Squeezing of the squared field amplitude by an anharmonic oscillator

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Hillery [Opt. Commun. **62**, 135 (1987)] has recently shown that the squeezing of the square of the field amplitude is a nonclassical effect. In this paper we show that the interaction of coherent light with a nonlinear medium modeled as a nonabsorbing anharmonic oscillator gives rise to this type of squeezing. We also show that this squeezing can be interpreted in terms of the Lie algebra of SU(1,1).

I. INTRODUCTION

Recently there have been several attempts at generalizing the notion of squeezing of the electromagnetic field. For instance Fisher *et al.*[1] discuss higher-order generalizations of the two-photon unitary operators that generate the usual squeezed states discussed by Yuen[2] and others.[3] Various other kinds of multiboson squeezed states have been considered.[4–6] On the other hand, it is possible to extend the notion of squeezing to higher-order moments of the field quadratures. In particular, Hong and Mandel[7] consider the Nth order moments of the real part of the field amplitude X_1 where $(\Delta X_1)^N$ is said to be squeezed if it takes values less than its coherent state value. This is uniquely nonclassical only for even values of N. A number of systems which exhibit the usual second-order squeezing also exhibit higher-order squeezing. Indeed, the degree of the squeezing may be even greater than that of second order.

More recently, Hillery[8] has shown that the squeezing of the square of the field amplitude (SFA) is also a nonclassical effect. This kind of squeezing is not equivalent to that of Hong and Mandel. Hillery showed that SFA squeezing occurs in the fundamental mode in second-harmonic generation and that the normal squeezing of the harmonic depends on this SFA squeezing of the fundamental.

In this paper we do two things. First we show that coherent light interacting with a nonlinear nonabsorbing medium modeled as anharmonic oscillator can also give rise to the amplitude-squared squeezing effect. This model system has previously been shown to give rise to usual second-order squeezing in terms of the field amplitude[9] and also to higher-order squeezing[10] (at least to sixth order) in the sense of Hong and Mandel. (The degree of squeezing increases with order for this system.) The second point we wish to make is that the algebra of the operators for the SFA is isomorphic to the Lie algebra of the noncompact group SU(1,1). This group has been shown to be of utility in quantum optics as coherent states over SU(1,1) are essentially the two-photon squeezed states mentioned earlier.[11,12] However, one may also consider the fluctuations in the components of the Lie algebra. The squeezing condition one obtains there[11] (which is also a generalized squeezing) will be shown to be equivalent to the squeezing of the SFA. This gives a physical interpretation to the Lie algebraically defined squeezing relations for SU(1,1).

II. SQUEEZING OF THE FIELD AMPLITUDE SQUARED

We shall follow the standard procedure of defining the slowly varying bose operators

$$A = ae^{i\omega t}, \quad A^\dagger = a^\dagger e^{-i\omega t} . \tag{2.1}$$

In terms of these operators the usual real and imaginary components of the field amplitude are

$$X_1 = \tfrac{1}{2}(A + A^\dagger), \quad X_2 = \frac{1}{2i}(A - A^\dagger) . \tag{2.2}$$

They satisfy the commutation relation

$$[X_1, X_2] = \frac{i}{2} , \tag{2.3}$$

which in turn leads to the uncertainty relation

$$(\Delta X_1)^2(\Delta X_2)^2 \geq \tfrac{1}{16} . \tag{2.4}$$

As usual, we say that squeezing exists if $(\Delta X_1)^2 < \frac{1}{4}$ or $(\Delta X_2)^2 < \frac{1}{4}$.

Writing the field amplitude as

$$E = X_1 + iX_2 , \tag{2.5}$$

the square of the amplitude becomes

$$E^2 = Y_1 + iY_2 , \tag{2.6}$$

where

$$Y_1 = \tfrac{1}{2}[A^2 + (A^\dagger)^2], \quad Y_2 = \frac{1}{2i}[A^2 - (A^\dagger)^2] . \tag{2.7}$$

These operators obey the commutation relation

$$[Y_1, Y_2] = i(2N+1) , \tag{2.8}$$

where $N = A^\dagger A$ is the usual number operator. The commutator of Eq. (2.8) leads to the uncertainty relation

$$(\Delta Y_1)^2(\Delta Y_2)^2 \geq \langle N + \tfrac{1}{2} \rangle^2 . \tag{2.9}$$

Squeezing of the squared field amplitude is said to exist if

$(\Delta Y_1)^2 < \langle N+\frac{1}{2}\rangle$ or $(\Delta Y_2)^2 < \langle N+\frac{1}{2}\rangle$. By expressing the variance in terms of the P representation of the state it is possible to show that the above conditions occur only when the P representation is nonpositive definite, corresponding to a nonclassical state.[8] It is easy to show that the uncertainty relations are minimized by coherent states, i.e., $(\Delta Y_1)^2 = (\Delta Y_2)^2 = \langle N+\frac{1}{2}\rangle$.

III. APPLICATION TO THE ANHARMONIC OSCILLATOR

The Hamiltonian for the nonabsorbing anharmonic oscillator is taken to be

$$H = \omega a^\dagger a + \frac{\lambda}{2}(a^\dagger)^2 a^2 , \tag{3.1}$$

where λ is related to the third-order susceptibility of the medium. It is known that the anharmonic term leads to optical bistability.[13]

The equation of motion for the annihilation operator $a(t)$ is

$$\begin{aligned}\dot a(t) &= -i[a(t), H] \\ &= -i(\omega + \lambda a^\dagger a)a .\end{aligned} \tag{3.2}$$

Since $[a^\dagger a, H]=0$, the number operator $N = a^\dagger a = A^\dagger A$ is a constant of the motion. Thus the solution of Eq. (3.2) is simply

$$\begin{aligned}a(t) &= \exp\{-i[\omega + \lambda a^\dagger(0)a(0)]t\}a(0) \\ &= \exp\{-i[\omega + \lambda N(0)]t\}a(0) ,\end{aligned} \tag{3.3}$$

or, in terms of the operators A and $A^\dagger$,

$$A(t) = \exp[-iN\tau]A(0) , \tag{3.4}$$

where we have set $\lambda t = \tau$. Assuming an initial coherent state $|\alpha\rangle$ with average photon number $n = \langle\alpha|N|\alpha\rangle = |\alpha|^2$, it is easy to show that

$$\langle\alpha|a^2(t)|\alpha\rangle = \alpha^2 e^{-i\tau}\exp[|\alpha|^2(e^{-2i\tau}-1)] , \tag{3.5a}$$

$$\langle\alpha|a^4(t)|\alpha\rangle = \alpha^4 e^{-6i\tau}\exp[|\alpha|^2(e^{-4i\tau}-1)] . \tag{3.5b}$$

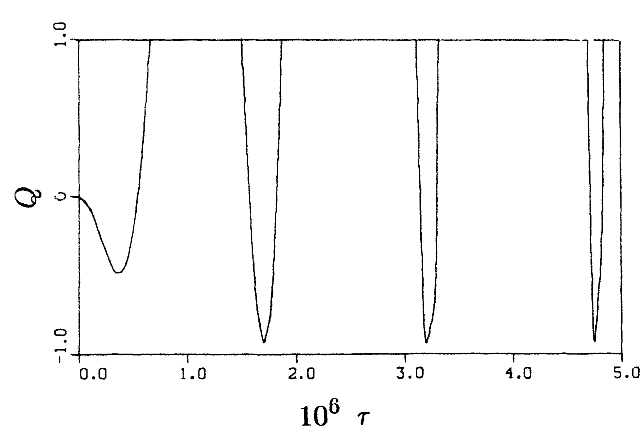

FIG. 1. Q vs τ for $n = 10^{+6}$.

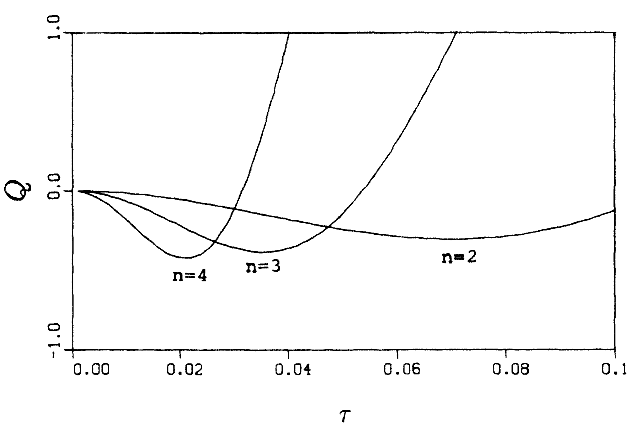

FIG. 2. Q vs τ for $n=4$, 9, and 16.

Thus,

$$\langle\alpha|Y_1|\alpha\rangle = n\,\mathrm{Re}\{e^{i\tau}\exp[n(e^{2i\tau}-1)]\} \tag{3.6}$$

and

$$\langle\alpha|Y_1^2|\alpha\rangle = \frac{n^3}{2}\mathrm{Re}\{e^{-6i\tau}\exp[n(e^{-4i\tau}-1)]\} + \frac{n^2}{2} + n + \frac{1}{2} . \tag{3.7}$$

The variance is, of course, given as $(\Delta Y_1)^2 = \langle Y_1^2\rangle - \langle Y_1\rangle^2$.

Now Tanás[9] has shown that the system in question here gives rise to the usual squeezing of the field amplitude for $n \sim 10^6$ and $\tau \sim 10^{-6}$, while Gerry and Rodrigues[10] have obtained the higher-order squeezing also in this range of parameters. In Fig. 1 we show that the real part of the SFA is also squeezed in this range. Actually, we find it convenient to plot the quantity

$$Q = \frac{(\Delta Y_1)^2 - \langle N+\frac{1}{2}\rangle}{\langle N+\frac{1}{2}\rangle} , \tag{3.8}$$

which has the range $-1 < Q < 0$ for squeezing. $|Q|$ is the fractional squeezing. The fractional squeezing at the first minimum, $|Q| \approx 0.49$, while at successive minima $|Q| \approx 0.93$. As in the usual case of field amplitude squeezing, squeezing of the SFA is found at lower field intensities and over longer time intervals. This feature is illustrated in Fig. 2.

IV. RELATION TO SU(1,1)

By making the following identifications:

$$\begin{aligned}\tilde K_1 &= \tfrac{1}{2}Y_1 , \\ \tilde K_2 &= -\tfrac{1}{2}Y_2 , \\ \tilde K_0 &= \tfrac{1}{2}N + \tfrac{1}{4} .\end{aligned} \tag{4.1}$$

it is possible to obtain the commutation relations

$$[\tilde K_1, \tilde K_2] = -i\tilde K_0 , \tag{4.2a}$$

$$[\tilde{K}_2, \tilde{K}_0] = i\tilde{K}_1 \ , \tag{4.2b}$$

$$[\tilde{K}_0, \tilde{K}_1] = i\tilde{K}_2 \ , \tag{4.2c}$$

which is the SU(1,1) Lie algebra. Defining $\tilde{K}_\pm = \tilde{K}_1 \pm i\tilde{K}_2$ this may be cast into the form

$$\begin{aligned} &[\tilde{K}_0, \tilde{K}_\pm] = \pm\tilde{K}_\pm \ , \\ &[\tilde{K}_-, \tilde{K}_+] = 2\tilde{K}_0 \ . \end{aligned} \tag{4.3}$$

From Eq. (2.1) it is easy to see that these operators are related to the usual boson realization of the SU(1,1) algebra according to

$$\begin{aligned} &K_+ = e^{2i\omega t}\tilde{K}_+ = \tfrac{1}{2}a^{\dagger 2} \ , \\ &K_- = e^{-2i\omega t}\tilde{K}_- = \tfrac{1}{2}a^2 \ , \\ &K_0 = \tilde{K}_0 = \tfrac{1}{4}(aa^\dagger + a^\dagger a) \ , \end{aligned} \tag{4.4}$$

which also satisfy the commutation relations of (4.3). In particular, Eq. (4.2b) leads to the uncertainty relation

$$(\Delta K_1)^2(\Delta K_2)^2 \geq \tfrac{1}{4}\,|\langle K_0\rangle|^2 \tag{4.5}$$

With the condition for squeezing being $(\Delta K_1)^2 < \frac{1}{2}\,|\langle K_0\rangle|$ or $(\Delta K_2)^2 > \frac{1}{2}\,|\langle K_0\rangle|$. It is apparent that these relations are equivalent to the condition for the squeezing of the SFA.

Consider now the Hamiltonian for a single-mode degenerate parametric amplifier with classical pumps

$$H = H_0 + \tfrac{1}{2}i\gamma[a^2 e^{2i(\omega t - \psi)} - (a^\dagger)^2 e^{-2i(\omega t - \psi)}] \ , \tag{4.6}$$

where $H_0 = \omega a^\dagger a$ and γ and ψ are real constants. In terms of the SU(1,1) generators this Hamiltonian may be written as

$$H = H_0 + i\gamma(K_- e^{2i(\omega t - \psi)} - K_+ e^{-2i(\omega t - \psi)}) \ , \tag{4.7}$$

where $H_0 = 2\omega K_0$. The time evolution operator can be shown to be[11,12]

$$U(t,0) = e^{-iH_0 t}U_I(t,0) \ ,$$

where

$$U_I(t,0) = \exp(-\gamma t e^{2i\psi}K_+ + \gamma t e^{-2i\psi}K_-) \ . \tag{4.8}$$

Acting on the ground state $|0\rangle$, we produce

$$\begin{aligned} |t\rangle &= U(t,0)\,|0\rangle \\ &= e^{-iH_0 t}\,|\xi(t)\rangle \ , \end{aligned} \tag{4.9}$$

where $|\xi(t)\rangle$ is an SU(1,1) coherent state and where $\xi(t) = -\tanh(\gamma t)e^{2i\psi}$. Upon calculating the variance with $|t\rangle$, we obtain

$$(\Delta K_1)^2 = \tfrac{1}{8}[1 + \sinh^2(2\gamma t)\cos^2(2\psi)] \ , \tag{4.9$'$}$$

$$(\Delta K_2)^2 = \tfrac{1}{8}[1 + \sinh^2(2\gamma t)\sin^2(2\psi)] \ , \tag{4.9$''$}$$

and for K_0 we have

$$\langle K_0\rangle = \tfrac{1}{4}\cosh(2\gamma t) \ . \tag{4.10}$$

Thus we obtain squeezing for K_1 if

$$[1 + \sinh^2(2\gamma t)\cos^2(2\psi)] \langle \cosh(2\gamma t) \tag{4.11}$$

or for K_2 if

$$[1 + \sinh^2(2\gamma t)\sin^2(2\psi)] < \cosh(2\gamma t) \ . \tag{4.12}$$

Apparently for the phase angle $\psi = 0$, K_2 is always squeezed for $t > 0$ while for $\psi = \pi/4$, K_1 is always squeezed for $t > 0$. It is curious that for $\psi = 0$, there is the usual squeezing in X_1 but for $\psi = \pi/4$, there is no squeezing in either X_1 or X_2. For other values of ψ there is squeezing in K_1 if

$$\cos^2(2\psi) \leq \frac{1}{[\cosh(2\gamma t) + 1]} \tag{4.13}$$

and squeezing in K_2 if

$$\sin^2(2\psi) \leq \frac{1}{[\cosh(2\gamma t) + 1]} \ . \tag{4.14}$$

These relations have been discussed previously but without reference to their physical interpretation.[11] From the above analysis it appears that they may be interpreted in terms of the squeezing of the squared field amplitude.

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