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SOLVING THE INVERSE PROBLEM FOR FUNCTION AND IMAGE APPROXIMATION USING ITERATED FUNCTION SYSTEMS

Bruno Forte[1,2] and Edward R. Vrscay[2]

[1]Facoltà di Scienze MM. FF. e NN. a Cà Vignal
Università Degli Studi di Verona
Strada Le Grazie
37134 Verona, Italy
e-mail: forte@biotech.univr.it

[2]Department of Applied Mathematics
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
e-mail: ervrscay@links.uwaterloo.ca

Abstract This paper is concerned with function approximation and image representation using a new formulation of Iterated Function Systems (IFS) over the general function spaces $\mathcal{L}^p(X,\mu)$: An N-map IFS with grey level maps (IFSM), to be denoted as $(\mathbf{w},\Phi)$, is a set $\mathbf{w}$ of N contraction maps $w_i : X \to X$ over a compact metric space (X,d) (the "base space") with an associated set Φ of maps $\phi_i : \mathbf{R} \to \mathbf{R}$. Associated with each IFSM is an operator T which, under certain conditions, may be contractive with unique fixed point $\overline{u} \in \mathcal{L}^p(X,\mu)$. A rigorous solution to the following inverse problem is provided: Given a target $v \in \mathcal{L}^p(X,\mu)$ and an $\epsilon > 0$, find an IFSM whose attractor satisfies $\| \overline{u} - v \|_p < \epsilon$. An algorithm for the construction of IFSM approximations of arbitary accuracy to a target set in $\mathcal{L}^2(X,\mu)$, where $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$ (Lebesgue measure), is also given. The IFSM formulation can easily be generalized to include the "local IFSM" (LIFSM) which considers the actions of contraction maps on *subsets* of X to produce smaller subsets. Some applications to function approximation on [0,1] and image representation on $[0,1]^2$ are presented.

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1. Introduction

This paper deals with the approximation of functions to a specified accuracy using the method of Iterated Function Systems (IFS). The main theoretical results along with some numerical computations involving functions $u : [0,1] \to [0,1]$ and images were presented in [14]. Here we provide the details and proofs of our method as well the results of more recent computations. The use of IFS-type methods for image compression has received much interest [4, 9, 10, 11, 19, 20]. The objective is to approximate images which normally require megabytes of storage by the fixed points/attractors of IFS-type operators. The representation or *coding* of such images by IFS parameters can require much less storage space, thus providing a means of data compression. The reconstruction (or *decompression*) of the image from the IFS parameters can be performed quickly (in "real time"). Nevertheless, the *compression* of images into IFS parameters can still require a significant amount of computer time. More research is needed to understand how to reduce this time without significantly affecting the accuracy of the approximation.

In light of the great interest in approximating measures and functions as well as the many efforts to devise effective IFS-based image encoding schemes, we present a systematic theoretical formulation and solution to an inverse problem of function/image approximation. We then provide an algorithm to construct the IFS-based approximations of arbitrarily small accuracy to a target function. In $\mathbf{R}^2$, this represents a method of image data compression.

An N-map contractive IFS on a compact metric space (X,d) is a set of contraction maps, $\mathbf{w} = \{w_1, w_2, ...w_N\}$, $w_i : X \to X$. Associated with the IFS $\mathbf{w}$ is a set-valued mapping $\widehat{\mathbf{w}}$ which acts on nonempty compact subsets of X. The theory of IFS with probabilities (IFSP), that is, a set of IFS maps $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, p_2, ...p_N\}$, was introduced in [18] and developed independently in [2]. Associated with an IFSP is a contractive "Markov" operator which works with probability measures, using the p_i as multiplicative weights. The approximation of measures by invariant measures of IFSP has received much interest - for a survey with references, see [22]. Recently, the inverse problem for measure approximation was formulated in terms of moments. A solution to this problem as well as an algorithm was provided [13]. In fact, Ref. [13] may be considered a stepping stone to the work reported in this paper.

Here, however, we formulate an IFS-type method over function spaces. A principal motivation arises from the problem of image representation - it is desirable to have more pointwise control of approximations to an image than can be achieved with measures. An image will be represented by a function $u : [0,1]^2 \to R_g$, where $R_g \subseteq \mathbf{R}^+$, is the *grey level range*, to be discussed below. The value $u(x)$ at a point or pixel $x \in X$ may then be interpreted as a nonnegative grey level or brightness value. We then consider an N-map IFS on the *base space* (X,d) along with a set of associated maps, $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$, $\phi_i : \mathbf{R} \to \mathbf{R}$. Such a system $(\mathbf{w}, \Phi)$ will be referred to as an *Iterated Function System with Maps* (IFSM). As in the usual IFS-type approaches, we seek to approximate a target function or image by the unique fixed point (attractor) of a contractive operator associated with an IFSM.

It now remains to identify an appropriate space of functions for an IFS-type approach. In earlier studies [5], the set-valued mapping $\widehat{\mathbf{w}}$ was considered to operate on the level sets of a function u. It thus seemed natural to consider $u \in \mathcal{F}(X)$, the class of fuzzy sets on X [24]. In order that the level sets be compact and nonempty, u was an element of the subspace $\mathcal{F}^*(X) \subset \mathcal{F}(X)$ of normalized upper semicontinuous fuzzy sets with the d_∞ metric [7] which involves Hausdorff distances between their respective level sets. Such a system $(\mathbf{w}, \Phi)$ of IFS maps and associated grey level maps was referred to as an Iterated Fuzzy Set System (IFZS). In this particular case, the grey level range is $R_g = [0, 1]$. Associated with each IFZS $(\mathbf{w}, \Phi)$ is a contractive operator $T_s : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$, with a unique fixed point u, the *attractor* of the IFZS. The contraction mapping T_s involves the *supremum* operator.

However, as we shall discuss in Section 2 below, there are some serious difficulties with the IFZS approach, especially with regard to the inverse problem. First, the Hausdorff metric d_∞ is too restrictive, both from the practical aspect of image processing as well as from some theoretical perspectives, notably continuity [12]. Second, the appearance of the *supremum* in the IFZS contractive operator T_s complicates the inverse problem in the case of "overlapping" IFSM, i.e. when some of the sets $w_i(X)$ overlap. In order to bypass these difficulties, we have introduced two fundamental modifications to the IFZS method, as summarized below:

(a) We define another distance function between two functions $u, v \in \mathcal{F}^*(X)$, as follows. First, replace the Hausdorff distance between level sets $h([u]^\alpha, [v]^\alpha)$ by $\mu([u]^\alpha \triangle [v]^\alpha)$, where μ denotes a measure on $\mathcal{B}(X)$, the σ-algebra of Borel subsets of X, and $\triangle$ denotes the symmetric difference operator. Then replace the *sup* in the d_∞ metric by an integration with respect to a measure ν on $\mathcal{B}(R_g)$. The result is a pseudometric on $(\mathcal{F}^*(X), d_\infty)$ that reduces to the $\mathcal{L}^1(X, \mu)$ distance when ν is the Lebesgue measure on $\mathcal{B}(R_g)$. By extending the grey level range, R_g, to the nonnegative real line $\mathbf{R}^+$, it is then natural to formulate an IFSM over $\mathcal{L}^1(X, \mu)$. It appears that only the space $\mathcal{L}^1(X, \mu)$ can be generated from the IFZS with such a procedure. However, there is nothing which prevents us from formulating IFSM over the general spaces $\mathcal{L}^p(X, \mu)$ for $p \geq 1$.

(b) We then introduce a new contractive operator T associated with an IFSM $(\mathbf{w}, \Phi)$ on $\mathcal{L}^p(X, \mu)$ which facilitates the solution of the inverse problem even when the subsets $w_i(X)$ overlap. This is particularly important with regard to (c) below. From the structure of the T operator, the fixed point equation $\overline{u} = T\overline{u}$ may be regarded as a "mixed" Fourier-type expansion of $\overline{u}(x)$ in terms of two sets of functions: (i) the set $\{\chi_k(x) = I_{w_k(X)}(x), k = 1, 2, ..., N\}$ of piecewise constant functions and (ii) the set $\{\psi_k(x) = \overline{u}(w_k^{-1}(x)), k = 1, 2, ..., N\}$, which are dilatations and translations of the "mother function" $\overline{u}(x)$ itself, reminiscent of scaling functions in wavelet theory.

The inverse problem for function approximation using IFSM may now be posed as follows:

Given a target function or image $v \in \mathcal{L}^p(X,\mu)$ and an $\epsilon > 0$, find an IFSM $(\mathbf{w}, \Phi)$ whose attractor $\overline{u} \in \mathcal{L}^p(X,\mu)$ satisfies $\| \overline{u} - v \|_p < \epsilon$.

From the "Collage Theorem" the inverse problem may be rephrased as follows:

Given a target function or image $v \in \mathcal{L}^p(X,\mu)$ and a $\delta > 0$, find an IFSM $(\mathbf{w}, \Phi)$ with operator T such that $\| v - Tv \|_p < \delta$.

As in [13], our formal solution to the inverse problem is unique in the following aspects:

(c) We begin with an infinite set $\mathcal{W} = \{w_1, w_2, ...\}$ of *fixed* affine contraction maps $w_i : X \to X$ which must satisfy a kind of density condition on X with respect to the measure μ. From this set, we construct sequences of N-map IFSM $(\mathbf{w}^N, \Phi^N), N = 1, 2, ...$, with corresponding operators T^N. Given a target function $v \in \mathcal{L}^p(X,\mu)$, we then prove that

$$\lim_{N\to\infty} \inf \| v - T^N v \|_p = 0,$$

which represents a formal solution to the inverse problem.

(d) Our algorithm to construct IFSM approximations works with functions in the space $\mathcal{L}^2(X,\mu) \subset \mathcal{L}^1(X,\mu)$. For each N, the optimal grey level maps in Φ^N minimize the squared $\mathcal{L}^2$ collage distance $\Delta^N = \| v - T^N v \|_2^2$. A further simplification results by considering only affine grey level maps $\phi_i(t) = \alpha_i t + \beta_i,\ t \in \mathbf{R}^+$. The minimization of Δ^N becomes a quadratic programming (QP) problem in the α_i and β_i over an appropriately defined compact subset $\Pi^{2N} \subset \mathbf{R}^{2N}$. Such problems can be solved numerically in a finite number of steps. In many cases, the minimum collage distance is achieved on a boundary point of the simplex Π^{2N}. In such cases, if $(\alpha_{k^*}, \beta_{k^*}) = (0,0)$ for some $k^* \in \{1, 2, ..., N\}$, then $\phi_{k^*}(t) = 0$ for all $t \in R_g$, which implies that the grey level map ϕ_{k^*} is superfluous. The automatic detection and elimination of such superfluous maps increases the *data compression factor*.

The layout of this paper is as follows. In Section 2 we review the basic features of the Iterated Fuzzy Set System and look at some of the disadvantages of the d_∞ metric. The new distance function described in (a) is then introduced. An IFSM over the space $\mathcal{L}^p(X,\mu)$ is then formulated with the introduction of a new contractive operator T. In Section 3 we present our formal solution to the inverse problem for IFSM on $\mathcal{L}^p(X,\mu)$. In Section 4, we extend our method to the case of "local IFSM". Section 5 is devoted to applications which include the approximation of computer images.

Finally, we conclude this introduction with some words on the image processing aspect of this study. The use of a fixed set of IFS maps is a rather standard procedure in most fractal encoding methods. Intuitively (and, in fact, from theory), one expects that the accuracy of the IFS approximation increases with the number of maps N used in the IFS. However, both (i) the computer time required to determine the optimal grey level maps Φ^N, as well as (ii) the time needed to generate the IFSM fixed point $\overline{u}$ by iteration of T, increase very quickly with N. Unlike the mathematical problem of function approximation, the object of image representation is not to obtain approximations of arbitrarily small accuracy but rather to obtain an acceptable approximation (of course, the smaller the better) with the fewest number of parameters and the least amount of computer time. Most current methods employ only "nonoverlapping" IFS maps w_i. As stated in (b) above, our new contractive operator T easily accounts for cases where the sets $w_i(X)$ overlap. In fact, our algorithm can choose the best IFS maps, overlapping or not. This may allow a greater accuracy to be achieved for a given number of maps.

2. Iterated Function Systems on Function Spaces

2.1 Glossary of Notation

Throughout this paper, the following notation will be employed:

$\mathbf{R}^+ = [0, \infty)$

(X, d)**:** a compact metric space. In most applications, where X is the "base space" of the IFS, X is typically a compact subset of $\mathbf{R}^n$, e.g. $[0,1], [0,1]^2$.

$Con(X) = \{w : X \to X | d(w(x), w(y)) \leq cd(x, y)$ for some $c \in [0,1), \forall x, y \in X\}$: the set of contraction maps on X. We define the *contractivity factor* of $w \in Con(X)$ to be

$$c \equiv \sup_{x,y \in X, x \neq y} d(w(x), w(y))/d(x, y).$$

$d_{Con(X)}$**:** a metric on the function space $Con(X)$. For $f, g \in Con(X)$,

$$d_{Con(X)}(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

(Note that the metric space $(Con(X), d_{Con(X)})$ may not be complete.)

$Con_1(X) = \{w \in Con(X) | w$ is one-to-one $\}$.

$\mathcal{H}(X)$**:** the set of nonempty compact subsets of X.

h**:** the Hausdorff metric on $\mathcal{H}(X)$. Let the distance between a point $x \in X$ and a set $A \in \mathcal{H}(X)$ be given by

$$d(x, A) = \inf_{y \in A} d(x, y).$$

Then for $A, B \in \mathcal{H}(X)$, define

$$h(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}.$$

$(\mathcal{H}(X), h)$ is a complete metric space [8].

$\mathcal{B}(X)$: the σ-algebra of Borel subsets of X.

$\mathcal{M}(X)$: the set of finite measures on $\mathcal{B}(X)$. In the special case that $X \subset \mathbf{R}^D$, let $m^{(D)} \in \mathcal{M}(X)$ denote the Lebesgue measure on $\mathcal{B}(X)$.

$I_A(x)$: the indicator function of a set $A \subseteq X$. $I_A(x) = 1$ if $x \in A$. $I_A(x) = 0$ otherwise.

$\mathcal{L}^p(X,\mu) = \{f : X \to \mathbf{R} \mid \ \| f \|_p \equiv [\int_X |f(x)|^p d\mu]^{1/p} < \infty\}$, $1 \le p < \infty$. For $f, g \in \mathcal{L}^p(X,\mu)$, define $d_p(f,g) \equiv \| f - g \|_p$. Note that since $\mu(X) < \infty$, it follows that $\mathcal{L}^q \subset \mathcal{L}^p$ for $1 \le p \le q$.

$< f, g > = \int_X f(x)g(x)d\mu$ for $f, g \in \mathcal{L}^2(X,\mu)$.

$\mathcal{L}^p_+(X,\mu) = \{f \in \mathcal{L}^p(X,\mu) \mid f(x) \ge 0, \forall x \in X\}$.

$R_g \subset \mathbf{R}$: the *grey level range* for image functions $u : X \to R_g$. (In practical applications, R_g is nonnegative and bounded.)

$Lip(Y) = \{\phi : Y \to Y, Y \subseteq \mathbf{R} \mid |\phi(t_1) - \phi(t_2)| \le K|t_1 - t_2|, \forall t_1, t_2 \in R_g$ for some $K \in [0,\infty)\}$.

IFS methods are based upon Banach's Fixed Point Theorem or Contraction Mapping Principle (CMP) as well as two simple consequences. For convenience, we state these important results below.

Theorem 1 *(CMP) Let (Y, d_Y) be a complete metric space. Suppose there exists a mapping $f \in Con(Y)$ with contractivity factor $c \in [0,1)$. Then there exists a unique $\overline{y} \in Y$ such that $f(\overline{y}) = \overline{y}$. Moreover, for any $y \in Y$, $d_Y(f^n(y), \overline{y}) \to 0$ as $n \to \infty$.*

The following result is often referred to in the IFS literature as the "Collage Theorem":

Theorem 2 *Let (Y, d_Y) be a complete metric space. Given a $y \in Y$ suppose that there exists a map $f \in Con(Y)$ with contractivity factor $c \in [0,1)$ such that $d_Y(y, f(y)) < \epsilon$. If $\overline{y}$ is the fixed point of f, i.e. $f(\overline{y}) = \overline{y}$, then $d_Y(y, \overline{y}) < \epsilon/(1-c)$.*

Finally, the following result establishes the continuity of fixed points of contraction maps on (Y, d_Y).

Theorem 3 *Let (Y, d_Y) be a metric space and $f, g \in Con(Y)$ with fixed points $\overline{y}_f$ and $\overline{y}_g$ and contraction factors c_f and c_g, respectively. Then*

$$d_Y(\overline{y}_f, \overline{y}_g) < \frac{1}{1 - \min(c_f, c_g)} d_{Con(Y)}(f, g). \tag{1}$$

This result was used to derive continuity properties of IFS attractors and IFSP invariant measures [6] as well as IFZS attractors [12].

2.2 Iterated Fuzzy Set Systems (IFZS)

We first briefly describe the basic features of IFZS, referring the reader to [5] for details. An N-map IFZS, denoted as $(\mathbf{w}, \Phi)$ $(N \leq \infty)$ has an IFS component, $\mathbf{w}$, on a compact metric space (X, d) and a grey-level component, Φ. In this particular case, $R_g = [0, 1]$.

The IFS Component: Let $\mathbf{w} = \{w_1, w_2, ..., w_N\}$, $w_i \in Con(X)$ denote a contractive N-map IFS, where $N \leq \infty$. The contractivity factor of the IFS is given by

$$c = \sup_{1 \leq i \leq N} \{c_i\} < 1. \tag{2}$$

For the IFS to be contractive, we must have $c < 1$, which will be assumed throughout the paper. (This condition that the IFS be strictly contractive may be relaxed to "eventually contractive", which is important in the case where X is discrete, e.g. pixels. At this point, for simplicity of discussion, we omit this technicality, but will return to it later in the paper.) Associated with the IFS $\mathbf{w}$ is a set-valued mapping $\widehat{\mathbf{w}} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined as follows. For an element $S \in \mathcal{H}(X)$, denote $\widehat{w}_i(S) = \{w_i(x), x \in S\}, i = 1, 2, ..., N$ and let

$$\widehat{\mathbf{w}}(S) \equiv \bigcup_{i=1}^{N} \widehat{w}_i(S). \tag{3}$$

As is well known [1, 2, 18], there exists a unique compact set $A \in \mathcal{H}(X)$, the *attractor* of the IFS $\mathbf{w}$, such that

$$A = \widehat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} \widehat{w}_i(A). \tag{4}$$

This follows from the fact that $\widehat{\mathbf{w}}$ is a contraction mapping on $(\mathcal{H}(X), h)$ with contractivity factor c [18].

We shall be primarily concerned with IFS whose maps on X are affine. For example, on $\mathbf{R}$, these maps have the general form

$$w_i(x) = s_i x + a_i, \quad c_i = |s_i| < 1, \quad s_i, a_i \in \mathbf{R}, \; i = 1, 2, ..., N. \tag{5}$$

In higher dimensions, e.g. $X \subset \mathbf{R}^D, D = 2, 3, ...$, it will be convenient (although not necessary) to consider the special class of contractive *similitudes*, e.g. rotations, inversions, reflections followed by translations. In all such cases, the contractivity relations become equalities. We denote this set of maps as $Sim(X) \subset Con(X)$, i.e.

$$Sim(X) = \{w : X \to X \mid d(w(x), w(y)) = cd(x, y) \text{ for a } c \in [0, 1), \forall x, y \in X\}.$$

In our construction of the IFSM, it will also be necessary to ensure that the IFS maps are one-to-one, so we define

$$Sim_1(X) = Sim(X) \cap Con_1(X).$$

The Grey Level Component: Let $\mathcal{F}(X)$ denote the class of functions $u : X \to [0, 1]$, often denoted as the class of *fuzzy sets* on X. Define the α-level set $[u]^\alpha$ of $u \in \mathcal{F}(X)$ as follows:

$[u]^\alpha = \{x \in X : u(x) \geq \alpha\}$, for $0 < \alpha \leq 1$ and

$[u]^0 = \overline{\{x \in X : u(x) > 0\}}$ (the bar denotes closure).

We consider the special subclass $\mathcal{F}^*(X) \subset \mathcal{F}(X)$: $u \in \mathcal{F}^*(X)$ if and only if

1. $u \in \mathcal{F}(X)$,
2. u is upper semicontinuous on (X, d) and
3. u is normalized, i.e. $u(x_0) = 1$ for some $x_0 \in X$.

From properties 2 and 3, $[u]^\alpha \in \mathcal{H}(X), \alpha \in [0, 1]$ for all $u \in \mathcal{F}^*(X)$. We now consider the following metric on $\mathcal{F}^*(X)$:

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} \{h([u]^\alpha, [v]^\alpha)\}, \quad \forall u, v \in \mathcal{F}^*(X). \tag{6}$$

The metric space $(\mathcal{F}^*(X), d_\infty)$ is complete [7].

Let $\mathcal{G}^+([0, 1])$ denote the set of all functions $\phi : [0, 1] \to [0, 1]$ such that

(i) ϕ is nondecreasing on [0,1],

(ii) ϕ is right continuous on [0,1) and

(iii) $\phi(0) = 0$.

Given an N-map IFS $\mathbf{w}$, let $\Phi = (\phi_1, \phi_2, ..., \phi_N)$ denote a set of associated *grey level maps* which satisfy the following conditions:

(a) $\phi_i \in \mathcal{G}^+([0, 1]), i \in \{1, 2, ..., N\}$ and

(b) $\phi_{i^*}(1) = 1$ for at least one $i^* \in \{1, 2, ..., N\}$.

The pair of vectors $(\mathbf{w}, \Phi)$ constitutes an N-map IFZS.

Associated with an IFZS $(\mathbf{w}, \Phi)$ is an operator $T_s : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$, defined as

$$(T_s u)(x) = \sup_{1 \le i \le N} \{\phi_i(\tilde{u}(w_i^{-1}(x)))\}, \quad x \in X, \tag{7}$$

where, for $B \subset X$,

$\tilde{u}(B) = \sup_{y \in B}\{u(y)\}$ if $B \neq \emptyset$

$\tilde{u}(\emptyset) = 0.$

Properties (i), (ii) and (b) guarantee that the operator T_s, defined below, maps $\mathcal{F}^*(X)$ into itself. The T_s operator is a quite natural extension of the IFS set-valued mapping $\widehat{\mathbf{w}}$ since it leads to the following relation involving the union of α-level sets:

$$[T_s u]^\alpha = \bigcup_{i=1}^{N} w_i([\phi_i \circ u]^\alpha), \quad \alpha \in [0, 1]. \tag{8}$$

This relation is characteristic of the *supremum* operator in T_s. In the special case where the sets $w_i(X)$ are disjoint, i.e. $w_i(X) \cap w_j(X) = \emptyset$ when $i \neq j$, it follows from Property (iii) that

$$(T_s u)(x) \equiv \phi_k(u(w_k^{-1}(x))), \quad x \in w_k(X), \ k = 1, 2, ..., N. \tag{9}$$

In [5], Property (iii) was also considered to be a natural assumption for grey level functions: If the grey level of a point or pixel $x \in X$ is zero, then it should remain zero after being acted upon by the ϕ_i maps.

T_s is a contraction mapping on the space $(\mathcal{F}^*(X), d_\infty)$, i.e.

$$d_\infty(Tu_1, Tu_2) \le c d_\infty(u_1, u_2) \ \forall u_1, u_2 \in \mathcal{F}^*(X), \tag{10}$$

where c is the contractivity factor of the IFS $\mathbf{w}$. Thus there exists a unique function $\overline{u} \in \mathcal{F}^*(X)$, the *attractor* of the IFZS $(\mathbf{w}, \Phi)$, such that $T_s \overline{u} = \overline{u}$. (Note that the normality of $\overline{u} \in \mathcal{F}^*(X)$ implies that $\overline{u}$ is not identically zero on X.) From Eq. (8), the α-level sets of the attractor $\overline{u}$ obey the following generalized self-tiling property:

$$[\overline{u}]^\alpha = \bigcup_{i=1}^{N} w_i([\phi_i \circ \overline{u}]^\alpha), \quad \alpha \in [0, 1]. \tag{11}$$

The IFZS approach represents a systematic method of constructing functions $u \in \mathcal{F}^*(X)$. There are some serious drawbacks regarding its applicability to the inverse problem:

1. The practicality of the T_s operator, because of the presence of the supremum, is limited to the case where the sets $w_k(X)$ are either disjoint or where they overlap on sets of measure zero, cf. Eq. (9).

2. The d_∞ metric is too restrictive, from both a theoretical perspective as well as the practical viewpoint of image processing. In order to illustrate the former restriction, consider the following two-map IFZS on $X = [0,1]$, where the IFS maps are $w_1(x) = \frac{1}{2}x$ and $w_2(x) = \frac{1}{2}x + \frac{1}{2}$. Now define a family of grey-level map vectors, $\Phi_n = (\phi_{n1}, \phi_{n2})$, where

$$\phi_{n1}(t) = \begin{cases} t, & 0 \leq t < \frac{1}{n} \\ \frac{1}{n}, & \frac{1}{n} \leq t \leq 1 \end{cases} \qquad \phi_{n2}(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{n} \\ \frac{1}{n}, & \frac{1}{n} \leq t < 1 \\ 1, & t = 1. \end{cases} \tag{12}$$

Let $\overline{u}_n$ denote the attractors of the IFZS $(\mathbf{w}, \Phi_n)$, $n = 1, 2, ...$ Then

$$\overline{u}_n(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{n}, & 0 < x < 1 \\ 1, & x = 1. \end{cases} \tag{13}$$

Thus, $[\overline{u}_n]^0 = [0,1]$. Now define $\Phi^* = (\phi_1^*, \phi_2^*)$, where

$$\phi_1^*(t) = 0, 0 \leq t \leq 1, \quad \phi_2^*(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & t = 1. \end{cases} \tag{14}$$

Note that $\| \phi_{ni} - \phi_i^* \|_\infty \to 0$ as $n \to \infty$ for $i = 1, 2$, where $\| \,.\, \|_\infty$ denotes the $\mathcal{L}^\infty(X, \mu)$ norm. Let $\overline{u}^*$ denote the attractor of the IFZS $(\mathbf{w}, \Phi^*)$. Then

$$\overline{u}^*(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1. \end{cases} \tag{15}$$

Clearly, $[\overline{u}^*]^0 = \{1\}$. Therefore, $d_\infty(\overline{u}, \overline{u}_n) = 1$ for all $n \geq 1$, from which it follows that $\lim_{n\to\infty} d_\infty(\overline{u}, \overline{u}_n) \neq 0$. On the other hand, $\| \overline{u} - \overline{u}_n \|_\infty \to 0$ as $n \to \infty$. In other words, convergence of the grey level maps in the $\mathcal{L}^\infty$ norm does not guarantee convergence of fixed points in the d_∞ metric.

Consider two identical photographs, "A" and "B", which contain a region of light shading, e.g. a white shirt. Now place a small black dot on this shirt in photograph B. The d_∞ distance between A and B can now be quite large, even though the photographs are still nearly identical "visually". This is a simple example of the more general problem of determining appropriate distance functions for vision.

3. Again due to the use of the d_∞ metric, there are some strong restrictions on the grey level maps ϕ_i (e.g. to ensure that all level sets $[u]^\alpha, \alpha \in [0,1]$ are nonempty and compact) which can severely limit the degree to which a function can be approximated. The previous "natural assumption" that $\phi_i(0) = 0, i = 1, 2, ..., N$ is, in fact, such a restriction.

4. From a more theoretical perspective, the restrictions on the ϕ_i limits the continuity properties of IFZS fixed point attractors $\overline{u}$ with respect to variations in the ϕ_i [12].

2.3 From IFZS to IFSM on $\mathcal{L}^p(X,\mu)$

2.3.1 Replacing the Hausdorff Metric

We now intend to replace the d_∞ metric involving Hausdorff distances between α-level sets with a weaker metric for the reasons given above. This will enable us to work with larger classes of functions, namely, the $\mathcal{L}^p$ spaces. As a result, our problem of image representation on the fuzzy-set grey level range [0,1] is generalized to that of function approximation on an arbitrary range R_g. From the assumption that $\mu(X) < \infty$,

1. $u \in \mathcal{F}^*(X)$ implies that $u \in \mathcal{L}^p(X,\mu)$ for $p \geq 1$ and
2. $\mathcal{L}^q(X,\mu) \subset \mathcal{L}^p(X,\mu)$ for $1 \leq p \leq q$.

Thus, for the remainder of this section, we relax the restriction that our functions are fuzzy sets, i.e. elements of $\mathcal{F}^*(X)$, and assume that we are working in the space $\mathcal{L}^1(X,\mu)$. Without loss of generality we consider nonnegative functions $u : X \to R_g$, where $R_g \subseteq \mathbf{R}^+$.

Let $\mu \in \mathcal{M}(X)$. For $u, v \in \mathcal{L}^1_+(X,\mu)$, define

$$\begin{aligned} G(u,v;\alpha) &= \int_X |I_{[u]^\alpha}(x) - I_{[v]^\alpha}(x)| d\mu(x) \\ &= \mu([u]^\alpha \bigtriangleup [v]^\alpha), \end{aligned} \tag{16}$$

where $\bigtriangleup$ denotes the symmetric difference operator: For $A, B \subseteq X$, $A \bigtriangleup B = (A \cup B) \setminus (A \cap B)$. Now let ν be a measure on $\mathcal{B}(R_g)$ and define

$$\begin{aligned} g(u,v;\nu) &= \int_{R_g} G(u,v;\alpha) d\nu(\alpha) \\ &= \int_{R_g} \int_X |I_{[u]^\alpha}(x) - I_{[v]^\alpha}(x)| d\mu(x) d\nu(\alpha), \end{aligned} \tag{17}$$

an integration over the product measure space $(X, \mathcal{B}(X), \mu) \times (R_g, \mathcal{B}(R_g), \nu)$. Both spaces are assumed to be σ-finite and, by Fubini's Theorem, the order of integration can be reversed [17], i.e.

$$g(u,v;\nu) = \int_X \int_{R_g} |I_{[u]^\alpha}(x) - I_{[v]^\alpha}| d\nu(\alpha) d\mu(x). \tag{18}$$

The net result is

$$\begin{aligned} g(u,v;\nu) = \nu(\{0\})\mu([u]^0 \bigtriangleup [v]^0) \quad &+ \quad \int_{X_u} \nu(\ (u(x), v(x)]\ ) d\mu(x) \\ &+ \quad \int_{X_v} \nu(\ (v(x), u(x)]\ ) d\mu(x), \end{aligned} \tag{19}$$

where $X_u = \{x \in X : u(x) < v(x)\}$ and $X_v = \{x \in X : v(x) < u(x)\}$. From the triangular inequality property involving symmetric differences of sets (hence measures of these sets), it follows that $g(u, v; \nu)$ is a pseudometric on $\mathcal{L}^1_+(X, \mu)$. In the particular case that $\nu = m^{(1)}$, the Lebesgue measure on R_g, $\nu\{0\} = 0$ and $g(u, v; \nu)$ reduces to

$$g(u, v; \nu) = \int_X |u(x) - v(x)| d\mu(x) = \| u - v \|_1, \tag{20}$$

the $\mathcal{L}^1(X, \mu)$ distance between u and v.

The rather restrictive Hausdorff metric d_∞ over α-level sets has been replaced by a weaker metric involving integrations over X and R_g. In principle, the measure ν may be used to define various types of *greyscales*, e.g (i) quantized grey levels, where ν consists of a finite set of Dirac measures, (ii) nonuniform distributions which model the varying sensitivities of the human eye to different regions of the grey level spectrum. For the remainder of this paper, we shall assume that $\nu = m^{(1)}$. While it appears that only the $\mathcal{L}^1$ distance can be generated by a measure ν, it will be worthwhile to consider $\mathcal{L}^p$ distances in general, $p \geq 1$.

2.3.2 IFSM on $\mathcal{L}^p(X, \mu)$

It now remains to formulate an Iterated Function System with Grey Level Maps (IFSM) - to be distinguished from the IFZS - on the function spaces $\mathcal{L}^p(X, \mu)$. As before the IFSM will consist of two components:

1. an **IFS component**, $\mathbf{w} = \{w_1, w_2, ..., w_N\}$, $w_i \in Con_1(X)$ (note the requirement that the w_i be one-to-one) and
2. a **grey level component**, $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$, $\phi_i : \mathbf{R} \to \mathbf{R}$, with conditions different from the IFZS case, as discussed below.

The distinguishing feature of the IFSM will be a new form for the "Markov" operator $T : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$, designed to easily handle cases where the sets $w_i(X)$ overlap. In the special "nonoverlapping" case where the sets $w_i(X)$ are disjoint, i.e. $w_i(X) \cap w_j(X) = \emptyset$ when $i \neq j$, there is little problem (cf. Eq. (9) for the IFZS case). Each point $x \in X_k$ has only one preimage $y = w_k^{-1}(x)$. We can then define a "nonoverlapping-case" operator T_{non} (which, up to some changes in the ϕ-maps, coincides with the T_s operator) as follows: For $u \in \mathcal{L}^p(X, \mu)$,

$$(T_{non}u)(x) \equiv \begin{cases} \phi_k(u(w_k^{-1}(x))), & for \;\; x \in w_k(X), k \in \{1, 2, ..., N\} \\ 0, & for \;\; x \notin \cup_{l=1}^N w_l(X). \end{cases} \tag{21}$$

Some conditions on the w_i and the ϕ_i which guarantee that T_{non} maps $\mathcal{L}^p(X, \mu)$ into itself will be established in Proposition 2 below.

The condition of disjointness of the $w_i(X)$ may also be weakened to one of nonoverlapping with respect to the measure μ as defined below.

Definition 1 *Let $\mu \in \mathcal{M}(X)$. A set of maps $w_i \in Con(X), i = 1, 2, ..., N$ is said to satisfy a nonoverlapping condition on X with respect to μ if $\mu(X_i \cap X_j) = 0$ whenever $i \neq j$, where $X_k = w_k(X), k = 1, 2, ..., N$.*

In this μ-nonoverlapping case, we may redefine the action of the operator T_{non} as follows: For $u \in \mathcal{L}^p(X, \mu)$,

$$(T_{non})(x) \equiv \begin{cases} \phi_k(u(w_k^{-1}(x))), & x \in w_k(X) - \cup_{l \neq k}^N w_k(X) \cap w_l(X), \\ 0, & x \in \cup_{l \neq k}^N w_k(X) \cap w_l(X). \end{cases} \quad (22)$$

The following estimate may then be obtained for this μ-nonoverlapping case:

Proposition 1 *Let $X \subset \mathbf{R}^D$, $D \in \{1, 2, ...\}$ and $\mu = m^{(D)}$. Let $(\mathbf{w}, \Phi)$ be an N-map IFSM such that*

1. *$w_i \in Sim_1(X)$ for $1 \leq i \leq N$,*
2. *the w_i satisfy a nonoverlapping condition on X with respect to μ,*
3. *$\phi_i \in Lip(\mathbf{R})$ for $1 \leq i \leq N$.*

Let T_{non} be the operator associated with this IFSM, as defined in Eq. (22). Then for a $p \geq 1$ and $u, v \in \mathcal{L}^p(X, \mu)$,

$$d_p(T_{non}u, T_{non}v) \leq C_{non}(D, p) d_p(u, v), \quad C_{non}(D, p) = [\sum_{i=1}^N c_i^D K_i^p]^{1/p}. \quad (23)$$

Proof: The nonoverlapping nature of the $w_k(X)$ allows us to write

$$\begin{aligned} \| T_{non}u - T_{non}v \|_p^p &= \sum_{k=1}^N \int_{X_k} |\phi_k(u(w_k^{-1}(x))) - \phi_k(v(w_k^{-1}(x)))|^p dx \\ &= \sum_{k=1}^N c_k^D \int_X |\phi_k(u(y)) - \phi_k(v(y))|^p dy \\ &\leq \sum_{k=1}^N c_k^D K_k^p \int_X |u(y) - v(y)|^p dy \\ &= [\sum_{k=1}^N c_k^D K_k^p] \| u - v \|_p^p . \quad \blacksquare \end{aligned} \quad (24)$$

Remark: If $C_{non}(D, p) < 1$, then the IFSM $(\mathbf{w}, \Phi)$ possesses a unique attractor $\overline{u} \in \mathcal{L}^p(X, m^{(D)})$. A sufficient, but not necessary, condition for contractivity is that all ϕ_i are contractive on R_g, i.e. $K_i < 1, i = 1, 2, ..., N$.

The use of nonoverlapping maps w_i (a standard procedure in the literature) is rather limited in scope. For greater flexibility, it would be desirable to use an operator T which can handle cases where the sets $w_i(X)$ overlap. However, when

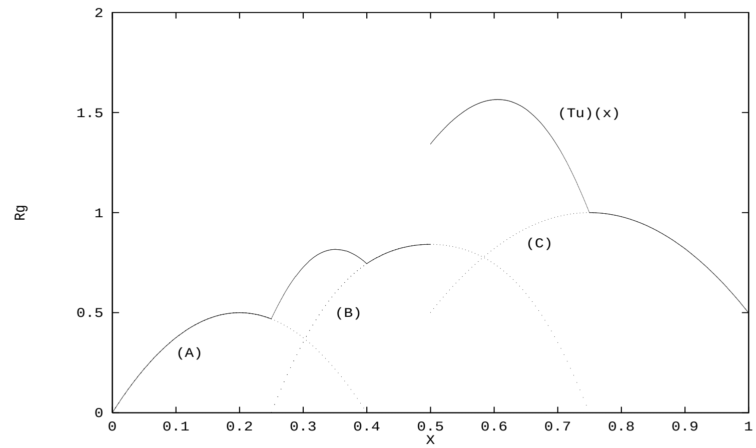

Figure 1: Graph of $(Tu)(x)$ where $u(x) = 4x(1-x)$ and T is the contractive operator for the following three-map IFSM on $X = [0,1]$: $w_1(x) = 0.4x$, $w_2(x) = 0.5x + 0.25$, $w_3(x) = 0.5x + 0.5$ $\phi_1(t) = 0.5t$, $\phi_2(t) = \sin t$, $\phi_3(t) = 0.5t + 0.5$. Also plotted (dotted graphs) are the "component" functions $\phi_i(u(w_i^{-1}(x)))$, $i = 1, 2, 3$, denoted as (A), (B) and (C), respectively.

a point $x \in X$ has more than one preimage, e.g. $\{w_{i_1}^{-1}(x), w_{i_2}^{-1}(x), ..., w_{i_K}^{-1}(x)\}$, $K > 1$, there exists the question of how to "combine" these values to produce the result $(Tu)(x)$. One of many possibilities is that $(Tu)(x)$ assume a value between the maximum and minimum of this set (e.g. convex combination). However, the use of the *sup* (or *inf*) as in the IFZS case is not practical. Since we will be working in $\mathcal{L}^p$ spaces, it is desirable to employ an operator which commutes with integration. As such, we adopt the following form for the operator T associated with an N-map IFSM $(\mathbf{w}, \Phi)$:

$$(Tu)(x) = {\sum_{k=1}^{N}}' \phi_k(u(w_k^{-1}(x))), \tag{25}$$

where the prime signifies that for a given $x \in X$, the summation is performed only over those k for which $w_k^{-1}(x)$ is defined. If $w_k^{-1}(x) = \emptyset, 1 \leq k \leq N$, then $(Tu)(x) \equiv 0$. The action of the operator T for an "overlapping" IFSM is illustrated in Figure 1.

We now establish some sufficient conditions on the IFS and grey level maps to ensure that the associated operator T maps $\mathcal{L}^p(X, \mu)$ into itself.

Proposition 2 *Let $(\mathbf{w}, \Phi)$ denote an N-map IFSM with associated operator T defined above. Assume that:*

1. For any $u \in \mathcal{L}^p(X,\mu)$, $u \circ w_k^{-1} \in \mathcal{L}^p(X,\mu)$, $1 \le k \le N$,

2. $\phi_k \in Lip(\mathbf{R})$, $1 \le k \le N$.

Then for $1 \le p < \infty$, $T : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu)$.

Proof: Let $u \in \mathcal{L}^p(X,\mu)$. Then for $1 \le k \le N$, we have

$$\begin{aligned} |(\phi_k \circ u \circ w_k^{-1})(x)|^p &= |(\phi_k \circ u \circ w_k^{-1})(x) - (\phi_k \circ u)(x) + (\phi_k \circ u)(x)|^p \\ &\le 2^{p-1}|(\phi_k \circ u \circ w_k^{-1})(x) - (\phi_k \circ u)(x)|^p \\ &\quad +2^{p-1}|(\phi_k \circ u)(x)|^p. \end{aligned} \tag{26}$$

From the Lipschitz condition on the ϕ_k,

$$|(\phi_k \circ u \circ w_k^{-1})(x)|^p \le \psi_k(x), \tag{27}$$

where

$$\psi_k(x) \equiv 2^{p-1}K_k|(u \circ w_k^{-1})(x) - u(x)|^p + 2^{p-1}|(\phi_k \circ u)(x)|^p. \tag{28}$$

From Assumption 1 and the fact that $\mathcal{L}^p(X,\mu)$ is a linear space, $u \circ w_k^{-1} - u \in \mathcal{L}^p(X,\mu)$. Moreover, from Assumption 2, $\phi_k \circ u \in \mathcal{L}^p(X,\mu)$. Therefore $\psi_k \in \mathcal{L}^p(X,\mu)$. From the inequality in (27), it follows that $\phi_k \circ u \circ w_k^{-1} \in \mathcal{L}^p(X,\mu)$. This, in turn, implies that $\sum_{k=1}^N \phi_k \circ u \circ w_k^{-1} \in \mathcal{L}^p(X,\mu)$. From our definition of the T operator, it follows that $Tu \in \mathcal{L}^p(X,\mu)$. ■

Remarks:

1. If the measure μ is regular, then weak conditions on the w_i (e.g. w_i affine) guarantee the property stated in Assumption 1.

2. The Lipschitz condition on the ϕ_i is probably subject to weakening but this is the subject of further work.

Proposition 3 *Let* $(\mathbf{w},\Phi)$ *be an* N*-map IFSM such that* $\phi_k(t) = \xi_k$, *where* $\xi_k \in \mathbf{R}, 1 \le k \le N$. *Then for any* $p \in [1,\infty)$ *and* $\mu \in \mathcal{M}(X)$, *the associated operator* T *is contractive on* $(\mathcal{L}^p(X,\mu), d_p)$, *with contraction factor* $C = 0$. *Furthermore, the fixed point* $\overline{u}$ *of* T *is given by the step function*

$$\overline{u}(x) = \sum_{k=1}^N \xi_k I_{w_k(X)}(x), \quad x \in X. \tag{29}$$

Proof: For $u, v \in \mathcal{L}^p(X)$,

$$\| Tu - Tv \|_p = [\int_X |\sum_{k=1}^{N}{}'[\phi_k(u(w_k^{-1}(x))) - \phi_k(v(w_k^{-1}(x)))]|^p d\mu]^{1/p}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{N}[\int_{X_k} |\phi_k(u(w_k^{-1}(x))) - \phi_k(v(w_k^{-1}(x)))|^p d\mu]^{1/p} \\
&= \sum_{k=1}^{N}\int_{X_k} |\xi_k - \xi_k|^p d\mu \\
&= 0. \qquad (30)
\end{aligned}$$

From the definition of T in Eq. (25), it follows that for any $u \in \mathcal{L}^p(X,\mu)$, $(Tu)(x) = \overline{u}(x)$, $\forall x \in X$. ∎

Proposition 4 *Let $X \subset \mathbf{R}^D$, $D \in \{1, 2, ...\}$, and $\mu = m^{(D)}$. Let $(\mathbf{w}, \Phi)$ be an N-map IFSM such that*

1. *$w_k \in Sim_1(X)$ and*
2. *$\phi_k \in Lip(\mathbf{R})$, $1 \leq k \leq N$.*

Then for a $p \in [1,\infty)$ and any $u, v \in \mathcal{L}^p(X,\mu)$,

$$d_p(Tu, Tv) \leq C(D,p) d_p(u,v), \quad C(D,p) = \sum_{k=1}^{N} c_k^{D/p} K_k. \qquad (31)$$

Proof: For $u, v \in \mathcal{L}^p(X,\mu)$,

$$\begin{aligned}
\| Tu - Tv \|_p &= [\int_X |\sum_{k=1}^{N}{}'[\phi_k(u(w_k^{-1}(x))) - \phi_k(v(w_k^{-1}(x)))]|^p dx]^{1/p} \\
&\leq \sum_{k=1}^{N}[\int_{X_k} |\phi_k(u(w_k^{-1}(x))) - \phi_k(v(w_k^{-1}(x)))|^p dx]^{1/p} \\
&= \sum_{k=1}^{N} c_k^{D/p}[\int_X |\phi_k(u(y)) - \phi_k(v(y))|^p dy]^{1/p} \\
&\leq \sum_{k=1}^{N} c_k^{D/p} K_k [\int_X |u(y) - v(y)|^p dy]^{1/p} \\
&= [\sum_{k=1}^{N} c_k^{D/p} K_k] \| u - v \|_p . \quad ∎ \qquad (32)
\end{aligned}$$

Remarks:

1. If $C(D,p) < 1$, then T is contractive over the space $(\mathcal{L}^p(X, m^{(D)}), d_p)$ and possesses a unique fixed point $\overline{u} \in \mathcal{L}^p(X, m^{(D)})$.
2. Note that for $C(D,p) < 1$, which implies that T is contractive, we can relax the restriction that all IFS maps be contractive, i.e. that $c_k < 1$ for $1 \leq k \leq N$.

3. In the special case that the w_i satisfy a nonoverlapping condition on X with respect to $m^{(D)}$, then an improved upper bound can be obtained from Proposition 2, namely,

$$d_p(Tu, Tv) \leq C_{non}(D,p) d_p(u,v), \quad C_{non}(D,p) = [\sum_{k=1}^{N} c_k^D K_k^p]^{1/p}. \tag{33}$$

Note that $C_{non}(D,p) \leq C(D,p)$. Because of the nonoverlapping condition, we also have

$$\sum_{k=1}^{N} c_k^D \leq 1, \tag{34}$$

which leads to the weaker inequality

$$C_{non}(D,p) \leq K, \quad K \equiv \max_{1 \leq k \leq N} K_k. \tag{35}$$

Examples:

1. The IFSM used in Figure 1 is contractive for $p \geq 1$. Its attractor is sketched in Figure 2.

2. $X = [0,1]$, $\mu = m^{(1)}$, $N = 3$, $w_i(x) = \frac{1}{3}(x+i-1), i = 1,2,3$, with grey level maps $\phi_1(t) = \frac{1}{2}t$, $\phi_2(t) = \frac{1}{2}$, $\phi_3(t) = \frac{1}{2}t + \frac{1}{2}$. The fixed point $\overline{u}(x)$ is (up to an equivalence class) the "Devil's staircase function" which is continuous at almost all $x \in X$ and differentiable for all $X \setminus C$, where C denotes the ternary Cantor set on [0,1].

3. $X = [0,1]$, $N = 3$, $\mu = m^{(1)}$, $w_i(x) = \frac{1}{3}(x+i-1), i = 1,2,3$, with grey level maps $\phi_1(t) = \frac{1}{4}t$, $\phi_2(t) = \frac{1}{4}t$, $\phi_3(t) = 2t$. Then $\overline{u} \equiv 0$ is a fixed point of T. However, T is contractive only on the space $(\mathcal{L}^1(X, m^{(1)}), d_1)$.

We now establish a rather simple continuity property of IFSM attractors. Since our applications to the inverse problem will involve only IFSM with fixed IFS maps w_k, we consider only continuity with respect to the grey level maps $\mathbf{\Phi}$. It will be convenient to denote two N-map IFSM with the same IFS maps $\mathbf{w}$ as $(\mathbf{w}, \mathbf{\Phi}_k), k = 1,2$, where $\mathbf{\Phi}_k = \{\phi_{k1}, ..., \phi_{kN}\}$. First define the following metric for grey level map vectors,

$$d_{\mathbf{\Phi}}^N(\mathbf{\Phi}_1, \mathbf{\Phi}_2) = \max_{1 \leq i \leq N} \sup_{t \in \mathbf{R}} |\phi_{1i}(t) - \phi_{2i}(t)|, \tag{36}$$

Proposition 5 *Let $(\mathbf{w}, \Phi_1)$ be an N-map contractive IFSM with fixed point $\overline{u}_1 \in \mathcal{L}^p(X,\mu)$. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all contractive N-map IFSM $(\mathbf{w}, \Phi_2)$ satisfying $d_{\mathbf{\Phi}}^N(\mathbf{\Phi}_1, \mathbf{\Phi}_2) < \delta_2$, it follows that $d_p(\overline{u}_1, \overline{u}_2) < \epsilon$, where $\overline{u}_2$ is the fixed point of the IFSM $(\mathbf{w}, \Phi_2)$.*

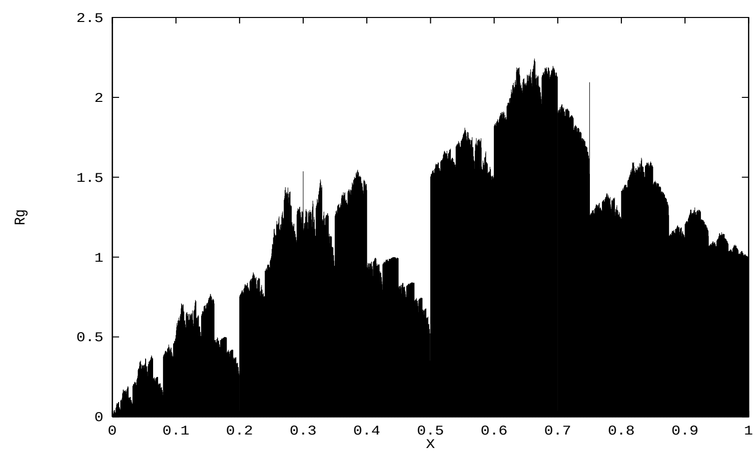

Figure 2: The fixed point attractor $\overline{u}(x)$ of the IFSM of Figure 1.

Proof: Let $Y = \mathcal{L}^p(X,\mu)$. Also let $T_i \in Con(Y), i = 1,2$, be the operators for the IFSM $(\mathbf{w}, \Phi_i)$, with contractivity factors $C_i < 1$, respectively. Then

$$d_{Con(Y)}(T_1, T_2) = \sup_{u \in Y} \| T_1 u - T_2 u \|_p \ . \tag{37}$$

For $1 \leq k \leq N$, define

$$f_k(x) = \begin{cases} \phi_{1k}(u(w_k^{-1}(x))) - \phi_{2k}(u(w_k^{-1}(x))), & x \in X_k, \\ 0, & x \notin X_k. \end{cases} \tag{38}$$

Then

$$\begin{aligned} d_{Con(Y)}(T_1, T_2) &= \sup_{u \in Y} \| \sum_{k=1}^{N} f_k \|_p \qquad (39) \\ &\leq \sup_{u \in Y} \sum_{k=1}^{N} \| f_k \|_p \\ &= \sup_{u \in Y} \sum_{k=1}^{N} [\int_{X_k} |\phi_{1k}(u(w_k^{-1}(x))) - \phi_{2k}(u(w_k^{-1}(x)))|^p d\mu]^{1/p} \\ &\leq M d_\Phi^N(\Phi_1, \Phi_2), \end{aligned}$$

where $M = \sum_{k=1}^{N} \mu(X_k)^{1/p}$. For a given $\epsilon > 0$, let

$$\delta = \epsilon[1 - C_1]M^{-1}[d_\Phi^N(\Phi_1, \Phi_2)]^{-1}, \tag{40}$$

From Theorem 3, it follows that $d_p(\overline{u}_1, \overline{u}_2) < \epsilon$. ∎

Remark: In the particular case that

1. $X \subset \mathbf{R}^D$, $D \in \{1, 2, ...\}$, $\mu = m^{(D)}$,

2. $w_i \in Sim_1(X)$, $1 \le i \le N$,

the constant $M = [m^{(D)}(X)]^{1/p} \sum_{k=1}^{N} c_k^{D/p}$.

2.3.3 Affine IFSM on $\mathcal{L}^p(X, \mu)$

In applications, it is convenient to employ affine IFS maps w_k as well as affine grey level maps ϕ_i. The latter have the form

$$\phi_k(t) = \alpha_k t + \beta_k, \quad t \in \mathbf{R}, \quad k = 1, 2, ..., N. \tag{41}$$

We refer to such a system $(\mathbf{w}, \Phi)$ as an *affine IFSM.* The action of the operator T associated with an affine IFSM may be written as follows: For $u \in \mathcal{L}^p(X, \mu)$,

$$(Tu)(x) = \sum_{k=1}^{N} [\alpha_k u(w_k^{-1}(x)) + \beta_k I_{w_k(X)}(x)]. \tag{42}$$

For affine IFSM on $X \subset \mathbf{R}^D$, it follows from Proposition 2 that for all $u, v \in \mathcal{L}^p(X, m^{(D)})$,

$$d_p(Tu, Tv) \le C(D, p) d_p(u, v), \qquad C(D, p) = \sum_{i=1}^{N} c_i^{D/p} |\alpha_i|. \tag{43}$$

If $C(D, p) < 1$ then T is contractive on $(\mathcal{L}^p(X, \mu), d_p)$ and possesses a unique fixed point $\overline{u}$, i.e. $T\overline{u} = \overline{u}$.

Remarks:

1. If $\beta_k = 0$ for $1 \le k \le N$, then $\overline{u}(x) \equiv 0$ is a fixed point of T.

2. Let $X = [0, 1]$, with $w_i(x) = s_i x + a_i, 1 \le i \le N$. If T is contractive with fixed point $\overline{u}$, then from Eq. (42),

$$\begin{aligned} \overline{u}(x) &= \sum_{k=1}^{N}{}' \alpha_k \overline{u}\left(\frac{x - a_k}{s_k}\right) + \beta_k I_{w_k(X)}(x) && (44) \\ &= \sum_{k=1}^{N}{}' [\alpha_k \psi_k(x) + \beta_k \chi_k(x)]. && (45) \end{aligned}$$

   In other words, $\overline{u}$ may be written as a linear combination of both piecewise constant functions $\chi_k(x)$ as well as functions $\psi_k(x)$ which are obtained by dilatations and translations of $\overline{u}(x)$ and $I_X(x) = 1$, respectively. This is reminiscent of the rôle of scaling functions in wavelet theory.

3. From Proposition 3, if $\alpha_k = 0$ for $1 \le k \le N$, then T is a contraction mapping with factor $C(D, p) = 0$.

Proposition 6 *Let $X \subset \mathbf{R}^D$, $D \in \{1, 2, ...\}$, and $\mu = m^{(D)}$. Let $(\mathbf{w}, \Phi)$ be an N-map IFSM such that*

1. *$w_k \in Sim_1(X)$ and*
2. *$\phi_k \in Lip(\mathbf{R})$, $1 \leq k \leq N$.*

Assume that for a $p \geq 1$, $C(D,p) < 1$, i.e. the operator T is contractive on $\mathcal{L}^p(X, \mu)$ with fixed point $\overline{u}$. Then

$$\| \overline{u} \|_p \leq \frac{B(D,p)}{1 - C(D,p)}, \qquad B(D,p) = \sum_{k=1}^{N} c_k^{D/p} |\beta_k|. \tag{46}$$

Proof: Taking the $\mathcal{L}^p$ norms of both sides of the fixed point relation $\overline{u} = T\overline{u}$ yields

$$\begin{aligned} \| \overline{u} \|_p &= \sum_{k=1}^{N} \| \alpha_k u(w_k^{-1}(x)) + \beta_k I_{w_k(X)}(x) \|_p \\ &\leq \sum_{k=1}^{N} \| \alpha_k u(w_k^{-1}(x)) + \beta_k I_{w_k(X)}(x) \|_p \\ &\leq \sum_{k=1}^{N} \| \alpha_k u(w_k^{-1}(x)) \|_p + \sum_{k=1}^{N} \| \beta_k I_{w_k(X)}(x) \|_p \\ &= \sum_{k=1}^{N} c_k^{D/p} |\alpha_k| \, \| \overline{u} \|_p + \sum_{k=1}^{N} c_k^{D/p} |\beta_k|. \end{aligned} \tag{47}$$

A rearrangement yields the desired result. ∎

Affine IFSM are used primarily because of their simplicity in practical calculations. The following result ensures that their use is sufficient from a theoretical perspective.

Theorem 4 *Let $X = \mathbf{R}^D$ and $\mu \in \mathcal{M}(X)$. For a $p \geq 1$, define $\mathcal{L}^p_A(X, \mu) \subset \mathcal{L}^p(X, \mu)$ to be the set of fixed points $\overline{u}$ of all contractive N-map affine IFSM $(\mathbf{w}, \Phi)$ for $1 \leq N < \infty$, on X. Then $\mathcal{L}^p_A(X, \mu)$ is dense in $(\mathcal{L}^p(X, \mu), d_p)$.*

Proof: We prove the theorem for the case $D = 1$. The same argument can be extended to higher dimensions.

Let $\mathcal{S}(X)$ denote the set of all step functions in X. Then for each $\sigma \in \mathcal{S}(X)$, there exists an N_σ, $1 \leq N_\sigma < \infty$, a set of numbers $\xi_k \in \mathbf{R}$ and a set of intervals $J_k = [a_k, b_k] \subset [0, 1]$, with $a_k \leq b_k$, $k = 1, 2, ..., N_\sigma$, such that

$$\sigma(x) = \sum_{k=1}^{N_\sigma} \xi_k I_{J_k}(x), \quad x \in X. \tag{48}$$

However, the function $\sigma(x)$ is the attractor for the N_σ-map affine IFSM $(\mathbf{w}, \Phi)$ given by

$$w_k(x) = (b_k - a_k)x + a_k, \quad \phi_k(t) = \xi_k, \quad k = 1, 2, ..., N_\sigma. \tag{49}$$

(From Proposition 4, the contractivity factor of this IFSM is $C(D,p) = 0$.) Thus $\mathcal{S}(X) \subset \mathcal{L}^p_A(X,\mu)$. Since $\mathcal{S}(X)$ is dense in $(\mathcal{L}^p(X,\mu), d_p)$ [17], the theorem is proved for $D = 1$. ∎

In Appendix A are given some elementary relations for integrals involving IFSM. In particular, we consider affine IFSM and relations involving moments of functions.

2.4 "Place-Dependent" IFSM

Before closing this section we mention that a more generalized "place-dependent" IFSM, or PDIFSM (in analogy with IFS with place-dependent probabilities [3]) on function spaces can be formulated with the following two components:

1. an **IFS component**, $\mathbf{w} = \{w_1, w_2, ..., w_N\}$, $w_i \in Con_1(X)$, as before, and
2. a **grey level component**, $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$, $\phi_i : \mathbf{R} \times X \to \mathbf{R}$, with suitable conditions.

The operator T associated with an N-map PDIFSM $(\mathbf{w}, \Phi)$ have the form

$$(Tu)(x) = \sum_{k=1}^{N}{}' \phi_k(u(w_k^{-1}(x)), w_k^{-1}(x)), \tag{50}$$

In other words, the ϕ_i are dependent both on the grey-level value at a preimage as well as the location of the preimage itself. Much of the theory developed above for IFSM extends to place-dependent IFSM and we outline the important points in Appendix C. The additional flexibility in this formulation and its effectiveness in coding images have been discussed in the literature [21, 23]. We present some results of computations in Section 5.

3. The Inverse Problem for IFSM on $\mathcal{L}^p(X,\mu)$

We now present a formal solution to the inverse problem followed by an algorithm to compute IFSM approximations of a target function v to arbitrary accuracy.

3.1 The Collage Theorem and a Formal Solution to the Inverse Problem

From the Collage Theorem (Theorem 2), the inverse problem for the approximation of functions in $\mathcal{L}^p(X,\mu)$ by IFSM may now be posed as follows:

Given a target function $v \in \mathcal{L}^p(X,\mu)$ and a $\delta > 0$, find an IFSM $(\mathbf{w}, \Phi)$ with associated operator T such that $d_p(v, Tv) < \delta$.

In our formal solution to the inverse problem, we shall be constructing sequences of N-map IFSM, denoted as $(\mathbf{w}^N, \Phi^N)$, with $N = 1, 2, 3, ...$, where the IFS maps in $\mathbf{w}^N$ are chosen from a fixed and infinite set $\mathcal{W}$ of contraction maps. It will be necessary to impose some conditions on this set, according to the following definitions.

Definition 2 *Let (X, d) be a compact metric space and $\mu \in \mathcal{M}(X)$. A family $\mathcal{A}$ of subsets $A = \{A_i\}$ of X is "μ-dense" in a family $\mathcal{B}$ of subsets B of X if for every $\epsilon > 0$ and any $B \in \mathcal{B}$ there exists a collection $A \in \mathcal{A}$ such that $A \subseteq B$ and $\mu(B \setminus A) < \epsilon$.*

Definition 3 *Let $\mathcal{W} = \{w_1, w_2, ...\}$, $w_i \in Con(X)$ be an infinite set of contraction maps on X. We say that $\mathcal{W}$ generates a "μ-dense and nonoverlapping" - to be abbreviated as "μ-d-n" - family $\mathcal{A}$ of subsets of X if for every $\epsilon > 0$ and every $B \subseteq X$ there exists a finite set of integers $i_k \geq 1, 1 \leq k \leq N$, such that*

1) $A \equiv \cup_{k=1}^N w_{i_k}(X) \subseteq B$,

2) $\mu(B \setminus A) < \epsilon$ and

3) $\mu(w_{i_k}(X) \cap w_{i_l}(X)) = 0$ if $k \neq l$.

A useful set of affine maps satisfying such a condition on $X = [0, 1]$ with respect to Lebesgue measure is given by the following "wavelet-type" functions:

$$w_{ij}(x) = 2^{-i}(x + j - 1), \; i = 1, 2, ..., \; j = 1, 2, ..., 2^i. \tag{51}$$

For each $i^* \geq 1$, the set of maps $\{w_{i^*j}, j = 1, 2, ..., 2^{i^*}\}$ provides a set of 2^{-i^*} contractions of [0,1] which tiles [0,1]. The set $\mathcal{W}$ provides N-map IFS with arbitrarily small degrees of refinement on (X, d).

Now let $\mathcal{W} = \{w_1, w_2, ...\}$, $w_i \in Con_1(X)$ be an infinite set of one-to-one contraction maps on X satisfying the μ-d-n property. Also let

$$\mathbf{w}^N = \{w_1, w_2, ..., w_N\}, \quad N = 1, 2, ..., \tag{52}$$

denote N-map truncations of $\mathcal{W}$. For each $N \geq 1$, let

$$\Phi^N = \{\phi_1, \phi_2, ..., \phi_N\}, \tag{53}$$

denote an associated N-vector of grey level maps with the restriction that the $\phi_i \in Lip(\mathbf{R})$. Now let $T^N : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$ be the operator associated with the N-map IFSM $(\mathbf{w}^N, \Phi^N)$. Given a target function $v \in \mathcal{L}^p(X, \mu)$, the following result ensures that the collage distance $\| v - T^N v \|_p$ can be made arbitrarily small.

Theorem 5 *Let $v \in \mathcal{L}^p(X, \mu)$, where $p \in [1, \infty)$. Assume that the infinite set of IFS maps $\mathcal{W} = \{w_1, w_2, ...\}, w_i \in Con_1(X)$, generates a μ-d-n family $\mathcal{A}$ of subsets of X. Then*

$$\lim_{N \to \infty} \inf \| v - T^N v \|_p = 0. \tag{54}$$

Proof: For $n > 0$, define

$$\begin{aligned} B_{ni} &= \{x \in X : \frac{i-1}{2^n} \leq v(x) < \frac{i}{2^n}\}, \quad 1 \leq i \leq 2^{2n}, \\ B_{n,2^{2n}+1} &= \{x \in X : v(x) \geq 2^n\}. \end{aligned} \tag{55}$$

Each set $B_{ni} \subseteq X$ is measurable in (X, μ). Now define the function,

$$v_n(x) = \sum_{i=1}^{2^{2n}} \frac{i}{2^n} I_{B_{ni}}(x) + 2^n I_{B_{2^{2n}+1}}(x). \tag{56}$$

From the definition of the Lebesgue integral, $\| v - v_n \|_p \to 0$ as $n \to \infty$. Now let an $\epsilon > 0$ be given. We choose n to be sufficiently large so that

$$\| v - v_n \|_p < \frac{\epsilon}{2}. \tag{57}$$

Define $\delta_{ni} = 0$ if $\mu(B_{ni}) = 0$ and $\delta_{ni} = 1$ otherwise, for $1 \leq i \leq 2^{2n} + 1$. Then

$$v_n(x) = \sum_{i=1}^{2^{2n}} \delta_{ni} \frac{i}{2^n} I_{B_{ni}}(x) + \delta_{n,2^{2n}+1} 2^n I_{B_{2^{2n}+1}}(x). \tag{58}$$

Define

$$\eta_{ni} = \frac{\epsilon}{2^{n+2} i}, \quad 1 \leq i \leq 2^{2n}, \quad \eta_{n,2^{2n}+1} = \frac{\epsilon}{2^{n+2}}. \tag{59}$$

From our μ-d-n assumption on the IFS maps $\mathbf{w}$ we can find, for each B_{ni}, $1 \leq i \leq 2^{2n} + 1$, a finite set of IFS maps $\mathbf{w}_i = \{w_{i_1}, w_{i_2}, ..., w_{i_{n_i}}\}$ such that $\mu(w_{i_k}(X) \cap w_{i_l}(X)) = 0$ for $k \neq l$ and

$$\mu(B_{ni} \setminus \widehat{\mathbf{w}}_i(X)) < \eta_{ni}, \quad 1 \leq i \leq 2^{2n} + 1, \tag{60}$$

where $\widehat{\mathbf{w}}_i(X) \equiv \cup_{k=1}^{n_i} w_{i_k}(X)$. Now define the function

$$u_n(x) = \sum_{i=1}^{2^{2n}} \delta_{ni} \frac{i}{2^n} I_{\widehat{\mathbf{w}}_i(X)}(x) + \delta_{n,2^{2n}+1} 2^n I_{\widehat{\mathbf{w}}_{2^{2n}+1}(X)}(x). \tag{61}$$

Clearly, $u_n(x) \in \mathcal{L}^p(X, \mu)$ and

$$\| v_n - u_n \|_p = \left[\int_X \left| \sum_{i=1}^{2^{2n}} \delta_{ni} \frac{i}{2^n} I_{B_{ni} \setminus \widehat{\mathbf{w}}_i(X)}(x) + \delta_{n,2^{2n}+1} 2^n I_{B_{n,2^{2n}+1} \setminus \widehat{\mathbf{w}}_{2^{2n}+1}(X)}(x) \right|^p d\mu\right]^{1/p} \tag{62}$$

By Minkowski's inequality, i.e.

$$\| a + b \|_p \leq \| a \|_p + \| b \|_p, \quad a, b \in \mathcal{L}^p(X, \mu), \tag{63}$$

it follows that

$$\begin{aligned}\| v_n - u_n \|_p &\leq \sum_{i=1}^{2^{2n}} \delta_{ni} \frac{i}{2^n} \mu(B_{ni} \setminus \widehat{\mathbf{w}}_i(X)) \\ &\qquad + \delta_{n,2^{2n}+1} 2^n \mu(B_{n,2^{2n}+1} \setminus \widehat{\mathbf{w}}_{2^{2n}+1}(X)) \\ &< \sum_{i=1}^{2^{2n}} \frac{i}{2^n} \eta_{ni} + 2^n \eta_{n,2^{2n}+1} \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2}. \end{aligned} \tag{64}$$

For each $i \in \{1, 2, ..., 2^{2n}\}$ such that $\delta_{ni} \neq 0$, i.e. $B_{ni} = v^{-1}[\frac{i-1}{2^n}, \frac{i}{2^n}) \neq \emptyset$ and $\mu(B_{ni}) > 0$, we can find an $x_i \in \widehat{\mathbf{w}}_i(X)$ such that $\xi_i \equiv v(x_i) \in [\frac{i}{2^n}, \frac{i+1}{2^n})$. As well, if $\delta_{n,2^{2n}+1} \neq 0$, we can find an $x_{2^{2n}+1} \in \widehat{\mathbf{w}}_{2^{2n}+1}(X)$ such that $\xi_{n,2^{2n}+1} \equiv v(x_{2^{2n}+1}) \in [2^n, \infty)$. Now define

$$\overline{u}_n(x) = \sum_{i=1}^{2^{2n}} \delta_{ni} \xi_{ni} I_{\widehat{\mathbf{w}}_i(X)}(x) + \delta_{n,2^{2n}+1} \xi_{n,2^{2n}+1} I_{\widehat{\mathbf{w}}_{2^{2n}+1}(X)}(x). \tag{65}$$

The function $\overline{u}_n$ is the fixed point for the IFSM composed of the IFS maps $w_{i_k}, 1 \leq k \leq n_i$ contained in $\mathbf{w}_i$, $1 \leq i \leq 2^{2n}+1$. Associated with each IFS map w_{i_k} is the (constant) grey level map $\phi_{i_k}(t) = \xi_{ni}, t \in \mathbf{R}$, which obviously belongs to $Lip(\mathbf{R})$. Let T_n denote the operator associated with this IFSM. From Proposition 4, its contraction factor is $C_n = 0$.

Let N be the smallest integer such that the truncation $\mathbf{w}^N$ of $\mathbf{w}$ contains all the IFS maps $\mathbf{w}_i = \{w_{i_1}, w_{i_2}, ..., w_{i_{n_i}}\}$ for $1 \leq i \leq 2^{2n}+1$. Now, for each IFS map $w_{i_k} \in \mathbf{w}_i$, $1 \leq i \leq n_i$, associate the (constant) grey level map $\phi_{i_k} = \xi_{nk}$. For all other $w_l \in \mathbf{w}^N$, $l \in \{1, 2, ..., N\}$ such that $w_l \notin \mathbf{w}_i$ for $1 \leq i \leq 2^{2n}+1$, let $\phi_l = 0$. Let T^N be the operator for the resulting IFSM $(\mathbf{w}^N, \Phi^N)$. Then $T^N \overline{u}_n = \overline{u}_n$. Furthermore, the contraction factor of this operator is $C^N = 0$. We then have the inequality

$$\| v - T^N v \|_p \leq \| v - v_n \|_p + \| v_n - \overline{u}_n \|_p + \| \overline{u}_n - T^N v \|_p . \tag{66}$$

Note that

$$\begin{aligned}\| \overline{u}_n - T^N v \|_p &= \| T^N \overline{u}_n - T^N v \|_p \\ &= 0 \end{aligned} \tag{67}$$

The net result is

$$\begin{aligned}\| v - T^N v \|_p &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{68}$$

Hence, given an $\epsilon = 2^{-k}, k \in \{1,2,3,...\}$, we can find an N_k and a finite IFSM $(\mathbf{w}^{N_k}, \Phi^{N_k})$, with ϕ_{i_k} constant (hence belonging to $Lip(\mathbf{R})$) such that $\| v - T^{N_k} v \|_p < 2^{-k}$. Thus $\lim_{N\to\infty} \inf \| v - T^N v \|_p = 0$. ∎

3.2 The Inverse Problem in $\mathcal{L}^2(X, \mu)$ as a Quadratic Programming Problem

We now describe an algorithm for the construction of IFSM approximations of arbitrary accuracy to a target set $v \in \mathcal{L}^2(X,\mu) \subset \mathcal{L}^1(X,\mu)$. Because of our primary interest in the problem of image representation, our discussion is restricted to the approximation of nonnegative (and bounded) *image functions*, $u : X \to \mathbf{R}^+$. Therefore, for the remainder of this section, we assume that $R_g = \mathbf{R}^+$, hence $v \in \mathcal{L}^2_+(X,\mu)$. (There is no loss of generality in this assumption since the contractivity of the T operator is unaffected by (i) reversals in sign of the α_i grey-level map parameters or (ii) shifts in the β_i grey-level map parameters.) For an N-map contractive IFSM $(\mathbf{w}, \Phi)$ on (X, d) with associated operator T, the squared $\mathcal{L}^2$ collage distance is given by

$$\begin{aligned} \Delta^2 &= \| v - Tv \|_2^2 \\ &= \int_X [\sum_{k=1}^{N}{}' \phi_k(v(w_k^{-1}(x))) - v(x)]^2 d\mu. \end{aligned} \tag{69}$$

Following our discussion in the previous section (and our strategy in [13]), we consider the IFS maps w_i to be fixed. The problem reduces to the determination of grey level maps ϕ_i which minimize the collage distance Δ^2. In the special "μ-nonoverlapping case, i.e.,

1. $\cup_{k=1}^N X_k = \cup_{k=1}^N w_i(X) = X$, i.e. the sets $X_k = w_k(X)$ "tile" X, and
2. $\mu(w_i(X) \cap w_j(X)) = 0$ for $i \neq j$, then

the squared collage distance Δ^2 becomes

$$\begin{aligned} \Delta^2 &= \sum_{k=1}^{N} \int_{X_k} [\phi_k(v(w_k^{-1}(x)) - v(x)]^2 d\mu \\ &= \sum_{k=1}^{N} \Delta_k^2, \end{aligned} \tag{70}$$

i.e. the sum of collage distances over the nonoverlapping subsets X_k. The minimization of each integral is a continuous version of "least squares" with respect to the measure μ: For each subset X_k, find the $\phi_k : R_g \to R_g$ which provides the best $\mathcal{L}^2(X,\mu)$ approximation to the graph of $v(x)$ vs. $v(w_k^{-1}(x))$ for $x \in X_k$.

Because of its simplicity, most, if not all, applications in the literature assume the μ-nonoverlapping property, with $\mu = m^{(D)}$ and $w_k \in Sim_1(X)$. Because of its widespread use, we examine this special case in more detail in Appendix B.

In the following discussion, however, we consider the more general case where the sets $w_i(X)$ can overlap on sets of nonzero μ-measure. We also assume the following:

1. $\cup_{k=1}^{N} w_k(X) = X$, i.e. the $w_i \in Con_1(X)$ "tile" X. Note that $w_i \in Con_1(X)$ implies that $c_i > 0$ for $1 \le i \le N$.

2. the grey level maps are affine, i.e. $\phi_i : \mathbf{R}^+ \to \mathbf{R}^+$, where $\phi_i(t) = \alpha_i t + \beta_i,\ \ t \in \mathbf{R}^+$. Thus, $\alpha_i, \beta_i \ge 0$ for $1 \le i \le N$.

The squared $\mathcal{L}^2$ collage distance then becomes

$$\begin{aligned}
\Delta^2 &= < v - Tv, v - Tv > \\
&= \sum_{k=1}^{N} \sum_{l=1}^{N} [< \psi_k, \psi_l > \alpha_k \alpha_l \ + \ 2 < \psi_k, \chi_l > \alpha_k \beta_l + \ < \chi_k, \chi_l > \beta_k \beta_l] \\
&\quad - 2 \sum_{k=1}^{N} [< v, \psi_k > \alpha_k + < v, \chi_k > \beta_k] + < v, v >, \qquad (71)
\end{aligned}$$

where

$$\psi_k(x) = u(w_k^{-1}(x)), \quad \chi_k(x) = I_{w_k(X)}(x), \quad x \in X. \qquad (72)$$

Note that Δ^2 is a quadratic form in the ϕ-map parameters α_i and β_i, i.e.

$$\Delta^2 = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c, \qquad (73)$$

where $\mathbf{x}^T = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N}$. The elements of the symmetric matrix $\mathbf{A}$ are given by

$$\begin{aligned}
a_{i,j} &= < \psi_i, \psi_j >, \\
a_{N+i,N+j} &= < \chi_i, \chi_j >, \\
a_{i,N+j} &= < \psi_i, \chi_j >, \quad 1 \le i \le N,\ 1 \le j \le N. \qquad (74)
\end{aligned}$$

As well,

$$\begin{aligned}
b_i &= -2 < v, \psi_i >, \\
b_{N+i} &= -2 < v, \chi_i >, \quad 1 \le i \le N. \qquad (75)
\end{aligned}$$

and $c = < v, v > = \| v \|_2^2$.

The minimization of Δ^2 is a quadratic programming (QP) problem in the parameters α_i and β_i, $i = 1, 2, ..., N$. In order to guarantee that a minimum of this quadratic form exists on a compact set of feasible parameters α_i, β_i, we impose the additional condition

$$\| Tv \|_1 \ \le \ \| v \|_1 . \qquad (76)$$

In terms of the grey level map parameters, this is a linear inequality constraint, i.e.

$$\sum_{k=1}^{N} (\alpha_k \| v \circ w_k^{-1} \|_1 + \beta_k \mu(X_k)) \ \le \ \| v \|_1 . \qquad (77)$$

For the case $X \subset \mathbf{R}^D$, $\mu = m^{(D)}$ and $w_i \in Sim_1(X)$, $1 \leq i \leq N$, which will be used in all applications, the above linear inequality constraint becomes

$$\sum_{k=1}^{N} c_k^D (\alpha_k \parallel v \parallel_1 + V_X \beta_k) \quad \leq \quad \parallel v \parallel_1, \tag{78}$$

where $V_X = m^{(D)}(X)$.

For a given target $v \in \mathcal{L}^p(X, \mu)$, assuming $\parallel v \parallel_1 \neq 0$, we denote the feasible set of N-map IFSM grey-level parameters as

$$\Pi_v^{2N} = \{(\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N} \mid \parallel Tv \parallel_1 \leq \parallel v \parallel_1, \alpha_i \geq 0, \beta_i \geq 0\}. \tag{79}$$

Note that Π_v^{2N}, which is compact in the natural topology on $\mathbf{R}^{2N}$, depends on the target function v.

The minimization of Δ^2 may now be written as the following QP problem:

$$\text{minimize} \quad \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c, \quad \mathbf{x}^T \in \Pi_v^{2N}. \tag{80}$$

The advantages of QP problems have been discussed in [13]. Briefly,

1. QP algorithms locate an absolute minimum of the objective function Δ^2 in the feasible region Π_v^{2N} in a finite number of steps and

2. in many problems, the minimum value Δ^2_{min} is achieved on a boundary point of the feasible region. In such cases, if $(\alpha_k, \beta_k) = (0,0)$ then $\phi_k(t) = 0$ which implies that the associated IFS map w_k is superfluous. QP (as opposed to gradient-type schemes) will locate such boundary points, essentially discarding such superfluous maps. The elimination of such maps represents an increase in the data compression factor. (This feature was observed with minimization of the collage distance involving IFS with probabilities [13].)

The following result guarantees that, with the exception of a degenerate case, the IFSM operator T corresponding to a a feasible N-map IFSM grey-level parameter $\mathbf{x}^T \in \Pi_v^{2N}$ is contractive in $\mathcal{L}^1(X, \mu)$.

Proposition 7 *Let $X \subset \mathbf{R}^D$, $\mu = m^{(D)}$ and $v \in \mathcal{L}^1(X, \mu)$, $\parallel v \parallel_1 \neq 0$. Assume that $w_i \in Sim_1(X)$ for $1 \leq i \leq N$ and $\mathbf{x}^T = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \Pi_v^{2N}$. Then the operator T corresponding to the N-map IFSM $(\mathbf{w}, \Phi)$ is contractive in $(\mathcal{L}^1(X, \mu), d_1)$ except possibly when $\beta_1 = \beta_2 = ... = \beta_N = 0$. In this special case $\overline{u} \equiv 0$ is a fixed point of T.*

Proof: From the proof of Proposition 5, if $u, v \in \mathcal{L}^1(X, \mu)$, then

$$\parallel Tu - Tv \parallel_1 \leq C(D,1) \parallel u - v \parallel_1, \quad C(D,1) = \sum_{k=1}^{N} c_k^D \alpha_k. \tag{81}$$

Then from Eq. (78),

$$\sum_{k=1}^{N} c_k^D (\alpha_k + V_X \beta_k \| v \|_1^{-1}) \leq 1. \tag{82}$$

Thus $C(D,1) < 1$, i.e. T is contractive in $(\mathcal{L}^1(X,\mu), d_1)$, except possibly if $c_k \beta_k = 0$ for $1 \leq k \leq N$. Since $w_i \in Sim_1(X)$, it follows that $c_k \neq 0$, $1 \leq k \leq N$ and so the latter condition holds only if $\beta_k = 0$ for all $1 \leq k \leq N$. It is easy to see that $\overline{u} \equiv 0$ is a fixed point of T in this special case. ∎

Remark: Note that $\mathbf{x}^T \in \Pi_v^{2N}$ does *not* guarantee that the operator T is contractive in $(\mathcal{L}^2(X,\mu), d_2)$. Hence, the Collage Theorem does not apply in $\mathcal{L}^2(X,\mu)$. Nevertheless, as we show below, our algorithm to approximate functions in $\mathcal{L}^2(X,\mu)$ exploits the contractivity of T in $(\mathcal{L}^1(X,\mu), d_1)$.

We now describe our algorithm. As before, let $\mathcal{W}$ be an infinite set of fixed affine contraction maps on $X \subset \mathbf{R}^D$ which generates a μ-dense and nonoverlapping family of subsets of X. Let

$$\mathbf{w}^N = \{w_1, w_2, ..., w_N\}, \quad N = 1, 2, ..., \tag{83}$$

denote N-map truncations of $\mathcal{W}$. Given a target function $v \in \mathcal{L}^p(X, m^{(D)})$, the region Π_v^{2N}, as defined in Eq. (79), contains all feasible points $\mathbf{x}^N = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbf{R}^{2N}$, each of which defines a unique N-vector of affine grey level maps Φ^N,

$$\Phi^N = \{\alpha_1 t + \beta_1, ..., \alpha_N t + \beta_N\}. \tag{84}$$

For an $\mathbf{x}^N \in \Pi^{2N}$, let $T^N : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu)$ be the operator associated with the N-map IFSM $(\mathbf{w}^N, \Phi^N)$. Let

$$\Delta_N^2 = \| v - T^N v \|_2^2 \tag{85}$$

denote the corresponding squared $\mathcal{L}^2$ collage distance. Since $\Delta_N^2 : \Pi^{2N} \to \mathbf{R}$ is continuous in the natural topology on $\mathbf{R}^{2N}$, it attains an absolute minimum value, $\Delta_{N,min}^2$ on Π^{2N}. For each N, we may determine this minimum value using QP. The following result ensures that $\Delta_{N,min}^2$ may be made arbitrarily small.

Theorem 6 $\Delta_{N,min}^2 \to 0$ *as* $N \to \infty$.

Proof: Since $\Pi^{2N} \subset \Pi^{2N+2}$ for $N = 1, 2, ...$, it follows that $\Delta_{2N+2,min}^2 \leq \Delta_{2N,min}^2$. Thus $\{\Delta_{N,min}^2\}_{N=1}^{\infty}$ is a nonincreasing sequence of nonnegative numbers. Hence there exists a limit, $L \geq 0$, of this sequence. We now show that $L = 0$.

The proof involves a minor modification of the proof of Theorem 5. As such, we employ all constructions made between Eqs. (55) and (64), inclusive. Then, for each $i \in \{1, 2, ..., 2^{2n} + 1\}$ such that $B_{ni} = v^{-1}[(i-1)2^{-n}, i2^{-n}) \neq \emptyset$ and $\mu(B_{ni}) > 0$ (i.e. $\delta_{ni} \neq 0$), define $\zeta_{ni} = (i-1)2^{-n}$. Now define

$$\overline{u}_n(x) = \sum_{i=1}^{2^{2n}} \delta_{ni}\zeta_i I_{\widehat{\mathbf{w}}_i(X)}(x) + \delta_{n,2^{2n}+1}\zeta_i I_{\widehat{\mathbf{w}}_{2^{2n}+1}(X)}(x). \tag{86}$$

The function $\overline{u}_n$ is the fixed point for the IFSM composed of the IFS maps in $\mathbf{w}_i$, $1 \leq i \leq 2^{2n} + 1$. Associated with each IFS map w_{i_k} is the (constant) grey level map $\phi_{i_k}(t) = \zeta_i$, $t \in \mathbf{R}^+$ Note that by construction

$$\| \overline{u}_n \|_1 \;\; \leq \;\; \| v \|_1 \, . \tag{87}$$

Now let T_n denote the operator associated with this IFSM. Then $T_n\overline{u}_n = \overline{u}_n$ and the contraction factor of T_n is $C_n = 0$.

As in the proof of Theorem 6, let N be the smallest integer such that the truncation $\mathbf{w}^N$ of $\mathcal{W}$ contains all the IFS maps in $\mathbf{w}_i$ for $1 \leq i \leq 2^{2n}$. Let T^N be the operator for the IFSM $(\mathbf{w}^N, \Phi^N)$, where $\phi_{i_k^*} = \zeta_k$, $k = 1, 2, ..., n_\sigma$ and all other $\phi_j(t) = 0$. Then $T^N\overline{u}_n = \overline{u}_n$. From Eq. (87), the IFSM grey-level parameters defining T^N lie in the region Π_v^{2N}.

Thus, proceeding in the same way as in Eqs. (63)-(64), we have the inequalities

$$[\Delta^2_{N,min}]^{1/2} \;\; \leq \; \| v - T^N v \|_2 \; < \;\; \epsilon. \tag{88}$$

Since $\lim_{n\to\infty} \inf \| v - T^N v \|_2 = 0$, it follows that $L = 0$ and the theorem is proved. ∎

Our formal solution to the inverse problem is not yet complete, however, since an operator T corresponding to a grey-level vector $\mathbf{x}^T \in \Pi_v^{2N}$ is not necessarily contractive in $(\mathcal{L}^2, d_2)$. As a result, the Collage Theorem in $(\mathcal{L}^2, d_2)$, along with the result of Theorem 6, cannot be used to establish the approximation of a target v to an arbitary accuracy in $(\mathcal{L}^2, d_2)$. However, let us return to the proof of Theorem 6. For each $N = N(n)$ value, let $\mathbf{x}^T_{min} \in \Pi_v^{2N}$ denote the point at which Δ^2_N attains its minimum value $\Delta^2_{N,min}$. Let T^N_{min} denote the IFSM operator defined by the grey-level parameters in $\mathbf{x}^T_{min}$. Since T^N is contractive in $\mathcal{L}^1(X, \mu)$, it possesses a unique and attractive fixed point $\overline{u}^N_{min} \in \mathcal{L}^1(X, \mu)$. From Proposition 2, T maps $\mathcal{L}^2(X, \mu)$ into itself so $\overline{u}^N_{min} \in \mathcal{L}^2(X, \mu)$. From the relation [17]

$$\| u \|_1 \leq \; \mu(X)^{\frac{1}{2}} \, \| u \|_2, \quad \forall u \in \mathcal{L}^2(X, \mu), \tag{89}$$

we have the following result.

Corollary 1 $\| v - T^N_{min} v \|_1 \to 0$ *as* $N \to 0$.

Our algorithm is thus guaranteed to construct $\mathcal{L}^2$ approximations, $\overline{u}^N_{min}$, of the target v to arbitrary accuracy in $\mathcal{L}^1$ distance. Some numerical computations will be presented in Section 5.

4. The Inverse Problem With "Local IFSM" on $\mathcal{L}^p(X, m^{(D)})$

Our method can easily be generalized to incorporate the strategy of Jacquin [19], namely, that we consider the actions of contractive maps w_i on *subsets* of X (the "parent blocks") to produce smaller subsets of X (the "child blocks"). (This is also referred to as a "local IFS" (LIFS) in [4].) Rather than trying to approximate a target as a union of contracted copies of itself as in the IFS method, the local IFS method tries to express the target as a union of copies of *subsets* of itself. In this paper, only some simple constructions of Local IFSM are considered. A more general formulation will be presented elsewhere [15].

4.1 A Simple Nonoverlapping Local IFSM

It is convenient to first formulate a simple "local IFSM" (LIFSM) on $\mathcal{L}^p(X, \mu)$, where $\mu = m^{(D)}$ as follows. Let $J_k \subset X$, $k = 1, 2, ..., N$, with $N \geq 1$, such that

1. $\cup_{k=1}^N J_k = X$ (tiling condition) and
2. $\mu(J_j \cap J_k) = 0$ for $j \neq k$ (μ-nonoverlapping condition).

In addition, suppose that for each J_k, $1 \leq k \leq N$, there exists an $I_{j(k)} \subseteq X$ and a map $w_{i(k),k} \in Con(X)$, with contractivity factor $c_{i(k),k}$, such that $w_{i(k),k}(I_{i(k)}) = J_k$. In other words, for each "child block" J_k, there is a corresponding "parent block" $I_{i(k)}$.

For each map $w_{i(k)} : I_{i(k)} \to J_k$, let there be a grey level map $\phi_k : \mathbf{R} \to \mathbf{R}$. The vectors $\mathbf{w}_{loc} = \{w_{i(1),1}, ..., w_{i(N),N}\}$ and Φ comprise an N-map LIFSM $(\mathbf{w}_{loc}, \Phi)$. Now define an associated operator $T_{loc} : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$ as follows: For $u \in \mathcal{L}^p(X, \mu)$ and $x \in J_k$, $k \in \{1, 2, ..., N\}$,

$$(T_{loc}u)(x) \equiv \begin{cases} \phi_k(u(w_{i(k),k}^{-1}(x))), & x \in J_k(X) - \cup_{l \neq k}^N J_k(X) \cap J_l(X), \\ 0, & x \in \cup_{l \neq k}^N J_k(X) \cap J_l(X). \end{cases} \tag{90}$$

Proposition 8 *Let $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$. Let $(\mathbf{w}_{loc}, \Phi)$ be a local IFSM defined as above, with $\phi_k \in Lip(\mathbf{R})$ for $1 \leq k \leq N$. Then for $u, v \in \mathcal{L}^p(X, m)$,*

$$d_p(T_{loc}u, T_{loc}v) \leq C_{loc}(D, p) d_p(u, v), \quad C_{loc}(D, p) = [\sum_{k=1}^N c_{i(k),k}^D K_k^p]^{1/p}. \tag{91}$$

Proof: For $u, v \in \mathcal{L}^p(X, m)$,

$$\begin{aligned} \| T_{loc}u - T_{loc}v \|_p^p &= \sum_{k=1}^N \int_{J_k} |\phi_k(u(w_{i(k),k}^{-1}(x))) - \phi_k(v(w_{i(k),k}^{-1}(x)))|^p dx \\ &= \sum_{k=1}^N c_{i(k),k}^D \int_{I_k} |\phi_k(u(y)) - \phi_k(v(y))|^p dy \end{aligned}$$

$$\leq \sum_{k=1}^{N} c_{i(k),k}^{D} K_k^p \int_{I_k} |u(y) - v(y)|^p dy$$

$$\leq [\sum_{k=1}^{N} c_{i(k),k}^{D} K_k^p] \parallel u - v \parallel_p^p . \quad \blacksquare \tag{92}$$

Remarks:

1. If $C_{loc}(D,p) < 1$, then T_{loc} is contractive over the space $(\mathcal{L}^p(X, m^{(D)}), d_p)$ and possesses a unique fixed point $\overline{u}$.

2. The factor $C_{loc}(D,p)$ is similar in form to the "optimal" factor $C_{non}(D,p)$ of Eq. (23), due to the nonoverlapping property of the J_k. It is not necessary to impose the restriction that all ϕ_k maps be contractive. As before, it follows that

$$C_{loc}(D,p) \leq K, \qquad K = \max_{1 \leq k \leq N} K_k. \tag{93}$$

The weaker upper bound, K, which is independent of D or p, is identical to the result for the "Fractal Grey-Scale Transform" [4].

4.2 $\mathcal{L}^2$ Collage Distance

As above, let $X = [0,1]^D$, $\mu = m^{(D)}$ and $v \in \mathcal{L}^p(X, m^{(D)})$ be a target set. Given an N-map LIFSM as defined above, we now compute the squared $\mathcal{L}^2$ collage distance,

$$\begin{aligned}
\Delta^2 &= \parallel T_{loc} v - v \parallel_2^2 \\
&= \sum_{k=1}^{N} \int_{J_k} [\phi_i(v(w_{i(k),k}^{-1}(x)) - v(x)]^2 dx \\
&= \sum_{k=1}^{N} \Delta_k^2.
\end{aligned} \tag{94}$$

Again, because the child blocks are conveniently nonoverlapping, the problem reduces to the minimization of each squared collage distance Δ_k^2 over the block J_k, a "least squares" determination of ϕ_k. In the special case that the ϕ_k maps are affine, the minimization of each Δ_k^2 is, as before, a quadratic programming problem in the two parameters α_k and β_k.

4.3 Formal Solution and Algorithm for the Inverse Problem

Given a target set v, a formal solution of the inverse problem for the nonoverlapping LIFSM case is straightforward, following the ideas of Theorems 5 and 6. As such, we merely outline the constructions involved. Let $J_0 = X$ and $J_i \subset X, i = 1, 2, ...$ be an infinite sequence of closed subsets so that for each $x \in X$ and any $\epsilon > 0$, there exists an $i^* \in \{1, 2, ...\}$ such that $J_{i^*} \subset N_\epsilon(x)$. We now choose *both* child and parent blocks from this sequence of sets: Assume that for each $k \geq 1$ there exists a $i(k) \geq 1$ and a map $w_{i(k),k} \in Con(X)$ such that $J_k = w_{i(k),k}(J_{i(k)})$. Let $\mathcal{W} = \{w_{i(1),1}, w_{i(2),2}, ...\}$. Also let $\mathbf{w}_{loc}^N, N = 1, 2, ...$, be the N-map truncations of $\mathcal{W}$. For an $\mathbf{x}^N \in \Pi_v^{2N}$, define an associated contractive operator T_{loc}^N for the N-map local IFSM $(\mathbf{w}_{loc}^N, \Phi^N)$. Then, minimize the squared $\mathcal{L}^2$ collage distances Δ_N^2 over Π_v^{2N} to produce a sequence $\Delta_{N,min}^2, N = 1, 2, ...$ A theorem analogous to Theorem 6 is the result.

4.4 Local IFSM With "More Degrees of Freedom"

The Local IFSM discussed above represents only one of many possible ways in which parent blocks may be mapped to child blocks. Some additional possibilities, to each of which would correspond a particular T_{loc} operator, are listed below:

1. For a given child cell J_k, we may wish to consider more than one parent cell I_j at the same time.
2. It may be possible, and desirable, to consider more than one affine mapping from a given parent I_i to a given child J_k. For example, on [0,1], we can consider both the orientation-preserving and orientation non-preserving maps (e.g. $w_{ik} = s_{ik}x + a_{ik}$, with $s_{ik} = 1$ and -1, respectively). In $[0, 1]^2$, there are eight possible contraction maps from a larger parent square to a smaller child square and we may wish to employ some or all of them in our T_{loc} operator.
3. Combining 1 and 2 above.
4. Overlapping child cells.

Clearly, there are many possibilities. From a practical viewpoint, however, there are limitations. In this section, we formulate the inverse problem associated with (2) above. (The extension of this method to (3) above is rather straightforward.) Some numerical calculations using this strategy have been performed and will be reported in the next Section.

For simplicity, we assume equipartitions of $X \subset \mathbf{R}^D$ which produce regular parent and child blocks, i.e. squares, cubes. As well, we assume that the tiling and μ-nonoverlapping conditions of Section 4.1 are also satisfied by the child blocks J_k. Let $w_{i(k),k}^{(l)}, l = 1, 2, ..., M_D$ denote the set of all possible similitudes mapping a parent block $I_{i(k)}$ to a child block J_k, all having a common contraction factor $c_{i(k),k}$ $(M_1 = 2, M_2 = 8, ...)$. Associated with each IFS map $w_{i(k),k}^{(l)}$ will be a grey level map $\phi_k^{(k)} \in Lip(R_g)$. Then the operator $T_{loc} : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$ associated with such an LIFSM is given by

$$(T_{loc}u)(x) = \sum_{l=1}^{M_D} \phi_k^{(l)}(u([w_{i(k),k}^{(l)}]^{-1}(x))), \quad x \in J_k, \quad k = 1, 2, ..., N. \tag{95}$$

Since the child blocks J_k are nonoverlapping, the squared $\mathcal{L}^2$ collage distance separates into a sum of collage distances over each child cell J_k, i.e.

$$\begin{aligned}\Delta^2 &= < T_{loc}v - v, T_{loc}v - v > \\ &= \sum_{k=1}^{N} \Delta_k^2.\end{aligned} \tag{96}$$

In the case of affine grey level maps, i.e.

$$\phi_k^{(l)} = \alpha_k^{(l)} t + \beta_k^{(l)}, \quad 1 \leq l \leq M_D, \tag{97}$$

each collage distance is given by

$$\begin{aligned}\Delta_k^2 \;=\; & \sum_{l=1}^{M_D} \sum_{m=1}^{M_D} [< \psi_l, \psi_m > \alpha_k^{(l)} \alpha_k^{(m)} \;+\; 2 < \psi_l, \chi_m > \alpha_k^{(l)} \beta_k^{(m)} \\ & + \; < \chi_l, \chi_m > \beta_k^{(l)} \beta_k^{(m)}] \\ & - 2 \sum_{l=1}^{M_D} [< v, \psi_l > \alpha_k^{(l)} + < v, \chi_l > \beta_k^{(l)}] + < v_k, v_k >,\end{aligned} \tag{98}$$

where

$$\psi_l(x) = u([w_{i(k),k}^{(l)}]^{-1}(x)), \quad \chi_l(x) = I_{J_k}(x), \quad v_k(x) = v(x) I_{J_k}(x). \tag{99}$$

At first sight, it would appear that each Δ_k^2 is a quadratic form in the $2M_D$ parameters $\alpha_k^{(l)}$ and $\beta_k^{(l)}$, $1 \leq l \leq M_D$. However, the functions $\chi_l(x)$ are identical. Therefore, Δ_k^2 reduces to the following quadratic form in the $M_D + 1$ parameters $\alpha_k^{(l)}$, $1 \leq l \leq M_D$, and $\beta = \sum_{l=1}^{M_D} \beta_k^{(l)}$,

$$\Delta_k^2 = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c_k, \tag{100}$$

where $\mathbf{x}^T = (\alpha_k^{(1)}, ..., \alpha_k^{(M_D)}, \beta) \in \mathbf{R}^{M_D+1}$. The elements of the symmetric matrix $\mathbf{A}$ are given by

$$a_{i,j} =< \psi_i, \psi_j >, \;\; 1 \leq i, j \leq M_D, \tag{101}$$

and

$$a_{i,M_D+1} = a_{M_D+1,i} =< \psi_i >, \;\; 1 \leq i \leq M_D. \tag{102}$$

As well,

$$b_i = -2 < v, \psi_i >, \;\; 1 \leq i \leq N, \tag{103}$$

and $b_{M_D+1} =< v_k >$, $c_k =< v_k, v_k >=\| v_k \|_2^2$.

The feasible set of parameters is chosen to be

$$\begin{aligned}\Pi_{v_k}^{M_D+1} = \quad & \{(\alpha_k^{(1)}, ..., \alpha_k^{(M_D)}, \beta_k) \in \mathbf{R}^{M_D+1} \;|\; \| T_{loc}v_k \|_1 \leq \| v_k \|_1, \\ & \beta_k \geq 0 \;, \alpha_k^{(l)} \geq 0, 1 \leq l \leq M_D\}.\end{aligned} \tag{104}$$

5. Applications and Numerical Computations

In this section we present some results of our algorithm to construct IFSM and Local IFSM approximations to functions ($X = [0,1]$) and images ($X = [0,1]^2$). In all applications $\mu = m^{(D)}$ is the Lebesgue measure. All computations were done in double-precision FORTRAN using an IBM Model 355 POWERStation equipped with a RISC processor.

5.1 Function Approximation on [0,1]

5.1.1 Normal IFSM

We present some results for the "normal" IFSM method of Section 3.2 where the "wavelet"-type basis of affine IFS maps $\mathbf{w}$ in Eq. (51) has been employed. The N-map truncations $\mathbf{w}^N$ were constructed by arranging the $w_{i,j}$ as follows:

$$w_{1,1}, w_{1,2}, w_{2,1}, \ldots w_{2,4}, w_{3,1}, \ldots, w_{3,8}, w_{4,1}, \ldots . \tag{105}$$

The vector $\mathbf{w}^{N(i^*)}$, where $N(i^*) = \sum_{i=1}^{i^*} 2^i$, contains affine maps with contraction factors 2^{-i}, $i = 1, 2, \ldots, i^*$. (For $i^* = 1, 2, 3, 4$, $N(i^*) = 2, 6, 14, 30$, respectively.) In all cases, the contractivity factor of $\mathbf{w}^N$ is $c = 1/2$. For each N, the minimum squared collage distance $\Delta^2_{N,min}$ in the feasible region Π_v^{2N} was computed using the NAG quadratic programming (QP) algorithm E04NAF. The calculations were performed using a discretization of 5000 points on [0,1].

Example 1: The target function $v(x) = \sin(\pi x)$. Figures 3(a)-(c) show the approximations yielded by the truncations $N = 2, 6$ and 14, respectively. The target $v(x)$ has also been plotted for comparision. Some details of the numerical results are presented below. The geometric symmetry of the plot is manifested in the grey-level map coefficients.

$\mathbf{N = 2}$: This is a simple nonoverlapping case where the two IFS maps $w_{1,1}(x) = \frac{1}{2}x$ and $w_{1,2}(x) = \frac{1}{2}x + \frac{1}{2}$ are used. The solution can be obtained in closed form. The optimal grey-level maps are

$$\begin{aligned} \phi_{1,1}(t) = \phi_{1,2}(t) &= \frac{8\pi - 24}{3\pi^2 - 24}t + \frac{6\pi - 16}{3\pi^2 - 24} \\ &\cong 0.20243t + 0.50775. \end{aligned} \tag{106}$$

The fixed point attractor

$$\overline{u}(x) = \frac{2}{\pi} \cong 0.63662, \quad a.e.\ x \in [0,1]. \tag{107}$$

This value is the fixed point of the ϕ_i maps. The "spike" at $x = \frac{1}{2}$ is due to the overlap of the sets $w_i(X)$ at that point: $\overline{u}(\frac{1}{2}) \cong 1.27324$. Spikes occur at all

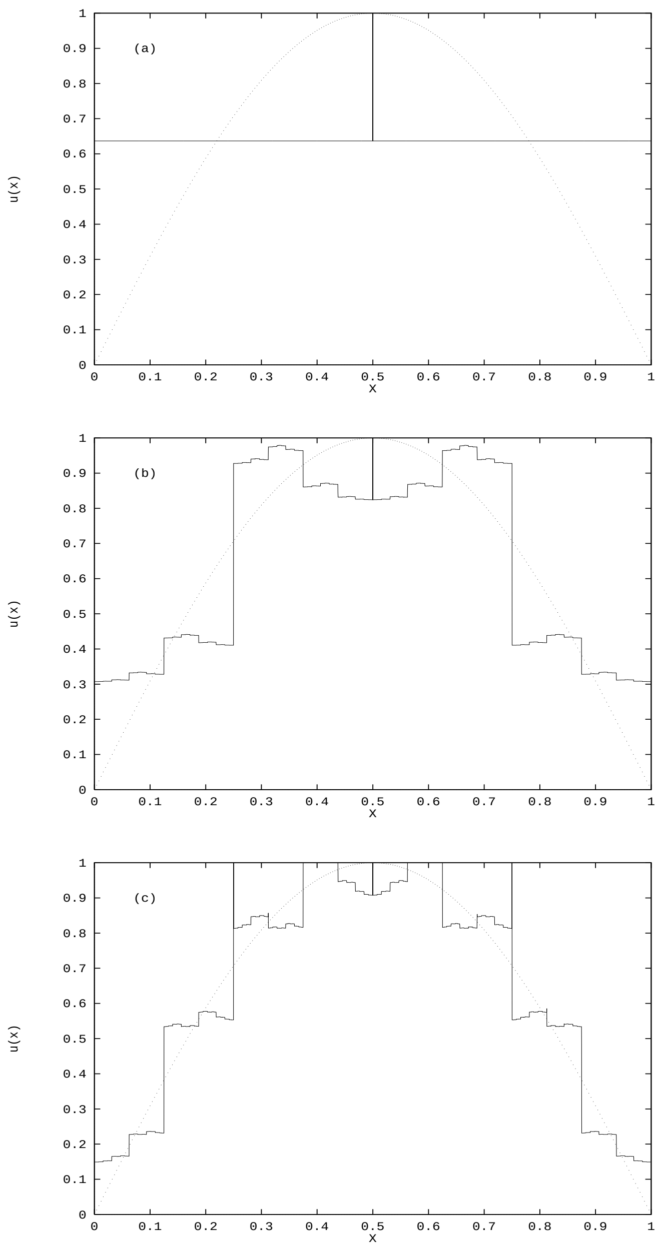

Figure 3: Approximations to the target set $v(x) = \sin(\pi x)$ on $X = [0, 1]$ yielded by the "normal" IFSM method of Section 3.2, using the wavelet-type basis of Eq. (51), with $N = 2, 6$ and 14 maps, respectively $(i^* = 1, 2, 3)$.

dyadic points $x = i2^{-j}$ in $(0, 1)$. The $\mathcal{L}^2$ error in approximation is $\| v - \overline{u} \|_2 = \frac{1}{2} - \frac{4}{\pi^2} \cong 0.30775$. Our numerical calculations agree with the above results.

N = 6 : There are only four non-zero grey-level maps:

$$\begin{aligned} \phi_{1,1}(t) &= \phi_{1,2}(t) \cong 0.19975t + 0.24593, \\ \phi_{2,2}(t) &= \phi_{2,3}(t) \cong 0.03052t + 0.50763. \end{aligned} \tag{108}$$

The $\mathcal{L}^2$ error in approximation is $\| v - \overline{u} \|_2 \cong 0.14943$.

N = 14 : There are eight non-zero grey-level maps:

$$\begin{aligned} \phi_{1,1}(t) &= \phi_{1,2}(t) \cong 0.20340t + 0.11870, \\ \phi_{2,2}(t) &= \phi_{2,3}(t) \cong 0.02924t + 0.50568. \\ \phi_{3,2}(t) &= \phi_{3,7}(t) \cong 0.00054t + 0.24968, \\ \phi_{3,4}(t) &= \phi_{3,5}(t) \cong 0.13236t + 0.24668. \end{aligned} \tag{109}$$

The $\mathcal{L}^2$ error in approximation is $\| v - \overline{u} \|_2 \cong 0.07526$.

Example 2: The target function $v(x) = \sqrt{x}$. Figures 4(a)-(c) show the approximations yielded by the truncations $N = 2, 6$ and 14, respectively ($i^* = 1, 2, 3$). The target $v(x)$ has also been plotted for comparision. The accuracy of the 2-map IFSM is rather striking.

5.1.2 Nonoverlapping Local IFSM

We now apply the simple nonoverlapping local IFSM method of Section 4.1 to Example 1. In the calculations below the child blocks J_k as well as the parent blocks I_j are the dyadic subintervals obtained by the action of the $w_{i,j}$ IFS maps on [0,1]. Following the strategy of Jacquin [19], for each child J_k we test all possible parent blocks. In principle, for each parent, we consider both possible similitude contraction maps (i.e. orientation preserving and nonpreserving) and choose the parent and map giving the minimum collage distance Δ_k^2. (The symmetry of the target $\sin(\pi x)$ makes many of these minima equal in magnitude.) Fig. 5 shows some approximations to $v(x) = \sin(\pi x)$ yielded by this simple LIFSM method. In Fig. 5(a), we have used two parents $I_j = w_{1,j}(X)$ and four child blocks $J_k = w_{2,k}(X)$. Fig. 5(b) is the result of using four parent blocks $I_j = w_{2,j}(X)$ and eight child blocks $J_k = w_{3,k}(X)$. Not surprisingly, for a given number of IFSM maps, N, the local IFSM method is seen to yield better results than the normal IFSM, since the former seeks to tile the target function $v(x)$ with copies of parts of itself. (In all of the above cases, the contraction factor of T_{loc} is $c = \frac{1}{2}$.) Some caution must be employed, however, as seen in Fig. 5(c), where two parents $I_j = w_{1,j}(X)$ and eight children $J_k = w_{3,k}(X)$ are used. The approximation is rather poor. The "halves" of the function $\sin(\pi x)$ provide poor collages of the rather straight portions J_k, $k = 1, 2, 3, 6, 7, 8$. As a result, it is necessary to employ more refined partitions for the parent cells.

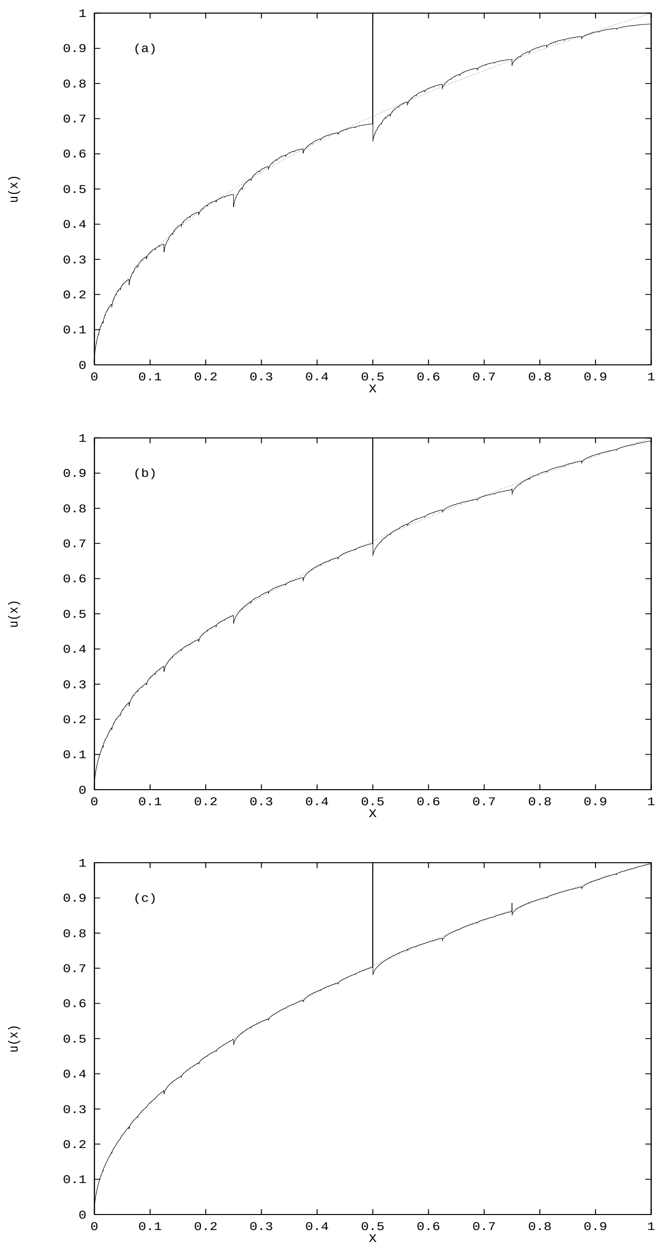

Figure 4: Approximations to the target set $v(x) = \sqrt{x}$ on $X = [0, 1]$ yielded by the "normal" IFSM method of Section 3.2, using the wavelet-type basis of Eq. (51), with $N = 2, 6$ and 14 maps, respectively.

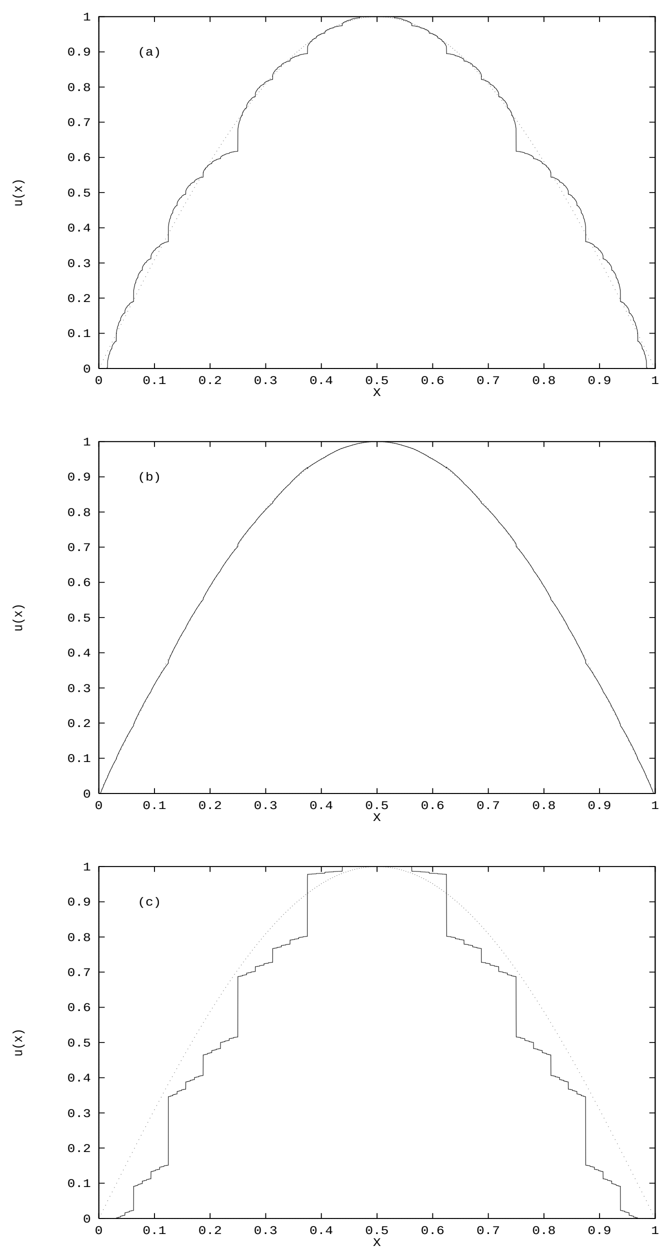

Figure 5: Approximations to the target set $v(x) = \sin(\pi x)$ on $X = [0, 1]$ yielded by the nonoverlapping Local IFSM method of Section 4.1 using N_I parent intervals and N_J child intervals. (a) $(N_I, N_J) = (2, 4)$. (b) $(N_I, N_J) = (4, 8)$. (c) $(N_I, N_J) = (2, 8)$.

5.2 Image Approximation And Coding Using Local IFSM

We now consider Local IFSM approximations to images, that is, discrete arrays of pixels which can be represented by bounded, nonnegative valued functions $f : [0,1] \to R_g \subset \mathbf{R}^+$. Unlike the case of function approximation, cf. Section 5.1, the primary goal of "fractal-based" image representation is not to approximate images to arbitrary accuracy but rather to approximate them to some acceptable accuracy with the fewest possible (IFSM) parameters, thus achieving a high *compression ratio.* In the rather simple LIFSM treatment which follows, no attempt is made to construct optimal partitions of parent and/or child blocks. The use of *adaptive* partitioning, e.g. quadtrees, in IFS-type methods has been discussed by others [9] and is beyond the scope of this paper.

As in [14], we consider the target image "Lena", a 512×512 pixel greyscale image shown in Figure 6. Each pixel in the image assumes one of 256 values between 0 and 255, representing one byte of storage. In computations, these values were rescaled to values in [0,1]. Thus, our target image v is represented by a step function on $[0,1]^2$. For each IFSM approximation $\overline{u}$ to the target v we give the $\mathcal{L}^1$ error $\| v - \overline{u} \|_1$ as well as the relative $\mathcal{L}^1$ error, $\| v - \overline{u} \|_1 \,/\, \| v \|_1$. We denote the execution time required to determine the IFSM parameters as the "coding time".

5.2.1 Nonoverlapping Local IFSM Method

It is convenient to construct "nonoverlapping" child and parent blocks of given resolutions by dividing the 512×512 pixel array into disjoint subsets of $2^m \times 2^m$ pixels, with $m_{child}, m_{parent} \in \{1, 2, ..., 8\}$. (Typically, we use $m_{child} = 2m_{parent}$ so that each parent block contains four child blocks. In this case, the contractivity factor for the affine maps between parents and children is then $c = \frac{1}{2}$.)

Let N_J and N_I denote the number of child and parent blocks, J_k and I_j, respectively. ($N_J = 2^{18-m_{child}}, N_I = 2^{18-m_{parent}}$.) There are eight possible IFS similitudes $w_{i,k}^{(l)}, 1 \leq l \leq 8$ which map a given parent J_k onto a given child I_j. Suppose that for each child J_k, we select a particular parent $J_{j(k)}$, an IFS map $w_{j(k),k}^{l(k)}$, and an associated affine grey level map $\phi_k(t) = \alpha_k t + \beta_k$. Following the discussion in Section 4.1, the operator T_{loc} associated with this N_J-map LIFSM will be defined as follows: For a $u \in \mathcal{L}^p(X,\mu)$ and $x \in J_k,\ k \in \{1, 2, ..., N_J\}$,

$$(T_{loc}u)(x) \equiv \begin{cases} \phi_k(u([w_{i(k),k}^{(l)}]^{-1}(x))), & x \in J_k(X) - \cup_{m \neq k}^{N} J_k(X) \cap J_m(X), \\ 0, & x \in \cup_{m \neq k}^{N} J_k(X) \cap J_m(X). \end{cases} \tag{110}$$

If

$$C_{loc} = \sum_{k=1}^{N_J} c_{i(k),k}^2 |\alpha_k| < 1, \tag{111}$$

Figure 6: The target image "Lena", a 512×512 pixel array, 8 bits (256 grey-level values) per pixel.

then T_{loc} is contractive on $(\mathcal{L}^1, d_1)$ and possesses a unique fixed point $\overline{u} \in \mathcal{L}^1$. If the grey level parameters α_k, β_k were obtained by the QP method outlined in Section 3.2, then T_{loc} is contractive except possibly when all β_k parameters are zero. Since the latter condition has not been encountered in any of the applications we have studied, we assume that the resulting operator T_{loc} is contractive in general. For a given partitioning, hence N_J, we naturally wish to find the best LIFSM, i.e. the LIFSM which minimizes the $\mathcal{L}^2$ collage distance $\| v - T_{loc} v \|_2$. As N_J increases, this minimal collage distance decreases. However, from the practical viewpoint of *data compression*, this increase in accuracy is countered by a decrease in the *compression ratio* as well as an increase in the amount of computer time required to determine the optimal IFS and grey level maps, as we outline below.

The $4N_J$ parameters,

$$\{i(k), l(k), \alpha_k, \beta_k,\ 1 \leq k \leq N_J\}, \tag{112}$$

comprise the *code* for the above LIFSM representation of the image. They define uniquely the T_{loc} operator whose fixed point $\overline{u}$ may then be computed numerically. It may be possible that the storage requirements for the indices $i(k)$ and $l(k)$ and the ϕ-map parameters α_k and β_k are different (i.e. low length vs higher length integers). In our very simplistic discussion, we ignore such differences and consider only the number of parameters used in representing an image (and then make the crude substitution: one parameter = one byte of storage). Now suppose, as stated earlier, that the child blocks are formed by $2^{m_{child}} \times 2^{m_{child}}$ arrays of pixels. Since each block will be represented by 4 LIFSM parameters, the compression ratio is given by

$$\begin{aligned} \text{compression ratio} &\equiv \frac{\text{no. of pixels in image}}{\text{no. of parameters representing image}} \\ &= 2^{2m_{child}-4} \\ &= 2^{14}[N_J]^{-1}. \end{aligned} \tag{113}$$

Figures 7(a) and 7(b) show the approximations to "Lena" using the strategy originally devised by Jacquin [19]. Given a child block J_k, $k \in \{1, 2, ..., N_J\}$, we considered each parent I_j, $1 \leq j \leq N_I$, testing in turn all 8 possible contraction maps $w_{j,k}^{(l)}$. The parent $I_{j(k)}$ and map $w_{j(k),k}^{(l(k))}$ which yielded the minimum collage distance $\Delta_{k,min}^2$ were chosen to form the T_{loc} operator. The error in approximation is observed to decrease as N_J increases, as expected. However, the coding time increases very rapidly with N_J. (The $(N_I, N_J) = (32^2, 64^2)$ case required over 4 hours of computer time for coding. The resulting $\mathcal{L}^1$ error of 0.02 represents a very small improvement for such a great increase in computer time.)

One possibility of reducing the computer time is to lessen the search for optimal parent blocks or even eliminate the search entirely. In order to investigate the latter idea, we considered only "nearest" parent blocks, i.e. given a child block J_k, we used the block the block I_j which contains J_k, but continued to test all eight possible contraction maps. The result, shown in Fig. 8, approximates the target with about the same error as that of Fig. 7(b). However, it was achieved in roughly 1/100th the computer time.

It is conceivable that an approximation of better accuracy could be achieved, but with a slightly increased coding time, if a search is performed over a relatively small set of parent blocks. Jacquin [19] already considered such an approach by classifying the parent and child blocks as one of four types according to a standard method of image block classification. Given a child J_k with a given property, the search for an optimal parent would only have to be performed over the subset of parent blocks sharing that property. We are currently investigating other methods of classification.

Some authors [21, 23] have shown that the use of more general place-dependent grey-level maps (which involve a greater number of parameters in the $\mathcal{L}^2$ fit between parent and child cell) can eliminate the need for a search of optimal parent blocks. Our own computations using PD-LIFSM support these claims and we report them in Section 5.2.3.

5.2.2 Overlapping Local IFSM Method

We now apply the more generalized LIFSM method of Section 4.2 to the target

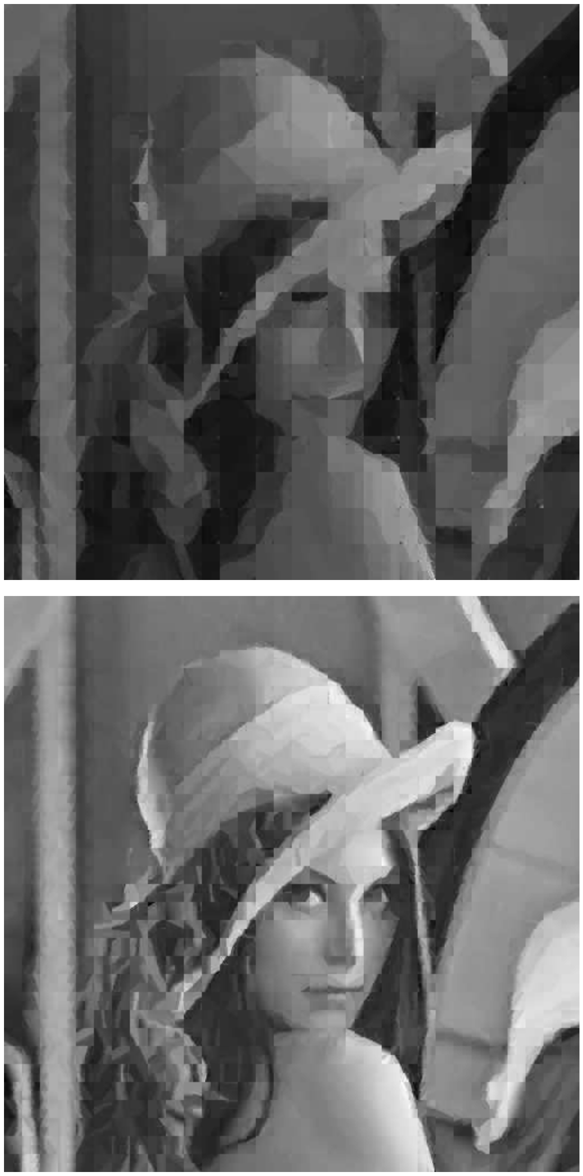

Figure 7: Approximations to target image "Lena" using the nonoverlapping Local IFSM method of Jacquin with N_I parent blocks and N_J child blocks. For a given child block, all possible parent blocks were tested. (a) $(N_I, N_J) = (8^2, 16^2)$. $\mathcal{L}^1$ error $\| v - \overline{u} \|_1 = 0.04$. Relative $\mathcal{L}^1$ error $= 0.093$. Coding time $= 223$ sec.. (b) $(N_I, N_J) = (16^2, 32^2)$. $\mathcal{L}^1$ error $\| v - \overline{u} \|_1 = 0.029$, Relative $\mathcal{L}^1$ error $= 0.068$. Coding time $= 3344$ sec..

Figure 8: Approximation to target image "Lena" using nonoverlapping Local IFSM method of Jacquin with $N_I = 32^2$ parent blocks and $N_J = 64^2$ child blocks. For each child block, only the parent block containing it was used. All 8 possible maps were tested. $\mathcal{L}^1$ error $\| v - \overline{u} \|_1 = 0.029$, Relative $\mathcal{L}^1$ error $= 0.068$. Coding time = 34 sec..

image "Lena". Following [14], for each child block J_k and each possible parent block I_j, we consider all eight IFS maps $w_{i,j}^{(l)}$, $1 \leq l \leq 8$, simultaneously. The minimization of the collage distance Δ_k^2 is a quadratic programming problem in 9 unknowns. Fig. 9(a) was produced by using $N_I = 32^2$ parents and $N_J = 64^2$ children. The compression ratio associated with this LIFSM is reduced since more maps are used per child block. However, in most cases, the QP algorithm located a minimum of Δ_k^2 on the boundary of the feasible region $\Pi_{v_k}^{2N}$ and no more than *two* grey level maps differed significantly from zero. The approximation in Fig. 9(b) was produced by eliminating the search for optimal parent blocks. As for the Jacquin case, given a child J_k, we chose the parent I_j which contained J_k, reducing the computer time. However, the accuracy of the approximation is no better than that of the nonoverlapping LIFSM method in Fig. 8. As well, the coding time is larger.

5.2.3 Nonoverlapping Place-Dependent LIFSM

In this section, we present the results of some computations using a local nonoverlapping IFSM method with place-dependent grey-level maps. We use the parent

Figure 9: Approximations to target image "Lena" using overlapping Local IFSM method with $N_I = 32^2$ parent blocks and $N_J = 64^2$ child blocks. For each parent-child pair (I_j, J_k), all eight possible IFS maps $w_{jk}^{(l)}, 1 \leq l \leq 8$, were used simultaneously. (a) For each child, all possible parents were tested. $\| v - \overline{u} \|_1 = 0.018$, Relative $\mathcal{L}^1$ error $= 0.042$. Coding time $= 3402$ sec.. (b) No search for optimal parent blocks. For each child, only the parent containing it was used. $\| v - \overline{u} \|_1 = 0.029$, Relative $\mathcal{L}^1$ error $= 0.068$. Coding time $= 71$ sec..

and child blocks of the previous section. The grey-level maps used were affine in the grey-level value as well as in the spatial coordinates, i.e.

$$\phi_k(t,x,y) = (a_{1k}x + a_{2k}y + a_{3k})t + (b_{1k}x + b_{2k}y + b_{3k}). \tag{114}$$

The squared $\mathcal{L}^2$ collage distance over each child cell J_k associated with the IFS map $w_k \equiv w^{(l)}_{i(k),k}$ and the grey level map ϕ_k is a quadratic form in the six ϕ-map parameters:

$$\Delta_k^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + < v^2 >_k, \tag{115}$$

where $\mathbf{x}^T = (a_{1k}, a_{2k}, a_{3k}, b_{1k}, b_{2k}, b_{3k})$. The matrix $\mathbf{A}$ is, up to the factor c_k^2, given by

$$\begin{bmatrix} < x^2v^2 >_k & < xyv^2 >_k & < xv^2 >_k & < x^2v >_k & < xyv >_k & < xv >_k \\ < xyv^2 >_k & < y^2v^2 >_k & < yv^2 >_k & < xyv >_k & < y^2v >_k & < yv >_k \\ < xv^2 >_k & < yv^2 >_k & < v^2 >_k & < xv >_k & < yv >_k & < v >_k \\ < x^2v >_k & < xyv >_k & < xv >_k & < x^2 >_k & < xy >_k & < x >_k \\ < xyv >_k & < y^2v >_k & < yv >_k & < xy >_k & < y^2 >_k & < y >_k \\ < xv >_k & < yv >_k & < v >_k & < x >_k & < y >_k & < 1 >_k \end{bmatrix} \tag{116}$$

and

$$\begin{aligned} \mathbf{b}^T \quad = \quad & c_k^2[< xv(v \circ w_k) >_k, < yv(v \circ w_k) >_k, < v(v \circ w_k) >_k, \\ & < x(v \circ w_k) >_k, \ < y(v \circ w_k) >_k, \ < (v \circ w_k) >_k]. \end{aligned} \tag{117}$$

We have used the notation

$$< f >_k \equiv \int_{J_k} f(x,y)dxdy. \tag{118}$$

(An advantage of this formulation of Δ_k^2 - see Appendix C - as compared to the general derivaion in Section 3.2 which uses the inverse IFS maps, is that the Hessian matrix A is dependent only upon the parent block I_j. Thus, these matrices do not have to be recomputed as we scan the child cells of the image.)

Our numerical calculations confirm the statements of some authors [21, 23] that there is little need for searching when place-dependent LIFSM are used. We have found experimentally that for most parent-child pairs, (I_j, J_k), the minimum collage distances yielded by each of the eight possible affine maps $w^{(l)}_{j,k}$, $1 \leq l \leq 8$, are equal to at least three figures of accuracy. As well, we have found that the collage distances yielded by the various parent blocks do not differ by much. Fig. 10 shows the PD-LIFSM approximation obtained when, for a given child block, we chose the parent block which contained it and the IFS map with contraction factor $\frac{1}{2}$ and zero rotation or inversion. The use of six ϕ-map parameters per child block J_k is offset somewhat by the elimination of two parameters, namely the parent index $i(k)$ and the map index $l(k)$ in (112). Thus the compression ratio for PD-LIFSM is a factor of $3/2$ higher than that for LIFSM in Eq. (113).

Figure 10: Approximation to target image "Lena" using place-dependent Local IFSM method with $N_I = 32^2$ parent blocks and $N_J = 64^2$ child blocks. No search for optimal parent blocks. For each child, only the parent containing it was used. Only the map producing no rotation or inversion was used. $\| v - \overline{u} \|_1 = 0.022$, Relative $\mathcal{L}^1$ error $= 0.05$. Coding time $= 27$ sec..

However, there is a tremendous saving in computer time, with very little sacrifice in accuracy. As expected, this PD-LIFSM method yielded a slightly better approximation than the non-optimal-parent LIFSM method of Fig. 8. If a search over all possible parent blocks is performed for each child block in this PD-LIFSM method (the coding time is 5709 sec.), there is no improvement in the approximation.

6. Concluding Remarks

We have presented the theoretical basis of approximating functions and images to arbitrary accuracy using a formulation of Iterated Function Systems over the general function spaces $\mathcal{L}^p(X, \mu)$. An algorithm for constructing IFSM approximations to target functions/images in $\mathcal{L}^2([0,1]^D, m^{(D)})$ has also been given. Our theory and algorithm can easily be extended to cover the cases of Local IFSM and Place-Dependent IFSM/LIFSM.

There remain many interesting and open theoretical questions for further research. For example, it would be desirable to establish weaker conditions on the grey level maps ϕ_i which guarantee that the "Markov" operator T associated with an IFSM $(\mathbf{w}, \Phi)$ maps $\mathcal{L}^p(X, \mu)$ into itself. There also arises the question of other possible forms of the operator T associated with an IFSM or LIFSM. We have examined this question in some detail and the results will appear elsewhere [15]. Our investigation on developing better image coding and compression schemes also continues. As in [13], no special attention was paid to the choice of IFS maps satisfying the μ dense and nonoverlapping property.

Finally, we have very recently provided a unifying link between IFS-type methods on function spaces, namely, IFZS and IFSM, and the method of IFS with probabilities (IFSP) on probability measure spaces [16]. This has been achieved by formulating a method of fractal transforms over $\mathcal{D}(X)$, the space of distributions on the base space (X, d). Special cases of this distributional fractal transform include IFSP and IFSM.

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Appendix A: Relations for Integrals Involving IFSM

In this section we derive some elementary relations for integrals involving IFSM. Given an N-map IFSM $(\mathbf{w}, \Phi)$ on $\mathcal{L}^p(X,\mu)$ with operator $T : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu)$, let $u \in \mathcal{L}^p(X,\mu)$ and $v = Tu$. Then from the definition of T in Eq. (25) in the main text, for any function $f \in \mathcal{L}^p(X,\mu)$,

$$\begin{aligned}\int_X f(x)v(x)d\mu(x) &= \int_X f(x)(Tu)(x)d\mu(x) \\ &= \sum_{i=1}^{N} \int_{X_i} f(x)(\phi_i \circ u \circ w_i^{-1}(x))d\mu(x). \end{aligned} \tag{119}$$

For the remainder of this Appendix, we assume that $X \subset \mathbf{R}$ (closed and bounded) and $\mu = m^{(1)}$. As well, we consider only affine IFSM with IFS and grey level maps having the form

$$\begin{aligned} w_i(x) &= s_i x + a_i, \quad c_i = |s_i| < 1, \\ \phi_i(t) &= \alpha_i t + \beta_i, \quad 1 \le i \le N. \end{aligned}$$

Then

$$\begin{aligned}\int_X f(x)v(x)dx &= \sum_{i=1}^{N} \alpha_i \int_X f(x)u(w_i^{-1}(x))dx + \sum_{i=1}^{N} \beta_i \int_X f(x) I_{X_i}(x)dx \\ &= \sum_{i=1}^{N} \alpha_i \int_{X_i} f(x)u(w_i^{-1}(x))dx + \sum_{i=1}^{N} \beta_i \int_{X_i} f(x)dx \qquad (120) \\ &= \sum_{i=1}^{N} \alpha_i c_i \int_X (f\circ w_i)(y)u(y)dy + \sum_{i=1}^{N} \beta_i c_i \int_X (f \circ w_i)(y)dy. \end{aligned}$$

In light of the use of moments of measures of Iterated Function Systems with Probabilities (IFSP), we let $f(x) = x^n$ for $n \geq 0$ and consider the following power moments of u and v,

$$g_n = \int_X x^n u(x)dx, \quad h_n = \int_X x^n v(x)dx, \quad n = 0, 1, 2, \dots \ . \tag{121}$$

From Eq. (120), we have

$$\int_X x^n v(x)dx = \sum_{i=1}^{N} \alpha_i c_i \int_X (s_i y + a_i)^n u(y)dy + \sum_{i=1}^{N} \beta_i c_i \int_X (s_i y + a_i)^n dy. \tag{122}$$

Expansion of the binomial terms yields the relation

$$h_n = \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{i=1}^{N} \alpha_i c_i s_i^k a_k^{n-k} \right] g_k + \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{i=1}^{N} \beta_i c_i s_i^k a_k^{n-k} \right] m_k, \tag{123}$$

where $m_k = \int_X x^k dx$, $k \geq 0$, are the moments with respect to Lebesgue measure on X.

Now suppose that T is contractive and $u = v = \overline{u} = T\overline{u}$, i.e. u is the fixed point or attractor of the IFSM $(\mathbf{w}, \Phi)$. Then $h_n = g_n$, $n \geq 0$, are the moments of the fixed point $\overline{u}$. A rearrangement of Eq. (123) gives

$$\left[1 - \sum_{i=1}^{N} \alpha_i c_i \right] g_0 = \left[\sum_{i=1}^{N} \beta_i c_i \right] m_0, \tag{124}$$

and

$$\begin{aligned} [1 - \sum_{i=1}^{N} \alpha_i c_i s_i^n] g_n \quad &= \quad \sum_{k=0}^{n-1} \binom{n}{k} \left[\sum_{i=1}^{N} \alpha_i c_i s_i^k a_k^{n-k} \right] g_k \\ &+ \quad \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{i=1}^{N} \beta_i c_i s_i^k a_k^{n-k} \right] m_k, \quad n \geq 1 \end{aligned} \tag{125}$$

Starting with Eq. (124), g_0 can be computed (without knowing $\overline{u}$ explicitly). The moments g_n of the fixed point $\overline{u}$ may then be computed recursively in terms of the IFSM parameters $s_i, a_i, \alpha_i, \beta_i$. This is analogous to the case involving moments of invariant measures for IFSP.

Example: $N = 2$, $X = [0, 1]$, $\mu = m^{(1)}$, $w_1(x) = \frac{1}{2}x$, $w_2(x) = \frac{1}{2}x + \frac{1}{2}$, with grey level maps $\phi_1(t) = \frac{1}{2}t$ and $\phi_2(t) = \frac{1}{2}t + \frac{1}{2}$. Then $\overline{u}(x) = x$ a.e.. The moments of $\overline{u}$ are $g_n = \int_0^1 x^{n+1}dx = \frac{1}{n+2}$, $n \geq 0$. From Eq. (124),

$$[1 - \frac{1}{4} - \frac{1}{4}] g_0 = \frac{1}{4} \tag{126}$$

which gives the correct result $g_0 = \frac{1}{2}$. From Eq. (125), with $n = 1$,

$$[1 - \frac{1}{8} - \frac{1}{8}]g_1 = \frac{1}{16} + \frac{1}{8} + \frac{1}{16}, \tag{127}$$

which gives $g_1 = \frac{1}{3}$.

If we set $\beta_k = 0$, $1 \leq k \leq N$ and define $p_k = \alpha_k c_k$ in Eq. (125), then the resulting equation is identical in form to the moment relations for IFSP. In order that the $p_k = \alpha_k c_k$ be considered as probabilities, then the constraint

$$\sum_{k=1}^{N} \alpha_k c_k = 1 \tag{128}$$

would have to be imposed. The computation of moments g_k would then begin with $g_0 = \int_X u(x)dx$. It is convenient to set $g_0 = 1$ by normalizing $\overline{u}$: Since $\beta_k = 0$ for $1 \leq k \leq N$, we may replace $u = v$ in Eq. (122) with Ku, $K \in \mathbf{R}$ constant.

Appendix B: The Inverse Problem With Nonoverlapping Affine Maps

Here we consider in more detail the minimization of the squared $\mathcal{L}^2_+$ collage distances Δ_i^2 of Eq. (70) with the following assumptions:

1. $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$.
2. $w_i \in Sim_1(X)$. As well, $X = \cup_{i=1}^N X_i$, where $X_i = w_i(X)$; in other words, the X_i "tile" the space X.
3. $\mu(X_i \cap X_j) = 0$ for $i \neq j$ (μ-nonoverlapping condition).
4. Affine grey-level maps $\phi_i : \mathbf{R}^+ \to \mathbf{R}^+$, where $\phi_i(t) = \alpha_i t + \beta_i$, $t \in \mathbf{R}^+$. Thus, $\alpha_i, \beta_i \geq 0$ for $1 \leq i \leq N$.

Each squared collage distance Δ_i^2 over X_i becomes

$$\begin{aligned} \Delta_i^2 &= \int_{X_i} [\alpha_i v((w_i^{-1}(x))) + \beta_i - v(x)]^2 dm^{(D)} \\ &= c_i^D \int_X [\alpha_i v(x) + \beta_i - v(w_i(x))]^2 dm^{(D)}. \end{aligned} \tag{129}$$

As in the main text, we assume the IFS maps w_i to be fixed and consider each Δ_i^2, $1 \leq i \leq N$, to be a function of the two grey level map parameters α_i, β_i. Δ_i^2 is a quadratic form in α_i, β_i:

$$\begin{aligned} c_i^{-D} \Delta_i^2 &= \| v \|_2^2 \, \alpha_i^2 + 2\alpha_i\beta_i \, \| v \|_1 + \beta_i^2 \\ &- 2 < v, v \circ w_i > \alpha_i - 2 \, \| v \circ w_i \|_1 \, \beta_i + \| v \circ w_i \|_2^2 \,. \end{aligned} \tag{130}$$

The minimization of Δ_i is a quadratic programming (QP) problem subject to the constraints defining Π_v^{2N}. From Proposition 7, if not all the β_i, $1 \leq i \leq N$ are zero, then the operator T is contractive in $(\mathcal{L}^1(X,\mu), d_1)$.

Least Squares Approach: In most applications to image representation in the literature the condition that the ϕ_i (hence T) map $\mathbf{R}^+$ into itself is relaxed. The following stationarity conditions are imposed,

$$\frac{\partial \Delta_i^2}{\partial \alpha_i} = \frac{\partial \Delta_i^2}{\partial \beta_i} = 0, \quad i = 1, 2, ..., N, \tag{131}$$

to yield the following set of linear equations in α_i and β_i:

$$\begin{aligned} \| v \|_2^2 \, \alpha_i + \| v \|_1 \, \beta_i &= < v \circ w_i, v >, & (132) \\ \| v \|_1 \, \alpha_i + \beta_i &= \| v \circ w_i \|_1, \quad i = 1, 2, ..., N. & (133) \end{aligned}$$

Provided that $D_v := \| v \|_1^2 - \| v \|_2^2 \neq 0$, the solutions are given by

$$\begin{aligned} \alpha_i &= D_v^{-1}[< v \circ w_i, v > - \| v \circ w_i \|_1 \| v \|_1], & (134) \\ \beta_i &= D_v^{-1}[\| v \|_2^2 \| v \circ w_i \|_1 - \| v \|_1 < v \circ w_i, v >], \quad i = 1, 2, ..., N. & (135) \end{aligned}$$

There is no guarantee, however, that the solutions α_i and β_i in Eqs. (134) and (135) will be nonnegative. This may not be a great problem in actual applications, since $\phi_i(v(x))$ may still assume nonnegative values for $x \in X$ (or negative values which can be rounded off to zero). As well, the actual fixed point $\overline{u}$ of the operator T may turn out to be nonnegative. Nevertheless, this detail is generally overlooked in the literature.

Note, however, that for $1 \leq i \leq N$ (since $c_i \neq 0$),

$$\| v \circ w_i \|_1 = c_i^{-D} \int_{X_i} v(y) dy, \tag{136}$$

Multiplying Eq. (133) by c_i^D and summing over $1 \leq i \leq N$ yields

$$\begin{aligned} \| v \|_1 &= \sum_{i=1}^{N} c_i^D (\alpha_i \| v \|_1 + \beta_i) \\ &\leq \sum_{i=1}^{N} c_i^D (|\alpha_i| \| v \|_1 + |\beta_i|) \\ &= \| Tv \|_1 & (137) \end{aligned}$$

The equality $\| v \|_1 = \| Tv \|_1$ holds if $\alpha_i, \beta_i \geq 0, 1 \leq i \leq N$.

Appendix C: Some Aspects of Place-Dependent IFSM

Here we outline some basic theory for N-map IFSM $(\mathbf{w}, \Phi)$ with place-dependent grey level maps, $\phi_k : \mathbf{R} \times X \to \mathbf{R}$. The operator T associated with this IFSM will act as follows: For $u \in \mathcal{L}^p(X, \mu)$,

$$(Tu)(x) = \sum_{k=1}^{N}{}' \phi_k(u(w_k^{-1}(x)), w_k^{-1}(x)), \tag{138}$$

As done in the main text for IFSM, we establish some sufficient conditions on the IFS and grey level maps to ensure that T maps $\mathcal{L}^p(X, \mu)$ into itself. Define the following set of uniformly Lipschitz functions,

$$\begin{aligned} Lip(Y, X) \quad = \quad & \{\phi : Y \times X \to Y \subseteq \mathbf{R} : |\phi(t_1, s) - \phi(t_2, s)| \leq K|t_1 - t_2|, \\ & \forall t_1, t_2 \in Y, \ \forall s \in X \text{ for some } K \in [0, \infty)\}. \end{aligned} \tag{139}$$

Proposition 9 *Now let $(\mathbf{w}, \Phi)$ denote an N-map IFSM with associated operator T defined above. Assume that:*

1. For any $u \in \mathcal{L}^p(X, \mu)$, $u \circ w_k^{-1} \in \mathcal{L}^p(X, \mu)$, $1 \leq k \leq N$,

2. $\phi_k \in Lip(\mathbf{R}, X)$, $1 \leq k \leq N$.

Then for $1 \leq p < \infty$, $T : \mathcal{L}^p(X, \mu) \to \mathcal{L}^p(X, \mu)$.

The proof of this Proposition is virtually identical to that of Proposition 2 in the main text.

Proposition 10 *Let $X \subset \mathbf{R}^D$, $D \in \{1, 2, ...\}$, and $\mu = m^{(D)}$. Let $(\mathbf{w}, \Phi)$ be an N-map IFSM such that*

1. $w_k \in Sim_1(X)$ and

2. $\phi_k \in Lip(\mathbf{R}, X)$, $1 \leq k \leq N$.

Then for a $p \in [1, \infty)$ and any $u, v \in \mathcal{L}^p(X, \mu)$,

$$d_p(Tu, Tv) \leq C(D, p) d_p(u, v), \quad C(D, p) = \sum_{k=1}^{N} c_k^{D/p} K_k. \tag{140}$$

Proof: This proof involves a minor modification of the proof for Proposition 4 in the main text. For $u, v \in \mathcal{L}^p(X, \mu)$,

$$\begin{aligned} \| Tu \quad - \quad & Tv \|_p \\ = \quad & [\int_X |\sum_{k=1}^{N}{}' [\phi_k(u(w_k^{-1}(x)), w_k^{-1}(x)) - \phi_k(v(w_k^{-1}(x)), w_k^{-1}(x))]|^p dx]^{1/p} \\ \leq \quad & \sum_{k=1}^{N} [\int_{X_k} |\phi_k(u(w_k^{-1}(x)), w_k^{-1}(x)) - \phi_k(v(w_k^{-1}(x)), w_k^{-1}(x))|^p dx]^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} c_k^{D/p} [\int_X |\phi_k(u(y), y) - \phi_k(v(y), y)|^p \, dy]^{1/p} \\
&\leq \sum_{k=1}^{N} c_k^{D/p} K_k [\int_X |u(y) - v(y)|^p \, dy]^{1/p} \\
&= [\sum_{k=1}^{N} c_k^{D/p} K_k] \parallel u - v \parallel_p . \quad \blacksquare \qquad (141)
\end{aligned}
$$

Some possible forms that the place-dependent grey level maps ϕ can assume are as follows:

1. $\phi(t,s) = \sum_{i=0}^{n} a_i(s) t^i$, where the $a_i : X \to \mathbf{R}$, bounded on X,
2. $\phi(t,s) = f(t) + g(s)$ ("separable") with suitable conditions on $f : \mathbf{R} \to \mathbf{R}$ in $Lip(\mathbf{R})$ and $g : X \to \mathbf{R}$, bounded on X.

It is convenient to work with ϕ maps which are only first degree in the grey-level variable t, i.e.

$$
\begin{aligned}
\phi(t,s) &= \alpha t + \beta + g(s), \quad g : X \to \mathbf{R}, \text{ bounded on } X, \qquad (142) \\
\phi(t,s) &= \alpha(s) t + \beta(s), \quad \alpha, \beta : X \to \mathbf{R}, \text{ bounded on } X. \qquad (143)
\end{aligned}
$$

The action of the first set of maps can be considered as a "place-dependent" shift in grey-level value. The second set of maps produce a more direct interaction between position and grey-level value.

The Inverse Problem in $\mathcal{L}^2(X, \mu)$ With Place-Dependent IFSM

The theory of Section 3.1 regarding a formal solution to the inverse problem can be applied to place-dependent IFSM. The structure of the expression for the squared $\mathcal{L}^2$ collage distance will depend upon the functional form assumed for the ϕ_k maps. As in Appendix B, we consider the following "nonoverlapping IFS" case:

1. $X \subset \mathbf{R}^D$ and $\mu = m^{(D)}$. For simplicity, we consider only the case $D = 1$ here, since the expressions involving the variable $s \in X$ become quite complicated.
2. $w_i \in Sim_1(X)$. As well, $X = \cup_{i=1}^{N} X_i$, where $X_i = w_i(X)$; in other words, the X_i "tile" the space X.
3. $\mu(X_i \cap X_j) = 0$ for $i \neq j$ (μ-nonoverlapping condition).

We assume that the grey-level maps ϕ_i assume the functional form in Eq. (142), both (for simplicity) with degree n polynomial place-dependent coefficients, i.e.

$$\phi_i(t,s) = \alpha_i(s) t + \beta_i(s), \quad t \in \mathbf{R}, \ s \in X, \qquad (144)$$

where

$$\alpha_i(s) = \sum_{j=0}^{n} a_{kj} s^j, \quad \beta_i(s) = \sum_{l=0}^{n} b_{kj} s^j. \tag{145}$$

(Note that in the special case $n = 0$, the ϕ_i become the affine maps of "normal IFSM".) Each squared collage distance Δ_i^2 over X_i becomes

$$\begin{aligned} \Delta_i^2 &= \int_{X_i} [\alpha_i(w_i^{-1}(x))v((w_i^{-1}(x))) + \beta_i(w_i^{-1}(x)) - v(x)]^2 dx \\ &= c_i^D \int_X [\alpha_i(x)v(x) + \beta_i(x) - v(w_i(x))]^2 dx \\ &= c_i^D \int_X [v(x)\sum_{j=0}^{n} a_{ij}x^j + \sum_{k=0}^{n} b_{ik}x^k - v(w_i(x))]^2 dx. \end{aligned} \tag{146}$$

Δ_i^2 is a quadratic form in the coefficients $a_{ij}, b_{ij}, 1 \leq j \leq n$. The coefficients of this quadratic form involve power moments of the functions v, v^2 and $vv \circ w_i$ as well as moments over X. The minimization of Δ_1^2 is a quadratic programming (QP) problem subject to suitable constraints.

The method of "least squares" could also be applied to this problem. By imposing the stationarity conditions,

$$\frac{\partial \Delta_i^2}{\partial a_{ij}} = \frac{\partial \Delta_i^2}{\partial b_{ij}} = 0, \quad j = 1, 2, ..., n, \tag{147}$$

to yield the following set of linear equations in α_i and β_i: one obtains a set of $2n$ linear equations in the place-dependent polynomial coefficients.

Such place-dependent grey level maps could also be considered, with much work, in the overlapping IFS case, cf. Eq. (71) in the main text. The coefficients of the quadratic form in the $a_{ij}, b_{i,j}, 1 \leq i \leq N, 1 \leq j \leq n$ would involve power moments.