). In the collage approach, the constraint $\lambda \in \Lambda$ guarantees that T_λ is a contraction and, therefore, replaces the effect of the regularisation term in the Tikhonov approach. The following numerical examples show that the method is stable and can be used as an alternative technique for solving inverse problems for different classes of initial and boundary value problems. The collage-based inverse problem can be formulated as an optimisation problem as follows:

$$\min_{\lambda \in \Lambda} d(x, T_\lambda x) \tag{2}$$

This is a non-linear and nonsmooth optimisation programme and the regularity of the objective function strictly depends on the term $d(x, T_\lambda x)$. However, as later sections show, many times the above model (2) can be reduced to a quadratic optimisation programme. Several algorithms can be used to solve it including, for instance, penalisation methods, particle swarm ant colony techniques, and so on. The paper is organised as follows: Section 2 presents how the method works for the case of differential equation, while Section 3 illustrates the case of different families of PDEs, namely elliptic, parabolic and hyperbolic equations. Finally, Section 4 shows an interesting applications to the Euler-Bernoulli beam equation.

2 Inverse problems for DEs by the collage theorem

In Kunze and Vrscay (1999) and subsequent works, the authors showed how collage coding could be used to solve inverse problems for differential equations having the form

$$\begin{cases} \dot{u} = f(t, u), \\ u(0) = u_0, \end{cases} \tag{3}$$

by reducing the problem to the corresponding Picard integral operator associated with it,

$$(Tu)(t) = u_0 + \int_0^t f(s, u(s))\, ds. \tag{4}$$

Let us recall the basic results in the case when f belongs to L^2. Let us consider the complete metric space $C([-\delta, \delta])$ endowed with the usual d_∞ metric and assume that $f(t, u)$ is Lipschitz in the variable u, that is there exists a $K \geq 0$ such that $|f(s, u) - f(s, v)| \leq K|u - v|$, for all $u, v \in \mathbb{R}$. For simplicity we suppose that $u \in \mathbb{R}$ but the same consideration can be developed for the case of several variables. Under these hypotheses T is Lipschitz on the space $C([-\delta, \delta] \times [-M, M])$ for some δ and $M > 0$.

Theorem 2 (Kunze and Vrscay, 1999): The function T satisfies

$$\|Tu - Tv\|_2 \leq c\|u - v\|_2 \tag{5}$$

for all $u, v \in C([-\delta, \delta] \times [-M, M])$ where $c = \delta K$.

Now let $\delta' > 0$ be such that $\delta' K < 1$. In order to solve the inverse problem for (4) we take the L^2 expansion of the function f. Let $\{\phi_i\}$ be a basis of functions in $L^2([-\delta', \delta'] \times [-M, M])$ and consider $f_\lambda(s, u) = \sum_{i=1}^{+\infty} \lambda_i \phi_i(s, u)$. Each sequence of coefficients $\lambda = \{\lambda_i\}_{i=1}^{+\infty}$ then defines a Picard operator T_λ. Suppose further that each function $\phi_i(s, u)$ is Lipschitz in u with constants K_i.

Theorem 3 (Kunze and Vrscay, 1999): Let $K, \lambda \in \ell^2(\mathbb{R})$. Then

$$|f_\lambda(s, u) - f_\lambda(s, v)| \leq \|K\|_2 \|\lambda\|_2 |u - v| \tag{6}$$

for all $s \in [-\delta', \delta']$ and $u, v \in [-M, M]$ where $\|K\|_2 = \left(\sum_{i=1}^{+\infty} K_i^2\right)^{\frac{1}{2}}$ and $\|\lambda\|_2 = \left(\sum_{i=1}^{+\infty} \lambda_i^2\right)^{\frac{1}{2}}$

Given a target solution u, we now seek to minimise the collage distance $\| u - T_\lambda u \|_2$. The square of the collage distance becomes

$$\Delta^2(\lambda) = \|u - T_\lambda u\|_2^2 = \int_{-\delta}^{\delta} \left| u(t) - u_0 - \int_0^t \sum_{i=1}^{+\infty} \lambda_i \phi_i(s, u(s)) ds \right|^2 dt$$

and the inverse problem can be formulated as

$$\min_{\lambda \in \Lambda} \Delta(\lambda), \tag{7}$$

where $\Lambda = \{\lambda \in \ell^2(\mathbb{R}) : \|\lambda\|_2 \|K\|_2 < 1\}$. To solve this problem numerically, let us consider the first n terms of the L^2 basis; in this case the previous problem can be reduced to:

$$\min_{\lambda \in \tilde{\Lambda}} \tilde{\Delta}^2(\lambda) = \min_{\lambda \in \tilde{\Lambda}} \int_{-\delta}^{\delta} \left| u(t) - u_0 - \int_0^t \sum_{i=1}^{n} \lambda_i \phi_i(s, u(s)) ds \right|^2 dt, \tag{8}$$

where $\tilde{\Lambda} = \{\lambda \in \mathbb{R}^n : \|\lambda\|_2 \|K\|_2 < 1\}$. This is a classical quadratic optimisation problem which can be solved by means of classical numerical methods.

Let $\tilde{\Delta}^n_{\text{min}}$ be the minimum value of $\tilde{\Delta}$ over $\tilde{\Lambda}$. This is a non-increasing sequence of numbers (depending on n) and as shown in Forte and Vrscay (1998) it is possible to show that $\liminf_{n \to +\infty} \tilde{\Delta}^n_{\text{min}} = 0$. This states that the distance between the target element and the unknown solution of the differential equation can be made arbitrarily small.

In Kunze et al. (2007), the authors considered the case of inverse problems for random differential equations. This kind of formulation allows to include, in a unique theoretical approach, the effects of noise/random perturbations on the solutions of differential equations and it can be formulated as

$$\begin{cases} \frac{d}{dt} u(\omega, t) = f(t, \omega, u(\omega, t)), \\ u(\omega, 0) = u_0(\omega), \end{cases} \tag{9}$$

where both the vector field f and the initial condition u_0 are random variables defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$. Analogous to the deterministic case, for $X = C([0,T])$ this problem can be reformulated by using the following random integral operator $T : \Omega \times X \to X$:

$$(T_\omega u)(t) = u_0(\omega) + \int_0^t f(s, \omega, u(s))\, ds. \tag{10}$$

Solutions to (9) are fixed points of (10), that is solution of the equation $T_\omega u = u$. We recall that a function $T : \Omega \times X \to X$ is called a *random operator* (in a strict sense) if for any $u \in X$ the function $T(\cdot, u)$ is measurable. The random operator T is said to be continuous/Lipschitz/contractive if, for a.e. $\omega \in \Omega$, we have that $T(\omega, \cdot)$ is continuous/Lipschitz/contractive. A measurable mapping $u : \Omega \to X$ is called a *random fixed point* of the random operator T if u is a solution of the equation

$$T(\omega, u(\omega)) = u(\omega), \quad a.e.\ \omega \in \Omega. \tag{11}$$

In order to study the existence of solutions to such equations, let us consider the space Y of all measurable functions $u : \Omega \to X$. If we define the operator $\tilde{T} : Y \to Y$ as $(\tilde{T}u)(\omega) = T(\omega, u(\omega))$ the solutions of this fixed point equation on Y are the solutions of the random fixed point equation $T(\omega, u(\omega)) = u(\omega)$. The space Y is a complete metric space with respect to the following metric (see Kunze et al., 2007):

$$d_Y(u_1, u_2) = \int_\Omega d_X(u_1(\omega), u_2(\omega)) dP(\omega). \tag{12}$$

Example 1: Suppose that the stochastic process X_t is believed to follow a geometric Brownian motion; then it satisfies the stochastic differential equation

$$dX_t = aX_t\, dt + bX_t\, dW_t, \tag{13}$$

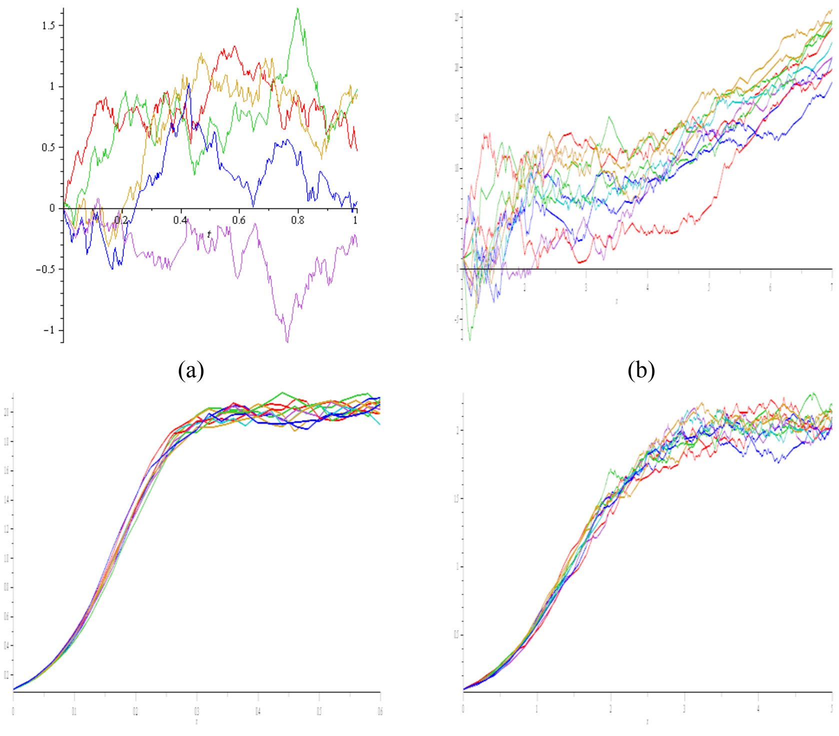
where W_t is a Wiener process and the constants a and b are the percentage drift and the percentage volatility, respectively. We consider the inverse problem: given realisations/paths X_t^i, $1 \le i \le N$, estimate the values a and b. Taking the expectation in (13), we see that $\mathbb{E}(X_t)$ satisfies the simple fixed point equation

$$\mathbb{E}(X_t) = T(\mathbb{E}(X_t)) = X_0 + \int_0^t a\mathbb{E}(X_r)\, dr.$$

Hence, to solve the inverse problem, we construct the mean of the realisations $X_t^* = \frac{1}{N}\sum_{i=1}^N X_t^i$ and use collage coding to determine the value of a that minimises the collage distance $d_2\left(X_t^*, TX_t^*\right)$. We can then estimate the value of b by using the known formula $var(X_t) = e^{2at}X_0^2\left(e^{b^2 t} - 1\right)$, approximating $var(X_t)$ from the realisations.

As an example, we set $a = 2$, $b = 4$, and $X_0 = 1$, and then generate N paths on $[0,1]$, dividing the interval into M subintervals in order to simulate the Brownian motion on $[0,1]$. Beginning with these paths, we seek estimates of a and b using collage coding. Figure 1(a) shows some paths for the Brownian motion and the process X_t.

Figure 1 (a) Different paths of the Brownian motion with M =1,000 and N =300 (b) Different paths of the mean-reverting process (c) Different paths of the stochastic logistic process (d) Different trajectories of the path-following stochastic process



(a) (b) (c) (d)

Table 1 Minimal collage distance parameters for different N and M, to five decimal places

N	M	b	a
100	300	3.78599	1.69618
100	600	4.33750	1.78605
100	1000	4.05374	1.85780
300	300	3.34231	1.80282
300	600	3.54219	1.81531
300	1000	3.84973	1.78323

Table 1 presents the numerical results of the example.

Example 2: Suppose that the stochastic process X_t is driven by a generalised mean-reverting process; then it satisfies the stochastic differential equation

$$dX_t = \left(\frac{g_t - X_t}{t}\right) dt + \frac{1}{t} dW_t, \tag{14}$$

where $g_t = g(t) = g^0 + g^1 t + \dots g^k t^k$ is a given polynomial in the variable t and W_t is a Wiener process. We consider the inverse problem: given realisations/paths X_t^i, $1 \le i \le N$, estimate g_t. Taking the expectation in (14), we see that $\mathbb{E}(X_t)$ satisfies the simple fixed point equation

$$\mathbb{E}(X_t) = T(\mathbb{E}(X_t)) = X_0 + \int_0^t \frac{(g_r - \mathbb{E}(X_r))}{r}\, dr.$$

Hence, to solve the inverse problem, we construct the mean of the realisations $X_t^* = \frac{1}{N}\sum_{i=1}^N X_t^i$ and use collage coding to determine the coefficients of g_t that minimise the collage distance $d_2(X_t^*, TX_t^*)$. As an example, we set $g^0 = 2$, $g^1 = 1$ and $g^2 = 1$, that is $g_t = 2 + t + t^2$, and $X_0 = 1$, and then generate N paths on $[1, 7]$, dividing the interval into M subintervals in order to simulate the stochastic process on $[1, 7]$. Beginning with these paths, we seek to estimate g_t in the form $g^0 + g^1 t + g^2 t^2$ using collage coding. Figure 1(b) shows some paths for this process X_t. Table 2 presents the numerical results of the example.

Table 2 Minimal collage distance parameters for different N and M, to five decimal places

N	*M*	g^0	g^1	g^2
100	300	1.97719	1.01806	0.99729
100	600	1.99763	1.00277	0.99962
100	900	1.99946	1.00163	0.99978
300	300	1.94818	1.02906	0.99638
300	600	1.97382	1.01434	0.99820
300	900	1.98366	1.00790	0.99922

Example 3: Suppose now that the stochastic process X_t is driven by a logistic stochastic process; then it satisfies the stochastic differential equation

$$dX_t = X_t(a - bX_t)\, dt + X_t\, dW_t, \tag{15}$$

where a and b are two parameters and W_t is a Wiener process. We consider the inverse problem: given realisations/paths X_t^i, $1 \le i \le N$, estimate a and b. As an example and following the same approach as above, we set $a = 20$, and $b = 10$ and $X_0 = 0.1$, and then generate paths on $[0, 1]$. Figure 1(c) shows ten paths for this process X_t. Table 3 presents the numerical results of the example.

Example 4: Suppose now that the stochastic process X_t is driven by the path-following stochastic process

$$dX_t = (a + b(\bar{X}_t - X_t))X_t\, dt + X_t\, dW_t, \tag{16}$$

where a and b are two parameters, $\bar{X}_t$ is a given function and W_t is a Wiener process. We consider the inverse problem: given a collection of realisations/paths X_t^i, $1 \le i \le N$, estimate the two parameters. As an example and following the same approach as above, we fix $\bar{X}_t = e^{0.01t}$, set $a = 1.001$, $b = 1$, and $X_0 = 0.1$, and then generate paths on $[0, 1]$. Figure 1(d) shows ten paths for this process X_t. Table 4 presents the numerical results of the example, in which we seek to recover a and b.

Table 3 Minimal collage distance parameters for different N and M, to five decimal places

N	M	a	b
100	300	20.03020	10.01753
100	600	19.99748	9.99778
100	900	20.00240	10.00031
300	300	20.00206	10.00162
300	600	19.99667	9.99889
300	900	19.98866	9.99262

Table 4 Minimal collage distance parameters for different N and M, to five decimal places

N	M	a	b
100	200	1.00326	1.00481
100	600	0.99650	0.99871
100	1000	1.00621	1.00395
300	200	0.98105	0.99478
300	600	0.99259	0.99819
300	1000	0.99526	0.99718

3 Inverse problems for PDEs through the generalised collage theorem

3.1 Elliptic equations

Let us consider the following variational equation,

$$a(u, v) = \phi(v), \qquad v \in H. \tag{17}$$

where $\phi(v)$ and $a(u, v)$ are linear and bilinear maps, respectively, both defined on a Hilbert space H. Let us denote by $\langle \cdot \rangle$ the inner product in H, $\|u\|^2 = \langle u, u \rangle$ and $d(u, v) = \|u - v\|$, for all $u, v \in H$. The existence and uniqueness of solutions to this kind of equation are provided by the classical *Lax-Milgram representation theorem*.

Theorem 4 (Lax-Milgram representation theorem): Let H be a Hilbert space and ϕ be a bounded linear nonzero functional and suppose that $a(u, v)$ is a bilinear form on $H \times H$ which satisfies the following:

1 there exists a constant $M > 0$ s.t. $|a(u, v)| \leq M\|u\|\|v\|$ for all $u, v \in H$

2 there exists a constant $m > 0$ s.t. $|a(u, u)| \geq m\|u\|^2$ for all $u \in H$.

Then there is a unique vector $u^* \in H$ such that $\phi(v) = a(u^*, v)$ for all $v \in H$.

The inverse problem may now be viewed as follows. Suppose that we have an observed solution u and a given (restricted) family of bilinear functionals $a_\lambda(u, v)$, $\lambda \in \mathbb{R}^n$. We now seek to find 'optimal' values of λ.

Suppose that we have a given Hilbert space H, a 'target' element $u \in H$ and a family of bilinear functionals a_λ. Then by the Lax-Milgram theorem, there exists a unique element u_λ such that $\phi(v) = a_\lambda(u_\lambda, v)$ for all $v \in H$. We would like to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|$ is small enough. The following theorem will be useful for the solution of this problem.

Theorem 5 [generalised collage theorem (Kunze et al., 2009a)]: Suppose that $a_\lambda(u, v) : \mathcal{F} \times H \times H \to \mathbb{R}$ is a family of bilinear forms for all $\lambda \in \mathcal{F}$ and $\phi_\lambda : H \to \mathbb{R}$ is a linear functional which satisfy the hypotheses of the Lax-Milgram theorem. Let u_λ denote the solution of the equation $a_\lambda(u, v) = \phi(v)$ for all $v \in H$ as guaranteed by the Lax-Milgram theorem. Given a target element $u \in H$ then

$$\|u - u_\lambda\| \leq \frac{1}{m_\lambda} F(\lambda), \tag{18}$$

where

$$F(\lambda) = \sup_{v \in H,\ \|v\|=1} |a_\lambda(u, v) - \phi(v)|. \tag{19}$$

In order to ensure that the approximation u_λ is close to a target element $u \in H$, we can, by the generalised collage theorem, try to make the term $F(\lambda)/m_\lambda$ as close to zero as possible. The appearance of the m_λ factor complicates the procedure as does the factor $1/(1-c)$ in standard collage coding, i.e., equation (1). If $\inf_{\lambda \in \mathcal{F}} m_\lambda \geq m > 0$ then the inverse problem can be reduced to the minimisation of the function $F(\lambda)$ on the space $\mathcal{F}$, that is,

$$\min_{\lambda \in \mathcal{F}} F(\lambda). \tag{20}$$

The choice of λ according to (20) for minimising the residual is, in general, not stabilising (see Engl and Grever, 1994). However, as the next sections show, under the condition $\inf_{\lambda \in \mathcal{F}} m_\lambda \geq m > 0$ our approach is stable. Following our earlier studies of inverse problems using fixed points of contraction mappings, we shall refer to the minimisation of the functional $F(\lambda)$ as a 'generalised collage method'.

Now let $\langle e_i \rangle \subset H$ be a basis of the Hilbert space H, not necessarily orthogonal, so that each element $v \in H$ can be written as $v = \sum_i \alpha_i e_i$. It can be easily proved that

$$\inf_{\lambda \in \mathcal{F}} \|u - u_\lambda\| \leq \frac{1}{m} \sup_{v \in H,\ \|v\|=1} \left[\sum_i \alpha_i^2 \right] \inf_{\lambda \in \mathcal{F}} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2 \right].$$

Let $V_n =< e_1, e_2, \ldots, e_n >$ be the finite dimensional vector space generated by e_i, $V_n \subset H$. Given a target $u \in H$, let $\Pi_{V_n} u$ be the projection of u on the space V_n and consider the following problem: find $u_\lambda \in V_n$ such that $\|\Pi_{V_n} u - u_\lambda\|$ is as small as possible. We have

$$\|\Pi_{V_n} u - u_\lambda\| \leq \frac{M}{m} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2 \right] \tag{21}$$

where $M = \max_{v = \sum_{i=1}^n \alpha_i e_i \in V_h, \|v\|=1} \sum_{i=1}^n \alpha_i^2$, so we have reduced the problem to the minimisation of the function

$$\inf_{\lambda \in \mathcal{F}} \|\Pi_{V_n} u - u_\lambda\| \leq \frac{M}{m} \inf_{\lambda \in \mathcal{F}} \sum_{i=1}^n |a_\lambda(u, e_i) - \phi(e_i)|^2 = \frac{M}{m} (F_n(\lambda))^2.$$

Example 5: As an application of the preceding method, we consider the following one-dimensional steady-state diffusion equation

$$-\frac{d}{dx}\left(\kappa(x)\frac{du}{dx}\right) = f(x),\ 0 < x < 1, \tag{22}$$

$$u(0) = 0, \tag{23}$$

$$u(1) = 0, \tag{24}$$

where the diffusivity $\kappa(x)$ varies in x. The inverse problem of interest is: given $u(x)$, possibly in the form of an interpolation of data points, and $f(x)$ on $[0,1]$, determine an approximation of $\kappa(x)$. In Vogel (2002) this problem is studied and solved via a regularised least squares minimisation problem. It is important to stress that the approach in Vogel (2002) seeks to directly minimise the error between the given $u(x)$ and the solutions $v(x)$ to equation (22). The collage coding approach allows us to perform a different minimisation to solve the inverse problem. We multiply equation (22) by a test function $\xi_i \in H_0^1([0,1])$ and integrate by parts to obtain $a(u,\xi_i) = \phi(\xi_i)$, where

$$a(u,\xi_i) = \int_0^1 \kappa(x)u'(x)\xi_i'(x)\,dx, \text{ and} \tag{25}$$

$$\phi(\xi_i) = \int_0^1 f(x)\xi_i(x)\,dx. \tag{26}$$

For a fixed choice of n, introduce the following partition of $[0,1]$, $x_i = \frac{i}{n+1}$, $i = 0,\ldots,n+1$, with n interior points, and define for $j = 0,1,2,\ldots$

$$V_n^r = \{v \in C[0,1] : v(0) = v(1) = 0 \text{ and } v \text{ is a polynomial of degree } r \text{ on } [x_{i-1},x_i]\,,\ i = 1,\ldots,n+1\}\,.$$

Denote a basis for V_n^r by $\{\xi_1,\ldots,\xi_n\}$. When $r = 1$, our basis consists of the hat functions

$$\xi_i(x) = \begin{cases} (n+1)\,(x-x_{i-1})\,, x_{i-1} \le x \le x_i \\ -(n+1)\,(x-x_{i+1})\,, x_i \le x \le x_{i+1}\,,\ i = 1,\ldots,n, \\ 0, \text{otherwise} \end{cases}$$

and when $r = 2$, our hats are replaced by parabolae, with

$$\xi_i(x) = \begin{cases} (n+1)^2\,(x-x_{i-1})\,(x-x_{i+1})\,, x_{i-1} \le x \le x_{i+1} \\ 0, \text{otherwise} \end{cases},$$

where $i = 1,\ldots,n$. Suppose that $\kappa(x) > 0$ for all $x \in [0,1]$. Then the m_λ in our formulation, which we denote by m_κ, can be chosen equal to $\inf_{x\in[0,1]} \kappa(x)$. In fact, we have $a(u,u) \ge m_\kappa \|u\|_{H_0^1}^2$, where the norm on H_0^1 is defined by the final equality. As a result, because we divide by m_κ, we expect our results will be good when $\kappa(x)$ is bounded away from 0 on $[0,1]$. Assume that we are given data points u_i measured at various x-values having no relation to our partition points x_i. These data points are interpolated to produce a continuous target function $u(x)$, a polynomial, say. Let us now assume a polynomial representation of the diffusivity, i.e., $\kappa(x) = \sum_{j=0}^{N} \lambda_j x^j$. In essence, this introduces a regularisation into our method of solving the inverse problem. Working on V_n^r, we have

$$a_\lambda(u, \xi_i) = \sum_{j=0}^{N} \lambda_j A_{ij}, \text{ with } A_{ij} = \int_{x_{i-1}}^{x_{i+1}} x^j u'(x) \xi_i'(x)\, dx. \tag{27}$$

Letting

$$b_i = \int_0^1 f(x)\xi_i(x)\, dx = \int_{x_{i-1}}^{x_{i+1}} f(x)\xi_i(x)\, dx, \ i = 1, \ldots, n, \tag{28}$$

we now minimise

$$(F_n(\lambda))^2 = \sum_{i=1}^{n} \left[\sum_{j=0}^{N} \lambda_j A_{ij} - b_i \right]^2. \tag{29}$$

Various minimisation techniques can be used; in this work we used the quadratic programme solving package in Maplesoft's Maple. As a specific experiment, consider $f(x) = 8x$ and $\kappa_{true}(x) = 2x + 1$, in which case the solution to the steady-state diffusion equation is $u_{true}(x) = x - x^2$. We shall sample this solution at 10 datapoints, add Gaussian noise of small amplitude ε to these values and then fit the data points to a polynomial of degree 2, to be denoted as $u_{target}(x)$. Given $u_{target}(x)$ and $f(x)$, we seek a degree 10 polynomial $\kappa(x)$ with coefficients λ_i so that the steady-state diffusion equation admits $u_{target}(x)$ as an approximate solution. We now construct $F_{30}(\lambda)$ and minimise it with respect to the λ_i. Table 5 presents the results. In all cases, the recovered coefficients for all terms of degree two and higher are zero to five decimal places, so we do not report them in the table. d_2 denotes the standard $\mathcal{L}^2$ metric on $[0, 1]$.

Example 6: We extend our work to an inverse problem for the two-dimensional steady-state diffusion equation. With $D = \{0 < x, y < 1\}$,

$$-\nabla \cdot (\kappa(x, y)\nabla u(x, y)) + q(x, y)u(x, y) = f(x, y), \ (x, y) \in D, \tag{30}$$

$$u(x, y) = 0, \ (x, y) \in \partial D, \tag{31}$$

where the diffusivity $\kappa(x, y)$ and radiativity $q(x, y)$ vary in both x and y. Given $u(x, y)$, $q(x, y)$, and $f(x, y)$ on $[0, 1]^2$, we wish to find an approximation of $\kappa(x, y)$. We multiply equation (30) by a test function $\xi_{ij}(x, y) \in H_0^1([0, 1]^2)$ and then integrate over D to get, suppressing the dependence upon x and y,

$$\int_D f\xi_{ij}\, dA = -\int_D (\nabla\kappa \cdot \nabla u)\xi_{ij}, dA - \int_D \kappa\xi_{ij}\nabla^2 u, dA + \int_D qu\xi_{ij}\, dA. \tag{32}$$

Table 5 Collage coding results when $f(x) = 8x$, $\kappa_{true}(x) = 1 + 2x$, data points = 10, number of basis functions = 30, and degree of $\kappa_{collage} = 10$. In the first four rows, we work on V^1_{30}; in the last four rows, we work on V^2_{30}. In each case, $F_{30}(\lambda)$ is equal to 0 to 10 decimal places

Noise ε	$d_2(u_{true}, u_{target})$	$\kappa_{collage}$	$d_2(\kappa_{collage}, \kappa_{true})$
0.00	0.00000	$1.00000 + 2.00000x$	0.00000
0.01	0.00353	$1.03050 + 2.05978x$	0.06281
0.05	0.01770	$1.17365 + 2.33952x$	0.35712
0.10	0.03539	$1.42023 + 2.81788x$	0.86213
0.00	0.00000	$1.00000 + 2.00000x$	0.00000
0.01	0.00353	$1.00832 + 2.03967x$	0.03040
0.05	0.01770	$1.03981 + 2.21545x$	0.16011
0.10	0.03539	$1.07090 + 2.48292x$	0.34301

Upon application of Green's first identity, with $\hat{n}$ denoting the outward unit normal to ∂D, (32) becomes

$$\int_D f\xi_{ij}\, dA = \int_D \kappa\nabla\xi_{ij}\cdot\nabla u dA - \int_{\partial D}\kappa\xi_{ij}(\nabla u\cdot\hat{n})\, ds + \int_D qu\xi_{ij} dA \tag{33}$$

Equation (33) can be written as $a(u, \xi_{ij}) = \phi(\xi_{ij})$, with

$$a(u,\xi_{ij}) = \int_D \kappa\nabla\xi_{ij}\cdot\nabla u\, dA - \int_{\partial D}\kappa\xi_{ij}(\nabla u\cdot\hat{n})\, ds + \int_D qu\xi_{ij}\, dA, \tag{34}$$

$$\phi(\xi_{ij}) = \int_D f\xi_{ij}\, dA. \tag{35}$$

For N and M fixed natural numbers, we define $h_x = \frac{1}{N}$ and $h_y = \frac{1}{M}$, as well as the $(N+1)(M+1)$ nodes in $[0,1]^2$:

$$(x_i, y_j) = (ih_x, jh_y),\ 0 \le i \le N,\ 0 \le j \le M.$$

The corresponding finite element basis functions $\xi_{ij}(x, y)$ are pyramids with hexagonal bases, such that $\xi_{ij}(x_i, y_j) = 1$ and $\xi_{ij}(x_k, y_l) = 0$ for $k \ne i$, $l \ne j$. If i or j is 0, the basis function restricted to D is only a portion of a such a pyramid. Now, if we expand $\kappa(x, y)$ in this basis, writing $\kappa(x,y) = \sum_{k=0}^{N}\sum_{l=0}^{M}\lambda_{kl}\xi_{kl}(x,y)$, then

$$a(u,\xi_{ij}) = \sum_{k=0}^{N}\sum_{l=0}^{M}\left[\lambda_{kl}\left(\int_D \xi_{kl}\nabla\xi_{ij}\cdot\nabla u\, dA - \int_{\partial D}\xi_{kl}\xi_{ij}(\nabla u\cdot\hat{n})\, ds\right) + \int_D qu\xi_{ij}\, dA\right].$$

Defining

$$A_{klij} = \int_D \xi_{kl} \nabla \xi_{ij} \cdot \nabla u \, dA - \int_{\partial D} \xi_{kl} \xi_{ij} (\nabla u \cdot \hat{n}) \, ds$$

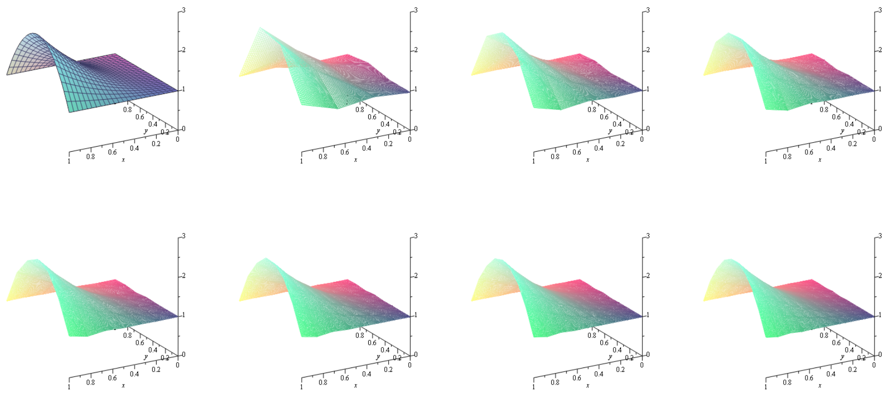
and

$$b_{ij} = \int_D f \xi_{ij} \, dA - \int_D q u \xi_{ij} \, dA$$

means that we must minimise

$$(F_{NM}(\lambda))^2 = \sum_{i=0}^{N} \sum_{j=0}^{M} \left[\sum_{k=0}^{N} \sum_{l=0}^{M} \lambda_{kl} A_{klij} - b_{ij} \right]^2 . \tag{36}$$

Figure 2 For two-dimensional Example 1, the graphs of our actual $\kappa(x, y)$ and the collage-coded approximations of κ with $N = M = 3$ through $N = M = 9$ (left to right, top to bottom)



As a first example, we set $u(x, y) = \sin(\pi x) \sin(\pi y)$, $q(x, y) = 0$, and use the function $\kappa(x, y) = 1 + 6x^2 y(1 - y)$ to determine $f(x, y)$ via equation (30). Now, given the functions $u(x, y)$, $f(x, y)$, and $q(x, y)$, we seek to approximate $\kappa(x, y)$. This inverse problem is treated as Example 3 in Keung and Zou (2000), using a modified Uzawa algorithm. We plug our known functions into equation (36) and find the minimising values of λ_{kl} using Maple's quadratic programme solver. In Figure 2, we present graphs of our actual $\kappa(x, y)$, as well as the results obtained by minimising equation (36) with $N = M = 3, 4, 5$.

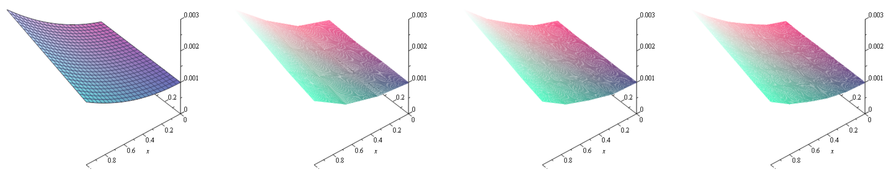
Next, we perturb the target function $u(x, y)$, leaving $f(x, y)$ and $q(x, y)$ exact. Table 6 presents the $\mathcal{L}^2$ error $\|u - u_{noisy}\|$ between the true solution u and the noised target u_{noisy} and the resulting error $\|\kappa - \kappa_{collage}\|$ between the true κ and the collage-coded approximation $\kappa_{collage}$ for numerous cases of N and M.

Table 6 Numerical results for the inverse problem with different levels of noise

$N = M$	$\|\kappa - \kappa_{collage}\|$		
	$u_{noisy} = u$	$\|u - u_{noisy}\| = 0.025$	$\|u - u_{noisy}\| = 0.05$
3	0.06306	0.09993	0.17050
4	0.03480	0.07924	0.15561
5	0.02246	0.07275	0.15128
6	0.01564	0.07118	0.15065
7	0.01160	0.07051	0.15039
8	0.00902	0.07008	0.15014
9	0.00733	0.06981	0.14996

As a second example, let us follow Example 7 of Li and Zou (2007), setting $u(x, y) = \sin(\pi x)\sin(\pi y)$, $q(x, y) = 4 + \cos(\pi xy)$, and $\kappa(x, y) = (1 + x^2 + xy)/1{,}000$. With these choices, we determine the function $f(x, y)$. The inverse problem is to estimate $\kappa(x, y)$ when given $u(x, y)$, $f(x, y)$, and $q(x, y)$. In Figure 3, we present graphs of the results obtained by minimising equation (36) with $N = M = 3$ through $N = M = 5$.

Figure 3 For two-dimensional Example 2, the graphs of our actual $\kappa(x, y)$ and the collage-coded approximations of κ with $N = M = 3, 4, 5$ (left to right)



3.2 Parabolic equations

Suppose that we have a given Hilbert space H and let us consider the following abstract formulation of a parabolic equation

$$\begin{cases} \langle \frac{d}{dt}u, v\rangle_H = \psi(v) + a(u, v) \\ u(0) = f \end{cases} \tag{37}$$

where $\psi : H \to \mathbb{R}$ is a linear functional, $a : H \times H \to \mathbb{R}$ is a bilinear form, and $f \in H$ is an initial condition. The existence and uniqueness of the solution to equation (37) is guaranteed under the same hypotheses on the bilinear form listed in the Lax-Milgram theorem [see Evans (2010) for more details].

The aim of the inverse problem for the above equation consists of getting an approximation of the coefficients and parameters starting from a sample of observations of a target $u \in H$. To do this, let us consider a family of bilinear functionals a_λ and let u_λ be the solution to

$$\begin{cases} \langle \frac{d}{dt} u_\lambda, v \rangle_H = \psi(v) + a_\lambda(u_\lambda, v) \\ u_0 = f \end{cases} \tag{38}$$

We would like to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|_H$ is small enough. To this end, Theorem 6 states that the distance between the target solution u and the solution u_λ of (38) can be reduced by minimising a functional which depends on parameters.

Theorem 6 (Capasso et al., 2009): Let $u : [0, T] \to H$ be the target solution which satisfies the initial condition in (37) and suppose that $\frac{d}{dt} u$ exists and belongs to H. Suppose that $a_\lambda(u, v) : \mathcal{F} \times H \times H \to \mathbb{R}$ is a family of bilinear forms for all $\lambda \in \mathcal{F}$ which satisfy the hypotheses of Lax-Milgram theorem. We have the following result:

$$\int_0^T \|u - u_\lambda\|_H dt \le \frac{1}{m_\lambda^2} \int_0^T \left(\sup_{\|v\|_H = 1} \left\langle \frac{d}{dt} u, v \right\rangle_H - \psi(v) - a_\lambda(u, v) \right)^2 dt$$

where u_λ is the solution of (38) s.t. $\|u - u_\lambda\|_H \Big|_{t=T} = \|u - u_\lambda\|_H \Big|_{t=0}$.

Whenever $\inf_{\lambda \in \mathcal{F}} m_\lambda \ge m > 0$ then the previous result states that in order to solve the inverse problem for the parabolic equation (37) one can minimise the following functional

$$\int_0^T \left(\sup_{\|v\|_H = 1} \left\langle \frac{d}{dt} u, v \right\rangle_H - \psi(v) - a_\lambda(u, v) \right)^2 dt \tag{39}$$

over all $\lambda \in \mathcal{F}$.

Example 7: Let us consider the following equation

$$u_t = (k(x) u_x)_x + g(x, t), \ 0 < x < 1, \tag{40}$$

$$u_0 = 0, \tag{41}$$

$$u_1 = 0, \tag{42}$$

where $g(x, t) = tx(1 - x)$, subject to $u(x, 0) = 10 \sin(\pi x)$ and $u(0, t) = u(1, t) = 0$. We set $k(x) = 1 + 3x + 2x^2$, solve for $u(x, t)$, and sample the solution at N^2 uniformly positioned grid points for $(x, t) \in [0, 1]^2$ to generate a collection of targets. Given this data and $g(x, t)$, we then seek an estimation of $k(x)$ in the form $k(x) = k_0 + k_1 x + k_2 x^2$. The results we obtain through the generalised collage method are summarised in Table 7. As for the elliptic case, the table shows that the method subject to noisy perturbations is stable.

Example 8: In Keung and Zou (1998), Example 1 considers the PDE

$$u_t = (k(x) u_x)_x + g(x, t), \ 0 < x < 1, \tag{43}$$

$$u(x, 0) = 0, \tag{44}$$

$$u(x, 1) = 0. \tag{45}$$

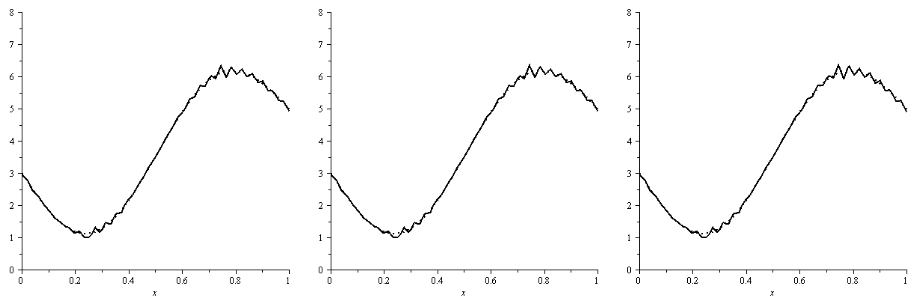
Table 7 Collage coding results for the parabolic equation in Example 6

Noise ε	N	k_0	k_1	k_2
0	10	0.87168	2.90700	0.21353
0	20	0.93457	2.97239	1.49201
0	30	0.94479	2.98304	1.76421
0	40	0.94347	2.97346	1.85572
0.01	10	0.87573	2.82810	0.33923
0.01	20	0.92931	2.91536	1.32864
0.01	30	0.92895	2.84553	0.59199
0.10	10	0.90537	1.97162	0.59043
0.10	20	0.77752	0.92051	–0.77746
0.10	30	0.60504	–0.12677	–0.14565

Table 8 Collage coding results for the parabolic equation in Example 8

Noise ε	$d_2(k(x), k_{collage}(x))$
0	0.0546405525
0.01	0.0551757332
0.1	0.0603784851

Figure 4 For Example 8, the graphs of our actual $k(x)$ and the collage-coded approximations of k with noise level $\varepsilon = 0$, 0.01, and 0.1, respectively (left to right)



Setting $k(x) = 3 + 2x^2 - 2\sin 2\pi x$ and choosing the exact solution $u(x,t) = e^{\sin \pi t} \sin 2\pi x$, the function $g(x,t)$ is determined by the PDE. The authors then use knowledge of u, possibly with noise added, and g to recover an estimate of $k(x)$. Here, we follow suit, sampling $u(x,t)$ at uniformly positioned points (x_i, t_j) in $[0,1]^2$, and adding noise with maximum amplitude ε. Given this data and $g(x,t)$, we then seek an estimation of $k(x)$ in the finite element 'hat' basis, V^1_{50}, defined in Example 5. We construct and minimise the generalised collage distance subject to the constraint that coefficients of k in the hat basis must be sufficiently non-negative (in order to guarantee that m_λ is bounded away from zero). We denote by $k_{collage}(x)$ the resulting piecewise linear k corresponding to the minimised collage distance. Note that we use the data values to approximate both first derivatives of u, as needed in the generalised collage distance.The results are presented in Table 8 and the graphs of $k(x)$ and $k_{collage}(x)$ for different values of ε are illustrated in Figure 4.

3.3 Hyperbolic equations

Let us now consider the following weakly-formulated hyperbolic equation

$$\begin{cases} \langle \frac{d^2}{dt^2}u, v\rangle_H = \psi(v) + a(u, v) \\ u(0) = f \\ \frac{d}{dt}u(0) = g \end{cases} \tag{46}$$

where $\psi : H \to \mathbb{R}$ is a linear functional, $a : H \times H \to \mathbb{R}$ is a bilinear form, and $f, g \in H$ are the initial conditions. Existence and uniqueness for equation (46) is guaranteed by assuming on the bilinear form a the same hypotheses of the Lax-Milgram theorem [see Evans (2010) for more details].

As in previous sections, the aim of the inverse problem for the above system of equations consists of reconstructing the coefficients starting from a sample of observations of a target $u \in H$. We consider a family of bilinear functionals a_λ and let u_λ be the solution to

$$\begin{cases} \langle \frac{d^2}{dt^2}u_\lambda, v\rangle_H = \psi(v) + a_\lambda(u_\lambda, v) \\ u_\lambda 0 = f \\ \frac{d}{dt}u_\lambda(0) = g \end{cases} \tag{47}$$

We would like to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|_H$ is small enough. Theorem 7 states that the distance between the target solution u and the solution u_λ of (47) can be reduced by minimising a functional which depends on parameters.

Theorem 7 (Kunze et al., 2012): Let $u : [0, T] \to H$ be the target solution which satisfies the initial condition in (46) and suppose that $\frac{du}{dt}$ and $\frac{d^2u}{dt^2}$ exist and belong to H. Suppose that $a_\lambda(u, v) : \mathcal{F} \times H \times H \to \mathbb{R}$ is a family of bilinear forms for all $\lambda \in \mathcal{F}$ which satisfy the hypotheses of Lax-Milgram theorem. We have the following result:

$$\int_0^T \|u - u_\lambda\|_H^2 dt \le \frac{1}{m_\lambda^2} \int_0^T \left(\sup_{\|v\|_H=1} \left\langle \frac{d^2}{dt^2}u, v \right\rangle_H - \psi(v) - a(u, v) \right)^2 dt$$

where u_λ is the solution of (47) s.t. $\frac{d}{dt}\|u - u_\lambda\|_H^2 \Big|_{t=T} = \frac{d}{dt}\|u - u_\lambda\|_H^2 \Big|_{t=0}$.

Example 9: We adjust Example 6, considering

$$u_{tt} - (k(x)u_x)_x = g(x, t) \tag{48}$$

where $g(x, t) = tx(1 - x)$, subject to $u(x, 0) = \sin(\pi x)$ and $u_t(x, 0) = 0$ and $u(0, t) = u(1, t) = 0$. We set $k(x) = 1 + 3x + 2x^2$ and construct target data as in Example 6 and then seek to recover $k(x) = k_0 + k_1 x + k_2 x^2$ given this data and $g(x, t)$. The results we get from the generalised collage method are summarised in Table 9, which again proves the stability of the method under noise.

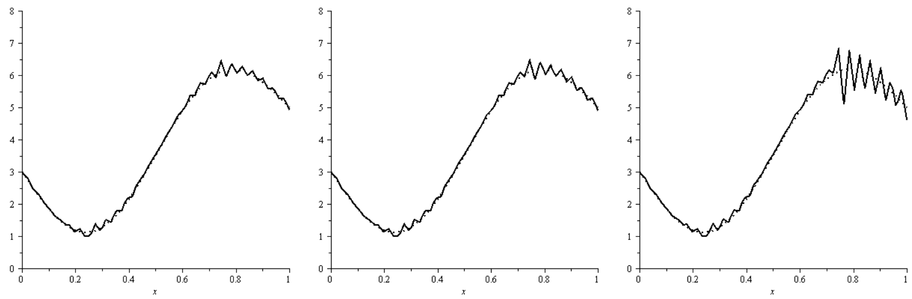
Table 9 Collage coding results for the hyperbolic equation in Example 7

Noise ε	N	k_0	k_1	k_2
0	10	0.87168	2.90700	0.21353
0	20	0.93457	2.97239	1.49201
0	30	0.94479	2.98304	1.76421
0	40	0.94347	2.97346	1.85572
0.01	10	0.87573	2.82810	0.33923
0.01	20	0.92931	2.91536	1.32864
0.01	30	0.92895	2.84553	0.59199
0.10	10	0.90537	1.97162	0.59043
0.10	20	0.77752	0.92051	–0.77746
0.10	30	0.60504	–0.12677	–0.14565

Table 10 Collage coding results for the hyperbolic equation in Example 10

Noise ε	$d_2(k(x), k_{collage}(x))$
0	0.0745607940
0.01	0.0833314175
0.1	0.1934637751

Figure 5 For Example 10, the graphs of our actual $k(x)$ and the collage-coded approximations of k with noise level $\varepsilon = 0$, 0.01, and 0.1, respectively (left to right)



Example 10: We consider the hyperbolic analog of Example 7, namely

$$u_{tt} = (k(x)u_x)_x + g(x,t),\ 0 < x < 1, \tag{49}$$
$$u(x,0) = 0, \tag{50}$$
$$u(x,1) = 0, \tag{51}$$

with $k(x) = 3 + 2x^2 - 2\sin 2\pi x$ and exact solution is $u(x,t) = e^{\sin \pi t} \sin 2\pi x$. We compute the function $g(x,t)$ using the PDE, and proceed as in Example 7. Since we now use the data values to approximate the *second* time derivatives of u, we expect that the quality of the results will decrease compared to Example 7. This change is not very significant, as evidenced by the results in Table 10 and the graphs of $k(x)$ and $k_{collage}(x)$ for different values of ε in Figure 5.

4 The Euler-Bernoulli beam equation with boundary measurements

In Elliot et al. (1999), Lesnic (2006) and Hasanov and Lesnie (2007), an inverse problem for the Euler-Bernoulli beam is considered. The model equation is

$$\begin{aligned}(k(x)u''(x))'' + q(x)u(x) &= f(x), \ x \in [0,1] \\ u(0) &= \alpha_0 \\ u'(0) &= \beta_0 \\ (ku'')(1) &= \gamma_1 \\ (ku'')'(1) &= \delta_1.\end{aligned}$$

In Elliot et al. (1999), the goal is to recover an estimate of the flexural rigidity $k(x)$ knowing $q(x)$, $f(x)$, and values of the deflection $u(x)$ across the beam. In Lesnic (2006), the goal is similar, but we are only given $q(x)$, $f(x)$, and the boundary measurements $u(1)$, $u'(1)$, $(ku'')(0)$, and $(ku'')'(0)$. Each of these inverse problems is treated in Vasiliadis (2010), using the collage theorem for ODEs and the generalised collage theorem. Here, we present the method for the inverse problem for the boundary measurements case using the generalised collage theorem. The PDE can be formulated weakly as $a(u,v) = \psi(v)$, where

$$a(u,v) = \int_0^1 k(x)u''(x)v''(x)\,dx + \int_0^1 q(x)u(x)v(x)\,dx$$

and

$$\begin{aligned}\psi(v) = \int_0^1 f(x)v(x)\,dx &- \left(\delta_0 + \int_0^1 (f(x) - q(x)u(x))\,dx\right) v(1) \\ &+ \left(\gamma_0 + \delta_0 + \int_0^1 (1-x)(f(x) - q(x)u(x))\,dx\right) v'(1).\end{aligned}$$

Given $u \in \bar{H}_2([0,1]) = \{u \in H^2([0,1]) | u(0) = u'(0) = 0\}$ and defining an appropriate family of bilinear functions $a_\lambda(u,v)$, each of the form as the above-given $a(u,v)$, the inverse problem is to find the parameter values λ that minimise the generalised collage distance $\sup_{v \in H^2([0,1]), \|v\|_{H^2([0,1])}=1} |a_\lambda(u,v) - \psi(v)|$. In the case that we are only given boundary measurements, the parameters λ include both the parameters determining the flexural rigidity $k(x)$ and the nuisance parameters for the representation of $u(x)$ on the interior of the beam. When we work in a finite-dimensional subspace $\bar{H}_2([0,1])$ with basis $\{x^i, \ i = 2, \ldots, N\}$, we must minimise

$$\begin{aligned}F_N(\lambda) = \sum_{i=2}^{N} \Bigg[&\int_0^1 k(x)u''(x)i(i-1)x^{i-2}\,dx \\ &+ \int_0^1 (q(x)u(x) - f(x))(x^i - 1 + i(1-x))\,dx + \delta_a(1-i) - \gamma_a i \Bigg]^2.\end{aligned}$$

In a specific example in Lesnic (2006), the true flexural rigidity is chosen as $k_{true}(x) = 1 + x^2$ and the inverse problem seeks to find an estimate of the rigidity of the form $k(x) = 1 + x^{k_0}$. The deflection takes the form $u(x) = \sum_{i=2}^{5} u^i x^i$, introducing four nuisance coefficients. These choices define the family $a_\lambda(u, v)$, with $\lambda = (k_0; u_2, u_3, u_4, u_5)$. We may add Gaussian noise to the four boundary measurements. In this case, the objective function $F_N(\lambda)$ is a complicated function of the parameters, so we choose to use particle swarm ant colony optimisation developed in Shelokar et al. (2007) to minimise it (with a swarm size of 300 ants and equal cognitive and social factors). The results are presented in Table 11.

Table 11 Collage coding results for the beam equation problem in Example 9, with $k_{true}(x) = 1 + x^2$

Relative noise	k_0	$d_2(k_{true}(x), k_{collage}(x))$
0%	1.9968	0.0004051
1%	1.9514	0.0062385
5%	1.7835	0.0292954
10%	1.5879	0.0595464

We repeat the process with $k_{true}(x) = 1 + x + 2x^2$, seeking a $k(x)$ of the form $k_0 + k_1 x + k_2 x^2$. The results of the experiment (also solved using particle swarm ant colony optimisation) are presented in Tables 12.

Table 12 Collage coding results for the beam equation problem in Example 9, with $k_{true}(x) = 1 + x + 2x^2$

Relative noise	k_0	k_1	k_2	$d_2(k_{true}(x), k_{collage}(x))$
0%	1.0232	0.8132	2.2174	0.0185801
1%	0.9801	1.2007	1.7769	0.0188624
5%	1.0401	0.8683	2.2053	0.0500894
10%	0.9777	1.5695	1.4413	0.0869026

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