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Generalized fractal transforms and self-similar objects in cone metric spaces

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ABSTRACT

We use the idea of a scalarization of a cone metric to prove that the topology generated by any cone metric is equivalent to a topology generated by a related metric. We then analyze the case of an ordering cone with empty interior and we provide alternative definitions based on the notion of quasi-interior points. Finally we discuss the implications of such cone metrics in the theory of iterated function systems and generalized fractal transforms and suggest some applications in fractal-based image analysis.

1. Introduction

Properties of contractive mappings are used throughout mathematics, usually to invoke Banach's theorem on fixed points of contractions. In the classical case of iterated function systems (IFSs), the existence of self-similar objects relies on this same theorem. Fundamental ingredients of the theory of IFS are the use of complete metric spaces and the notion of contractivity, which both depend on the definition of the underlying distance. Much recent work has focused on the extension of the notion of metric spaces and the related notion of contractivity. One such extension is the idea of *cone metric spaces*. In this context, the distance is no longer a positive number but a vector, in general an element of a Banach space which has been equipped with an ordering cone. In applications, it is often useful to compare two objects in multiple ways and combine these various comparisons together. Using a cone metric allows this and thus allows a better description of the complexity of the problem. Of course, this implies that many results of the classical theory of metric spaces need to be adapted.

For us, a motivating application of cone metric spaces is image processing, in particular when studying the structural similarity of images. This is a natural situation in which two images are compared using several different criteria, leading to vector-valued distances.

The paper is structured as follows: Section 2 provides the basic definitions and results of cone metric space, and Section 3 shows by scalarization techniques that the topology generated by any cone metric is equivalent to a topology generated by a related metric. In Section 3, we also discuss the case of an ordering cone with empty interior. Section 4 presents a construction of a Hausdorff cone-metric between compact subsets. Finally Section 5 illustrates a relevant application of cone metric spaces in fractal image analysis.

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2. Cone metric space: preliminary properties

We will use $\mathbb{E}$ to denote a Banach space and $P \subset \mathbb{E}$ will be a *pointed cone* in $\mathbb{E}$. This means that P satisfies

1. $0 \in P$,
2. $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$ and $x, y \in P$ implies $\alpha x + \beta y \in P$,
3. $P \cap -P = \{0\}$.

The cone P induces an order in $\mathbb{E}$ by $x \leq y$ if $y - x \in P$ or, said another way, there is some $p \in P$ so that $x + p = y$. The elements of P are said to be *positive* and the elements of the interior of P are *strictly positive*. We assume that P is closed and will also usually assume that $\mathrm{int}(P) \neq \emptyset$. However, in Section 3.2 we are particularly interested in the case when $\mathrm{int}(P)$ is empty. It is easy to show that $p + \mathrm{int}(P) \subseteq \mathrm{int}(P)$ for every $p \in P$. We say that $x \ll y$ if $y - x \in \mathrm{int}(P)$, so $0 \ll x$ means $x \in \mathrm{int}(P)$.

A *pointed wedge* satisfies properties 1 and 2, so every cone is a wedge but not conversely.

A *cone metric* on a set $\mathbb{X}$ is a function $d : \mathbb{X} \times \mathbb{X} \to P$ so that

1. $d(x, y) = 0$ iff $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{X}$.

The idea for defining a cone metric seems to have first appeared in [1] and again, independently, in [2]. The recent interest in this concept was triggered by the paper [3], which also investigated fixed point results in these spaces. Since [3] there has been a large number of other papers on cone metric spaces, extending various standard results and fixed point theorems to cone metric spaces. As we show in the next section (and as others have also shown), the topology of a cone metric space is given by a regular metric and thus the real novelty of cone metric spaces is not in the convergence structure. For us, the novelty is more in using cone metric spaces as a framework for thinking about multiobjective optimization problems.

We say that a sequence (x_n) in $\mathbb{X}$ *converges to x in the cone metric d* if for any $c \in \mathrm{int}(P)$ there is some $N \in \mathbb{N}$ so that for any $n \geq N$ we have $d(x_n, x) \ll c$. Notice the special use of points of the interior of P in this definition. By Proposition 1 we can weaken the requirement slightly and ask only that $d(x_n, x) \leq c$ rather than $d(x_n, x) \ll c$.

Proposition 1. *The sequence $x_n \to x$ in the cone metric d iff for every $c \in \mathrm{int}(P)$ there is some N so that $n \geq N$ implies that $d(x_n, x) \leq c$.*

Proof. One direction is clear since $a \ll b$ implies $a \leq b$.

For the other direction, we first notice that if $a \ll b \leq c$ then $b - a \in \mathrm{int}(P)$ and $c - b \in P$ so $c - a = (c - b) + (b - a) \in \mathrm{int}(P)$ and thus $a \ll c$. Let $c \in \mathrm{int}(P)$ and choose some $c' \in \mathrm{int}(P)$ with $c' \ll c$ (simply select $c' \in B_\delta(c) \subset P$ so that $c - c' \in \mathrm{int}(P)$). Then by assumption there is some N so that $n \geq N$ implies $d(x_n, x) \leq c' \ll c$ and thus $d(x_n, x) \ll c$. □

Example. The most natural example of a cone-metric on $\mathbb{R}^n$ uses the positive cone $P = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \ldots, n\}$. We use $\mathbb{X} = \mathbb{R}^n$ and define the cone metric d by

$$d(x, y) = (|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|). \tag{1}$$

The second and third properties of a cone metric are all clearly satisfied as each component of d satisfies each of these properties. Further $d(x, y) = 0$ iff $x_i = y_i$ for all i and thus $d(x, y) = 0$ iff $x = y$. For this cone metric, $x_n \to x$ in the cone metric d iff $x_n \to x$ in the usual norm on $\mathbb{R}^n$.

3. Convergence in cone metric space

In this section we define the notion of scalarization of a cone metric and investigate some properties. In particular, we show that each scalarization is a pseudometric (in the usual scalar-valued sense) and show that the topology generated by a cone metric is equivalent to that of a standard metric. This vastly simplifies the considerations of topological questions in a cone metric space. In particular, it also shows that, from the point of topology, the class of cone metric spaces is not new. However, this does not mean that there is no reason to consider cone metric spaces, as we discuss in Section 5.

Our results were discovered independently, but other researchers have established similar results [4–7]. In particular, the results in [5] use the same approach as we take leading up to Corollary 1. The paper [4] has a very nice approach to the problem in a general Hausdorff locally convex topological vector space. They use the "nonlinear scalarization function" associated with P, defined as

$$\xi_e(y) = \inf\{r : y \in re - P\},$$

where $e \in \mathrm{int}(P)$. This has the benefit of using the cone directly in the "scalarization" instead of indirectly (via the dual cone) as we do. The paper [6] can be viewed as a nice followup paper to [4].

Let $(\mathbb{X}, d)$ be a cone metric space which takes values in the cone $P \subset \mathbb{E}$. Recall that the *dual wedge* P^* is the set of all $p^* \in \mathbb{E}^*$ such that $p^*(q) \geq 0$ for all $q \in P$. Notice that by this definition P^* is always weak* closed as a subset of $\mathbb{E}^*$. It is possible for P^* not to be a cone, depending on P, but it is always a wedge. As an example, if $\mathbb{E} = \mathbb{R}^2$ and $P = \{(x, 0) : x \geq 0\}$ then $P^* = \{(x, y) : x \geq 0\}$ is not a cone. Let $S = \{p^* \in P^* : \|p^*\| = 1\}$ denote all those elements of P^* of norm one. We note that S is a weak* compact *base* for P^* in that $P^* = \{tp^* : t \geq 0, p^* \in S\}$. For each $p^* \in S$, define d^p by

$$d^p(x, y) = p^*(d(x, y)).$$

Proposition 2. *For each $p^* \in S$ we have d^p is a pseudometric on $\mathbb{X}$ and a metric on $\mathbb{X}$ if $p^* \in S \cap \operatorname{int}(P^*)$.*

Proof. It is immediate from the definition that $d^p(x, y) \geq 0$ and $d^p(x, y) = d^p(y, x)$. Furthermore, we have if $x, y, z \in \mathbb{X}$ then $d(x, y) \leq d(x, z) + d(y, z)$ which means that there is some $q \in P$ so that $d(x, y) + q = d(x, z) + d(z, y)$ so that $p^*(d(x, y)) + p^*(q) = p^*(d(x, z)) + p^*(d(z, y))$ but since $p^*(q) \geq 0$ this means that

$$p^*(d(x, y)) \leq p^*(d(x, z)) + p^*(d(z, y)) \Rightarrow d^p(x, y) \leq d^p(x, z) + d^p(z, y).$$

Thus d^p is a pseudometric for every $p^* \in S$. Note that the same argument works for any $p^* \in P^*$, but it is sufficient to consider $p^* \in S$ since $(\lambda p)^*(d(x, y)) = \lambda p^*(d(x, y))$ for any $\lambda \in \mathbb{R}^+$.

Take $p^* \in \operatorname{int}(P^*)$ (assuming $\operatorname{int}(P^*) \neq \emptyset$). Then we claim that $p^*(q) > 0$ for all nonzero $q \in P$. If not, then $p^*(q) = 0$ for some nonzero $q \in P$. However, then there is some $r^* \in \mathbb{E}^*$ with $r^*(q) = -1$ and $\|r^*\| = 1$. Now as $p^* \in \operatorname{int}(P^*)$, there is some $\delta > 0, \delta \in \mathbb{R}$ with $B_\delta(p^*) \subset P^*$. But then $p^* + \delta/2\, r^* \in P^*$. However, $(p^* + \delta/2\, r^*)(q) = -\delta/2$ which is a contradiction. Thus for all $p^* \in \operatorname{int}(P^*)$ we have $p^*(q) > 0$ for all nonzero $q \in P$. $\square$

Example. We continue the example introduced in the previous section.

In this case, the dual wedge P^* is naturally identified with P. It might be tempting to think of d^p as the "distance between points in the direction of p^*", but this is not correct. If it were correct then d^p would not be a metric for any p as any two points whose projection onto the line $\mathbb{R}\, p^* = \{rp^* : r \in \mathbb{R}\}$ were the same would have zero d^p distance. In fact, for $p^* \in P^*$, we have $p^* = (p_1, p_2, \ldots, p_n)$ with $p_i \geq 0$ and thus

$$d^p(x, y) = p_1|x_1 - y_1| + p_2|x_2 - y_2| + \cdots + p_n|x_n - y_n|. \tag{2}$$

Thus d^p is more correctly thought of as a weighted ℓ^1 norm. Notice that if all the weights are strictly positive (that is, each $p_i > 0$ and so $p^* \in \operatorname{int}(P^*)$), then (2) defines a metric.

Lemma 1. *Suppose that $q \notin P$. Then there is some $p^* \in P^*$ so that $p^*(q) < 0$.*

Proof. Since $q \notin P$, by the Hahn–Banach theorem there is some $r^* \in \mathbb{E}^*$ and $c \in \mathbb{R}$ with $r^*(q) < c$ and $r^*(p) \geq c$ for all $p \in P$. Now, $0 \in P$ which implies that $0 \geq c$, so in fact $r^* \in P^*$. $\square$

Notice that Lemma 1 implies that if $p^*(q) \geq 0$ for all $p^* \in P^*$ then we must have $q \in P$. This is a dual to the fact that $p^*(q) \geq 0$ for all $q \in P$ implies that $p^* \in P^*$.

Lemma 2. *If $x, y \in P$ with $p^*(x) \leq p^*(y)$ for all $p^* \in P^*$ then $x \leq y$.*

Proof. Let $c = y - x$. Then $p^*(c) \geq 0$ for all $p^* \in P^*$ so in fact $c \in P$ and so $x + c = y$ or $x \leq y$. $\square$

Lemma 3. *If $p^* \in P^*$ and $\epsilon > 0$ is such that $B_\epsilon(p^*) \subset P^*$ then $p^*(q) \geq \epsilon$ for all $q \in P$.*

Lemma 4. *Suppose that $c \in \operatorname{int}(P)$. Then for any $p \in P$ there is some $\epsilon > 0$ so that $\epsilon p \leq c$. In fact, we can even arrange it so that $\epsilon p \ll c$.*

Proof. Since $c \in \operatorname{int}(P)$, there is some $\epsilon > 0$ so that $B_{2\epsilon}(c) \subset P$. But then $c - \epsilon p \in B_{2\epsilon}(c) \subset P$ and thus there is some $p' \in P$ with $\epsilon p + p' = c$ or $\epsilon p \leq c$. In fact, $\epsilon p \ll c$. $\square$

Definition 1. We say that $x_n \to x$ in d^p uniformly over $p \in S$ if for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ with $n \geq N$ implying $d^p(x_n, x) < \epsilon$ for all $p \in S$.

Proposition 3. *The sequence $x_n \to x$ in the cone metric d iff $x_n \to x$ in d^p uniformly in p.*

Proof. Let $\epsilon > 0$ be chosen. Choose $c \in \operatorname{int}(P)$ with $\|c\| < \epsilon$. Then there is some N so that for all $n \geq N$ we have $d(x_n, x) \leq c$ which implies that $d^p(x_n, x) \leq p^*(c) \leq \|c\| < \epsilon$ for all $p \in S$.

Conversely, take $c \in \operatorname{int}(P)$. We know that $p^*(c) > 0$ for all $p^* \in P^*$ (by the same argument as in the proof of Proposition 2) in fact $\lambda = \inf_{p^* \in S} p^*(c) > 0$ since S is weak* compact. By the assumption of uniformity, there is some $N \in \mathbb{N}$ so $n \geq N$ implies that $d^p(x_n, x) < \lambda/2 < p^*(c)$ for all $p \in S$ which implies that $d(x_n, x) \leq c$ by Lemma 2. $\square$

Define $\rho(x, y) = \sup_{p \in S} d^p(x, y)$. Then ρ is a metric on $\mathbb{X}$ and convergence in ρ is equivalent to convergence in the cone metric d by Proposition 3. Notice that for $p \in P$ we have $\sup_{p^* \in S} p^*(p) \leq \|p\|$ and thus

$$\rho(x, y) \leq \|d(x, y)\|.$$

In general, the inequality can be strict. While $\sup_{\|q^*\|=1} q^*(p) = \|p\|$ the maximizing q^* is not necessarily in P. The metric ρ is the same metric as defined in [5].

Corollary 1. *For a sequence (x_n) in $(\mathbb{X}, d)$ we have $x_n \to x$ in the cone metric d iff $x_n \to x$ in the metric ρ.*

It is also possible to characterize convergence in a cone metric in a way which is more similar to convergence in a metric space. The next proposition does this by using a fixed $p \in \text{int}(P)$ and an arbitrary $\epsilon > 0$.

Proposition 4. *Let $p \in \text{int}(P)$. Then $x_n \to x$ in the cone metric d iff for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ so that $n \geq N$ implies that $d(x_n, x) \leq \epsilon p$.*

Proof. As $p \in \text{int}(P)$ and $\epsilon > 0$, we know that $\epsilon p \in \text{int}(P)$ as well. So, if $x_n \to x$ in d and $\epsilon > 0$ is given, then there must be some N so that $n \geq N$ implies that $d(x_n, x) \leq \epsilon p$.

For the converse, let $c \in \text{int}(P)$ be given. Then by Lemma 4 there is some $\epsilon > 0$ so that $\epsilon p \leq c$. Now by the assumption there is some $N \in \mathbb{N}$ so $n \geq N$ implies that $d(x_n, x) \leq \epsilon p \leq c$ and thus $x_n \to x$ in d. $\square$

Definition 2. We say that the cone P satisfies *property **B*** if there is some $p^* \in P^*$ so that the set $(p^*)^{-1}([0, 1]) \cap P$ is a norm bounded subset of P.

Notice that if property **B** is satisfied, then $(p^*)^{-1}([0, \lambda]) \cap P$ is norm bounded for any $\lambda \geq 0$ by the linearity of p^* and the fact that P is a cone. That is, because

$$\lambda(p^*)^{-1}([0, 1]) = (p^*)^{-1}([0, \lambda])$$

for any $\lambda \geq 0$. The so-called ''ice cream cones'' (see [8]) all satisfy property **B** as do any finite dimensional closed pointed cones.

Lemma 5. *Suppose that property **B** is satisfied for P and $c \in \text{int}(P)$. Then there is some $\epsilon > 0$ so that for all $q \in P$ with $p^*(q) < \epsilon$ we have $q \leq c$.*

Proof. Since $c \in \text{int}(P)$, there is some $\eta > 0$ so that $B_{2\eta}(c) \subset P$. By property **B**, the set $(p^*)^{-1}([0, 1])$ is bounded, say by $M > 0$. Then $(p^*)^{-1}([0, \epsilon])$ is bounded by ϵM. Choose ϵ so that $\epsilon M < \eta$. Then any $q \in (p^*)^{-1}([0, \epsilon])$ satisfies $\|q\| < s\eta$ and so $c - q \in P$ so $q \leq c$. $\square$

Proposition 5. *Suppose that P satisfies property **B**. Then $x_n \to x$ in the cone metric d iff $d^p(x_n, x) \to 0$.*

Proof. If $x_n \to x$ in the cone metric d then $d^p(x_n, x) \to 0$, as seen above.

Conversely, suppose that $d^p(x_n, x) \to 0$. That is, for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ so $n \geq N$ implies that $p^*(d(x_n, x)) \leq \epsilon$. Let $c \in \text{int}(P)$ be given. Then by Lemma 5 there is some $\epsilon > 0$ so that all $q \in P$ with $p^*(q) < \epsilon$ satisfies $q \leq c$. However, then by assumption there is some $N \in \mathbb{N}$ so that $n \geq N$ implies that $p^*(d(x_n, x)) \leq \epsilon$ which implies that $d(x_n, x) \leq c$. Thus $x_n \to x$ in the cone metric d. $\square$

Thus under the condition **B** on the cone P, convergence in the cone metric reduces to convergence in the one particular scalarization, d^p, so the cone metric topology is just a usual metric topology as given by d^p. This is a stronger statement than that of Corollary 1 where the metric involves the supremum of d^p for all $p \in S$, while Proposition 5 involves only one (albeit special) d^p.

Condition **B** is similar to the conditions discussed in the paper [9], in particular Property II, which requires the set $(p^*)^{-1}([0, 1]) \cap P$ to be relatively weakly compact. Norm bounded subsets of reflexive Banach spaces are always relatively weakly compact. Thus for reflexive spaces condition **B** and property II are equivalent.

3.1. Completeness and contractivity

The notion of a Cauchy sequence is defined in the obvious way in a cone metric space $(\mathbb{X}, d)$. It is easy to see that if (x_n) is Cauchy in $(\mathbb{X}, d)$ then for all $p^* \in S$ we have (x_n) is also Cauchy in the pseudometric d^p. We say that $(\mathbb{X}, d)$ is *complete* if every d-Cauchy sequence converges in the cone metric d. By similar arguments as in the proofs of Corollary 1 and Proposition 5, we get the following result.

Proposition 6. *A sequence is Cauchy in $(\mathbb{X}, d)$ iff it is Cauchy in $(\mathbb{X}, \rho)$. Furthermore, if P satisfies property **B** then (x_n) is Cauchy in $(\mathbb{X}, d)$ iff it is Cauchy in $(\mathbb{X}, d^p)$.*

*Thus, $(\mathbb{X}, d)$ is complete iff $(\mathbb{X}, \rho)$ is complete and if P satisfies property **B** then $(\mathbb{X}, d)$ is complete iff $(\mathbb{X}, d^p)$ is complete.*

Definition 3. We say that $T : (\mathbb{X}, d) \to (\mathbb{X}, d)$ is *contractive* if there is some $k \in [0, 1)$ so that for all $x, y \in X$ we have $d(T(x), T(y)) \leq kd(x, y)$.

Proposition 7. *Suppose that T is contractive with contractivity k. Then $\rho(T(x), T(y)) \leq k\rho(x, y)$ as well so that T is ρ-contractive. If P has property **B** with $p^* \in P^*$,then $d^p(T(x), T(y)) \leq kd^p(x, y)$ so T is contractive in the metric d^p as well.*

Proof. The proof is simple and follows easily from the fact that for any $q^* \in P^*$ we have that if $u, v \in \mathbb{E}$ with $u \leq v$ then $p^*(u) \leq p^*(v)$. $\square$

In fact more is true: T is contractive in the cone metric d with contractivity k if and only if T is contractive in ρ with contractivity k.

From the proposition, we easily get the following theorem (which is a slight strengthening of Theorem 1 in [3]). This result has also appeared in various forms in the literature (see [5] and the references therein).

Theorem 1. *Suppose that $(\mathbb{X}, d)$ is a complete cone metric space and $T : (\mathbb{X}, d) \to (\mathbb{X}, d)$ is a contraction. Then T has a unique fixed point.*

Proof. We simply see that T is also a contraction in the complete metric space $(\mathbb{X}, \rho)$ and thus has a unique fixed point. $\square$

A simple corollary to the proof of the Contraction Mapping theorem is the Collage theorem, which is used in the theory of IFS fractal image compression (and other IFS fractal-based methods in analysis) and will feature heavily in our applications in Section 5.

Theorem 2 (*The Collage Theorem*)**.** *Suppose that $(\mathbb{X}, d)$ is a complete cone metric space and T is a contraction on $\mathbb{X}$ with contractivity k and fixed point $\bar{x}$. Then for any $x \in X$, we have*

$$\frac{1}{k+1} d(Tx, x) \leq d(\bar{x}, x) \leq \frac{1}{1-k} d(Tx, x).$$

Proof. We simply see that

$$d(\bar{x}, x) \leq d(\bar{x}, Tx) + d(Tx, x) = d(T\bar{x}, Tx) + d(Tx, x) \leq kd(\bar{x}, x) + d(Tx, x),$$

which leads to the second inequality. The first inequality is obtained from

$$d(x, Tx) \leq d(x, \bar{x}) + d(\bar{x}, Tx) = d(x, \bar{x}) + d(T\bar{x}, Tx) \leq d(x, \bar{x}) + kd(x, \bar{x}). \quad \square$$

The Collage theorem is used in situations where we wish to find a contraction T whose fixed point $\bar{x}$ is "close" to some given x. Instead of minimizing $d(x, \bar{x})$ (which requires knowledge of $\bar{x}$), we can instead minimize $d(Tx, x)$, which is expressed entirely using the given data.

In most applications, the setup is a parameterized family of operators T_λ on $\mathbb{X}$, for $\lambda \in \Lambda \subset \mathbb{R}^s$. Given this and a fixed target element $x \in \mathbb{X}$, we wish to solve the following program

$$\min_{\lambda \in \Lambda} \psi(\lambda) := \min_{\lambda \in \Lambda} d(T_\lambda x, x). \tag{3}$$

Contrasting with the usual case of a real-valued distance, here the objective function $\psi(\lambda)$ assigns values in the Banach space $\mathbb{E}$ (ordered by the pointed cone $P \subset \mathbb{E}$). As is usually the case in vector optimization, a global solution to (3) is a vector $\lambda^* \in \Lambda$ such that $\psi(\lambda) \notin \psi(\lambda^*) - \mathrm{int}(P)$ (see [10]). There are two main approaches to dealing with multicriteria optimization programs, namely using the scalarization techniques and goal programming [10,11]. In this case, proceeding by scalarization, from the Collage theorem we get the estimates

$$\frac{1}{k+1} d^p(x, Tx) \leq d^p(x, \bar{x}) \leq \frac{1}{1-k} d^p(x, Tx)$$

for any $p^* \in S$ and thus

$$\frac{1}{k+1} \rho(x, Tx) \leq \rho(x, \bar{x}) \leq \frac{1}{1-k} \rho(x, Tx)$$

for the metric ρ.

3.2. The case of a cone with empty interior

In our application in Section 5 we will be interested in considering cone metric spaces where perhaps the natural cone P has no interior points. In this situation even the basic definition of convergence needs to be modified. Suppose that P is a closed and pointed cone but with $\mathrm{int}(P) = \emptyset$. As a motivating example, consider $\mathbb{E} = L^2[0, 1]$ and $P = \{f : f(x) \geq 0 \ a.e.\ x\}$. Then clearly $P = P^*$ and $\mathrm{int}(P) = \emptyset$.

As there are no interior points, we must replace the condition using interior points in the definition of convergence with something else. We choose to use the *quasi-interior* points. For this purpose, we assume that $\mathbb{E}$ is separable and reflexive. Then the set of quasi-interior points of P is non-empty and is characterized as (see [12, p. 17])

$$qi(P) = \{p \in P : q^*(p) > 0 \text{ for all } q^* \in P^* \setminus \{0\}\}. \tag{4}$$

We note that if $c \in qi(P)$ and $\epsilon > 0$ then $\epsilon c \in qi(P)$. The property expressed in (4) is the essential property that we used for our discussions in developing the results on scalarization. In fact, the proof of Proposition 2 shows that $\mathrm{int}(P) \subseteq qi(P)$.

Thus we say that $x_n \to x$ in the cone metric d if for all $c \in qi(P)$ we have that eventually $d(x_n, x) \leq c$.

Proposition 2 is independent of whether or not $\mathrm{int}(P) = \emptyset$, so again d^p is a pseudometric on $\mathbb{X}$ for any $p \in P^*$ and a metric if $p^* \in qi(P^*)$. We again define $\rho(x, y) = \sup_{p^* \in S} d^p(x, y)$ and notice that it is a metric.

Proposition 8. *$x_n \to x$ in the cone metric if and only if $\rho(x_n, x) \to 0$.*

Proof. Suppose that $x_n \to x$ in the cone metric and let $\epsilon > 0$ be given. Choose some $c \in qi(P)$, then $e = c(\epsilon/2\|c\|) \in qi(P)$ and so eventually $d(x_n, x) \leq e$ which implies that for all $p^* \in S$ we have

$$d^p(x_n, x) = p^*(d(x_n, x)) \leq p^*(e) = \epsilon/2 \Rightarrow \rho(x_n, x) < \epsilon.$$

Conversely, suppose that $\rho(x_n, x) \to 0$ and let $c \in qi(P)$. Then $p^*(c) > 0$ for all $p^* \in S$. In fact, since S is weak* compact, there is some $\lambda > 0$ so that $p^*(c) \geq \lambda > 0$ for all $p^* \in S$. Choose $\epsilon = \lambda/2$. Then eventually $\rho(x_n, x) < \epsilon$ which implies that

$$p^*(d(x_n, x)) \leq \rho(x_n, x) < \epsilon < p^*(c) \quad \text{for all } p^* \in S \Rightarrow d(x_n, x) \leq c. \quad \square$$

The notion of contraction does not change and so again we obtain the corresponding fixed point result.

Proposition 9. *Suppose that $(\mathbb{X}, d)$ is a complete cone metric space with $\mathrm{int}(P) = \emptyset$ and T is a contraction on $\mathbb{X}$. Then T has a unique fixed point.*

Clearly we also obtain the Collage theorem in this case, which we can either express in terms of the cone metric or in terms of ρ. As the ordering cone P in the Banach space $\mathbb{E}$ now has empty interior, we need to modify the notion of a global solution to the minimization problem (3). In this case we have: a vector $\lambda^* \in \Lambda$ is a global solution if $\psi(\lambda) \notin \psi(\lambda^*) - qi(P)$ for all $\lambda \in \Lambda$. That is, we replace the interior with the quasi-interior. Again we can approach such vector optimization problems by means of scalarization and thus solve a family of scalar optimization problems instead of the one vector optimization problem.

4. A Hausdorff cone metric

In this section we will again let $(\mathbb{X}, d)$ be a cone metric space which takes values in the cone $P \subset \mathbb{E}$. We continue our assumption that $qi(P^*) \neq \emptyset$ (so assuming that $\mathbb{E}$ is separable and reflexive is enough).

Let $\mathbb{H}(\mathbb{X})$ denote the collection of all nonempty and compact subsets of $\mathbb{X}$. Our purpose in this section is to define on $\mathbb{H}(\mathbb{X})$ a cone metric analogue of the usual Hausdorff distance between sets. The particular choice we make is natural and has some nice properties. The Banach space in which our new cone metric takes values is not $\mathbb{E}$.

Recalling that $S = \{p^* \in P^* : \|p^*\| = 1\}$, we define

$$\mathbb{F} = \{\text{all bounded } f : S \to \mathbb{R}\}$$

and note that $\mathbb{F}$ is a Banach space under the norm $\|f\| = \sup_x |f(x)|$. The cone $\mathbb{P} \subset \mathbb{F}$ is defined by $f \in \mathbb{P}$ whenever $f(p^*) \geq 0$ for all $p^* \in S$. The zero element of $\mathbb{F}$ is the zero function, which is included in $\mathbb{P}$. Notice that $\mathrm{int}(\mathbb{P})$ is not empty even if $\mathrm{int}(P)$ is empty.

Definition 4. Let $A, B \in \mathbb{H}(\mathbb{X})$. Define $d_H(A, B) \in \mathbb{F}$ by

$$d_H(A, B)(p^*) = d_H^p(A, B),$$

where d_H^p is the Hausdorff pseudometric on $\mathbb{H}(X)$ induced by d^p on $\mathbb{X}$.

It is easy to see that $d_H(A, B) \in \mathbb{P}$ for any A, B. Most of the cone metric properties of d_H follow from the fact that each d^p is a pseudometric. For any $p^* \in qi(P^*)$ we know that d^p is a metric which then means that d_H^p is actually a metric as well and thus $d_H^p(A, B) = 0$ if and only if $A = B$. This means that if $A \neq B$ we have $d_H(A, B) \neq 0$ and so d_H is a cone metric.

The following properties are all easy to verify

1. $A_1, A_2, B_1, B_2 \in \mathbb{H}(\mathbb{X})$ implies that
$$d_H(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{d_H(A_1, B_1), d_H(A_2, B_2)\},$$
where the maximum is taken in a pointwise fashion.
2. If $(\mathbb{X}, d)$ is complete then $(\mathbb{H}(\mathbb{X}), d_H)$ is also complete. If $\mathbb{X}$ is compact, then so is $\mathbb{H}(\mathbb{X})$.
3. If $T : \mathbb{X} \to \mathbb{X}$ is a contraction with contractivity k then $T : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ is also contractive with contractivity k. This is easy to see as $d^p(Tx, Ty) \leq kd^p(x, y)$ for each p^*.
4. If $T_i : \mathbb{X} \to \mathbb{X}, i = 1, 2, \ldots, N$, are contractive with contractivity k, then so is $T : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ defined by
$$T(A) = \bigcup_i T_i(A).$$

With this formalism it is possible to construct geometric fractals in cone metric spaces using the standard IFS theory and this new Hausdorff cone metric.

5. Applications to digital imaging

A fundamental task in image processing is comparing images and computing some measure of the "distance" between images. The choice of a suitable definition of distance is not at all easy; what is close in one metric can be very far in another one. This naturally leads to an environment in which many possible metrics can be considered simultaneously and cone metric spaces lend themselves to this requirement. One specific instance of this is in the analysis of the structural similarity (SSIM) index of images (see [13,14]). SSIM is used to improve the measuring of visual distortion between images and is also used in fractal-based approximation using entropy maximization and sparsity constraints (see [15]). In both of these contexts the difference between two images is calculated using multiple criteria, which leads in a natural way to consider vector-valued distances.

5.1. Structural similarity index

The structural similarity (SSIM) index is a recent successful image distance. In its original formulation it involves a product of three terms, each of which measures a particular aspect of two images (see [13,14]). As well, as highlighted in [13], SSIM can be reformulated by considering vector-valued distances which are a particular case of cone-metric distances in which the cone coincides with the positive orthant.

We give a brief review of this theory. In [13] the authors define the following vector-valued distance: given $x, y \in \mathbb{R}^N$,

$$d(x, y) = (d_1(x, y), d_2(x, y)) \in \mathbb{R}^2_+,$$

where d_1 describes the distance between the means of x and y (considered as lists of numbers) while d_2 measures the "distortion" between x and y. With P equal to the positive orthant $\mathbb{R}^2_+$, it can be proved that d is a cone-metric distance. For more details and numerical experiments see [13]. However, it is worth listing some comments which could be useful to extend this index to more general contexts and to create better compression algorithms. First, the assumption that the ordering cone coincides with the positive orthant can be too restrictive; in fact, roughly speaking, this corresponds to giving the same importance to all the vector components while, in some situations, it could be convenient to assume a lexicographic ordering cone which assigns priorities to vector components, or even more complicated cones for more delicate ordering of the importance of the components. This could imply that d is no longer a cone-metric and some modifications to its definition are required. Second, in order to use a cone-metric to analyze approximation problems, it could happen that one has to solve optimization programs which involve multi-objective functions. In the case of the positive orthant this can be done by scalarization techniques (which lead to some sort of weighed combination of d_1 and d_2) or goal programming algorithms. When different cones are assumed, these numerical techniques needed to be adapted by using elements of the dual cone (or its quasi-interior).

5.2. Fractal-based image representation and self-similarity

Many images exhibit approximate self-similarity and exploiting this structure, even in some generalized form, has proven useful in image compression, representation and analysis [16–20]. The basic idea is to break the image up into different size blocks and search for "small blocks" which are "similar" to larger blocks. This is illustrated in Fig. 1 on the "peppers" image. This block-encoding transformation was first proposed by Jacquin in [21] and these methods have since been very extensively explored (for example, see [22–24]).

This block-based method induces a type of IFS on functions [25] where the mapping from a larger block to a smaller block is represented by a contraction $w_i : \mathbb{X} \to \mathbb{X}$ and the distortion of the portion of the image on this block (usually just a contrast and brightness adjustment) is represented by a "grey-level" map $\phi_i : \mathbb{R} \to \mathbb{R}$. We give two examples to show how this IFS fractal formalism can be usefully extended to cone metric spaces.

Example. $L^\infty_+([1, +\infty])$ cone metric space.

As an illustrative example, let us consider a possible application in the area of image compression through generalized fractal transforms and, in particular, the case of Iterated Function Systems on Mapping (IFSM). The classical IFSM theory requires one to specify *a priori* in which L^p space the image is embedded; a cone metric-based approach allows an all-in-one environment which preserves the complexity of the problem. For simplicity, let $\mathbb{X} = [0, 1]$ and consider the following cone metric $d : L^\infty(\mathbb{X}) \times L^\infty(\mathbb{X}) \to L^\infty([1, +\infty])$, where $d(u, v)(p) = \|u - v\|_p$ for all $p \in [1, +\infty]$. Let $L^\infty_+([1, +\infty])$ denote the cone of all a.e. positive functions. It is easy to prove that $(L^\infty(\mathbb{X}), d)$ is a complete $L^\infty_+([1, +\infty])$-cone metric space. We can further define an IFSM on $L^\infty(\mathbb{X})$ in the usual manner, that is

$$Tu(x) = \sum_{i=1}^{n} \alpha_i u(w_i^{-1}(x)) + \beta_i, \tag{5}$$

where $w_i : \mathbb{X} \to \mathbb{X}$ is a set of non-overlapping maps with contractivity factors K_i. We have $T : L^\infty(\mathbb{X}) \to L^\infty(\mathbb{X})$ and a classical result (see [25]) shows that

$$\|Tu - Tv\|_p \le K\|u - v\|_p$$

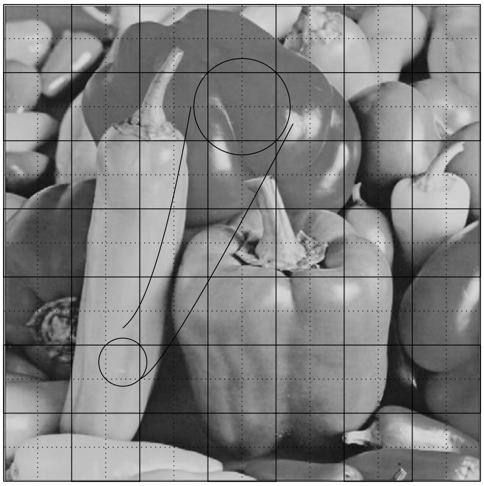

Fig. 1. Illustration of fractal-block coding.

for all $p \in [1, +\infty]$, where $K = \max_i K_i$. This implies that

$$d(Tu, Tv) \leq Kd(u, v)$$

with respect to the order induced by $L^\infty_+([1, +\infty])$.

As a numerical example let us compute the collage distance (the objective function ψ from (3)) for a simple illustrative case. Let $w_1(x) = \frac{1}{2}x$, $w_2(x) = \frac{1}{2}x + \frac{1}{2}$, $\phi_1(x) = \frac{1}{2}x + \frac{1}{4}$ and $\phi_2(x) = \frac{1}{2}x + \frac{1}{2}$. If we assume $u(x) = 1$ then $Tu(x) = \frac{3}{4}I_{[0,1/2]}(x) + I_{[1/2,1]}(x)$ where I is the indicator function. The collage distance $d(u, Tu)$ leads to the following result

$$d(u, Tu)(p) = \|u - Tu\|_p = \frac{1}{4}\left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Example. ℓ^2_+ cone metric space based on wavelets.

We now present a different type of example to illustrate the flexibility of the cone metric framework for image analysis problems. Our cone metric in this case will measure the difference between two images in each "resolution" level separately. To this end let $\psi_{i,j}$ be a wavelet basis for our image space (for example, either $L^2(\mathbb{R}^2)$ or $L^2([0, 1]^2)$ with periodic boundary conditions). The book [26] is a great source of information about wavelets (and many other beautiful topics).

For any image $f \in L^2$, we can expand f in the wavelet basis to obtain

$$f = \sum_{i,j} f_{i,j}\psi_{i,j}.$$

In this, the first subscript, i, represents the "scale" and the second subscript, j, represents the "location" at that scale. For two images f, g, we define our cone metric to be

$$d(f, g)(i) = \left(\sum_j (f_{i,j} - g_{i,j})^2\right)^{1/2}.$$

Thus $d(f, g)(i)$ represents the ℓ^2 distance between the images f and g on the wavelets at the scale i. Formally, $d : L^2 \times L^2 \to \ell^2_+$, as $d(f, g)(i) \geq 0$ for all i. We see that the cone ℓ^2_+ has an empty interior, so we must use the quasi-interior rather than the interior (as in Section 3.2).

Again we use an IFSM operator T, as in (5), for our image analysis. For appropriate maps w_k and ϕ_k, this induces a type of IFS operation on the wavelet coefficients (see [27]). If c_k is the Jacobian associated to w_k (which is strictly less than 1 as w_k is contractive) and K_k is the contractivity of the "grey-level map" ϕ_k, the one can show that (compare with Section 3.3 in [25])

$$d(Tf, Tg)(i) \leq \left(\sum_k |c_k| K_k^2\right)^{1/2} d(f, g)(i - 1),$$

which leads to conditions under which T is contractive in the cone metric d.

The benefit of using the cone metric framework is that we preserve the information about how well the approximation is at each distinct resolution level. Thus in an image recovery operation, we can either attempt to perform a vector optimization where we obtain a Pareto optimal point (non-dominated for all resolutions) or we can attempt the simpler problem of focusing on the visually important resolutions (usually the lower resolutions). It also allows one to truncate the expansion to any dimension.

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