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Iterated Fuzzy Set Systems: A New Approach to the Inverse Problem for Fractals and Other Sets

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Images with grey or colour levels admit a natural representation in terms of fuzzy sets, but without the usual probabilistic interpretation of the latter. We introduce a fuzzy set approach which incorporates, in part, the technique of iterated function systems (IFS) for the construction, analysis, and/or approximation of typically fractal sets and images. The method represents a significant departure from IFS, especially in the interpretation of the resulting image. The introduction of "grey-level maps," φ_i: $[0, 1] \to [0, 1]$ associated with the contractive maps w_i of the IFS affords much greater flexibility in the generation of images as well as in the inverse problem.

1. INTRODUCTION

The purpose of this paper is to introduce a fuzzy set approach for the construction, analysis, and/or approximation of sets and images which may exhibit fractal characteristics. Of particular concern is the *inverse problem* of encoding a target set or image with a relatively small number of parameters. In this regard, our method incorporates, at least in part, the technique of iterated function systems (IFS) in its underlying structure. IFS is the name given by Barnsley and Demko [1] (see also [2]) to a system consisting of a set of contraction maps $w_i: X \to X$, $i = 1, 2, ..., N$, and associated probabilities p_i, $i = 1, 2, ..., N$, $\sum_{i=1}^{N} p_i = 1$, where X denotes a compact metric space. For each set $\{w_i\}$ there exists a unique compact set $\mathscr{A} \subset X$, invariant under the "parallel action" $\bigcup_{i=1}^{N} w_i(\mathscr{A}) = \mathscr{A}$. As well, for

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a given set of probabilities $\{p_i\}$ there exists a unique invariant measure μ with support $\mathscr{A}$. (The geometric and measure theoretic aspects of such systems were, in fact, worked out earlier by J. Hutchinson [7].) The method described in this paper, however, represents a significant departure from IFS, especially in the interpretation of the resulting images. The novelty of our approach may be summarized in terms of the following two key points:

(1) The entire mathematical setting is provided by a subclass $\mathscr{F}^*(X)$ of the class $\mathscr{F}(X)$ of fuzzy sets on X [15], i.e., $\mathscr{F}(X) = \{\mathbf{u}: X \to [0, 1]\}$: all images are considered as fuzzy sets. This leads to two possible interpretations:

(a) in image representation the value $\mathbf{u}(x)$ of a fuzzy set at a point $x \in X$ may be interpreted as the normalized *grey level value* associated with that point,

(b) in pattern recognition, the value $0 \leqslant \mathbf{u}(x) \leqslant 1$ indicates the probability that the point x is in the foreground of an image.

(2) Associated with each map w_i, $i = 1, 2, ..., N$, is a *grey level map* $\varphi_i: I \to I$, where $I = [0, 1]$ is the grey level domain. The collection of maps $\{w_i, \varphi_i\}$ is used to define an operator $T_s: \mathscr{F}^*(X) \to \mathscr{F}^*(X)$, which is contractive with respect to a metric d_∞ on $\mathscr{F}^*(X)$. This metric is induced by the Hausdorff distance on the nonempty closed subsets of X. (The precise action of T_s on a fuzzy set $\mathbf{v} \in \mathscr{F}^*(X)$ is defined in terms of a suitable associative operation and will be discussed in the next section.) Starting with an (arbitrary) initial fuzzy set $\mathbf{u}_0 \in \mathscr{F}^*(X)$, the sequence $\mathbf{u}_n \in \mathscr{F}^*(X)$ produced by the iteration $\mathbf{u}_{n+1} = T_s \mathbf{u}_n$ converges in the d_∞ metric to a unique and invariant fuzzy set $\mathbf{u}^* \in \mathscr{F}^*(X)$, i.e., $T_s \mathbf{u}^* = \mathbf{u}^*$.

The collection of contractive maps $w_i(x)$, $i = 1, 2, ..., N, x \in X$, and associated grey level maps $\varphi_i(t)$, $t \in [0, 1]$ (the latter satisfying suitable conditions, to be given below) will be referred to as an *iterated fuzzy set system*, to be abbreviated as IFZS, and denoted compactly as $\{X, \mathbf{w}, \Phi\}$. The compact space X will be called the *base space* of the IFZS. The unique, invariant fuzzy set $\mathbf{u}^*(x)$ is an *attractor* for the IFZS. Moreover, the support of $\mathbf{u}^*(x)$ is a subset of the attractor $\mathscr{A} \subset X$ of the underlying IFS defined by the $\{w_i\}$ maps. The approximation of the target image is accomplished in procedure (2) outlined above: the fuzzy sets $\mathbf{u}_n$ represent grey level distributions on X, which converge to the grey level distribution $\mathbf{u}^*$.

Let us finally mention that the IFZS is a new tool for the inverse problem of fractal or image construction which, in our new setting, may be phrased as follows: Given a *target* fuzzy set (image) $\mathbf{v} \in \mathscr{F}^*(X)$, find an IFZS $\{X, \mathbf{w}, \Phi\}$ whose attractor $\mathbf{u}^* \in \mathscr{F}^*(X)$ approximates $\mathbf{v}$ to sufficient accuracy in the d_∞ metric. The inverse problem using the normal IFS has received much attention, with claims of significant data compression of images [3]. In the IFZS method, the introduction of the grey level maps φ_i (which, we emphasize, need not be contractive, nor even continuous) yields greater flexibility in the generation of images. Our preliminary studies indicate that the IFZS affords a considerable simplification in the treatment of the inverse problem for black and white as well as colour images.

The layout of this paper is as follows. Section 2 provides the mathematical basis for the IFZS approach. After our motivation for considering fuzzy sets is outlined, the exact mathematical setting on $\mathscr{F}^*(X)$ is discussed, along with a definition of the d_∞ metric (Section 2.1). Conditions on the grey level maps φ_i are then derived (Section 2.2), followed by a discussion of associative operators on fuzzy sets (Section 2.3). The net result is the use of the supremum to construct the operator $T_\vee : \mathscr{F}^*(X) \to \mathscr{F}^*(X)$ mentioned above. The contractivity of this operator is derived in Section 2.4, from which all the important properties of IFZS follow. In Section 3 are presented examples, some of which illustrate clearly the difference between IFS and IFZS, and demonstrate the generality of the latter. Section 4 provides a discussion of the parallelism of IFS and IFZS and where it stops, as well as some further statements about the use of the supremum as the associative operation.

2. The Iterated Fuzzy Set System (IFZS)

2.1. *Preliminaries: Images as Fuzzy Sets*

A black and white digitized image is a (finite) set $\mathbf{P}$ of points or pixels p_{ij}, usually in $\mathbb{R}^2$. Associated with each pixel p_{ij} is a nonnegative *grey level* or brightness value, t_{ij}. In what follows, we assume a normalized measure for grey levels, i.e., $0 \leqslant t_{ij} \leqslant 1$ ($0 =$ black: the background, $1 =$ white: the foreground). The function $h: \mathbf{P} \to [0, 1]$ defined by the grey level distribution of the image is called the *image function* [13]. The digitized image is fully described by its image function h. This is also the situation in the more theoretical case where grey levels are distributed continuously on the base space X. At this point, one can see that an image as described by an image function, is nothing but a *fuzzy set* [15] $\mathbf{u}: X \to [0, 1]$, even though *no* probabilistic meaning is attached to the values $\mathbf{u}(x)$ at each point $x \in X$.

It is usual to classify the pixels according to their grey levels in the following way: for each $\alpha \in (0, 1]$, we consider the set $\{x \in X: \mathbf{u}(x) \geqslant \alpha\}$, i.e., the set of all pixels whose grey levels exceed the *threshold value* α. In fuzzy set language, this set is called the α-level set of $\mathbf{u}$, and denoted $[\mathbf{u}]^\alpha$. (We also define $[\mathbf{u}]^0 = \text{closure}\{x \in X, \mathbf{u}(x) > 0\}$.) For $\alpha \in (0, 1]$, $[\mathbf{u}]^\alpha$ represents a thresholding of the grey level distribution at α. There is a *one-to-one* correspondence between an image function (fuzzy set) and the set of α-level sets $\{[\mathbf{u}]^\alpha\}$, $0 \leqslant \alpha \leqslant 1$.

A considerable amount of attention has been given to fuzzy set theory since 1965. Probably the only reason that it has not yet been applied to image representation is the lack of a probabilistic interpretation which has usually been used in dealing with fuzzy sets.

In what follows, (X, d) will denote a compact metric space. $\mathscr{F}(X)$ will denote the class of fuzzy sets on X, i.e., the class of all functions $\mathbf{u}: X \to [0, 1]$. Our attention is, in fact, restricted to a subclass $\mathscr{F}^*(X) \subset \mathscr{F}(X)$: namely, $\mathbf{u} \in \mathscr{F}^*(X)$ if and only if the following conditions are satisfied:

(1) $\mathbf{u} \in \mathscr{F}(X)$.

(2) $\mathbf{u}$ is uppersemicontinuous (u.s.c.) on (X, d).

(3) $\mathbf{u}$ is *normal*, that is, $\mathbf{u}(x_0) = 1$ for some $x_0 \in X$.

These properties yield the following results:

(a) For each $0 < \alpha \leqslant 1$, the α-level set, defined as $[\mathbf{u}]^\alpha := \{x \in X: \mathbf{u}(x) \geqslant \alpha\}$ is a nonempty compact subset of X.

(b) The closure of $\{x \in X: \mathbf{u}(x) > 0\}$, denoted by $[\mathbf{u}]^0$, is also a nonempty compact subset of X.

Let $\mathscr{H}(X)$ denote the set of all nonempty closed subsets of X, along with the Hausdorff distance function

$$h(A, B) := \max\{D(A, B), D(B, A)\}, \tag{1}$$

where

$$D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y). \tag{2}$$

Then $(\mathscr{H}(X), h)$ is a compact metric space [6]. In particular, it contains the α-level sets $[\mathbf{u}]^\alpha$, $0 \leqslant \alpha \leqslant 1$, of all $\mathbf{u} \in \mathscr{F}^*(X)$. At this point, a metric d_∞ (see [5]) on $\mathscr{F}^*(X)$ can be defined as

$$d_\infty(\mathbf{u}, \mathbf{v}) = \sup_{0 \leqslant \alpha \leqslant 1} \{h([\mathbf{u}]^\alpha, [\mathbf{v}]^\alpha)\} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathscr{F}^*(X). \tag{3}$$

The metric space $(\mathscr{F}^*(X), d_\infty)$ is complete. The metric d_∞ has been used in many applications of fuzzy set theory [8, 9, 12].

The IFS component of our IFZS will now be introduced. As in the usual IFS case [1], let there be given a set of N contraction maps $w_i: X \to X$ so that for some s, $0 \leqslant s < 1$,

$$d(w_i(x), w_i(y)) \leqslant s d(x, y), \qquad \forall x, y \in X, i = 1, 2, ..., N; \tag{4}$$

s is called the *contractivity factor*. From [1, 2, 7], there exists a unique set $\mathscr{A} \in \mathscr{H}(X)$, the *attractor* of the IFS, which satisfies:

$$\mathscr{A} = \bigcup_{i=1}^{N} w_i(\mathscr{A}), \tag{5}$$

where $w_i(\mathscr{A}) := \{w_i(x), x \in \mathscr{A}\}$. This represents the *self-tiling* property of IFS attractors. In other words the map $\mathbf{w}: \mathscr{H}(X) \to \mathscr{H}(X)$ defined by:

$$\mathbf{w}(S) := \bigcup_{i=1}^{N} w_i(S), \; S \in \mathscr{H}(X)$$

has an invariant set. In the literature, this property is sometimes referred to as the "parallel action" of the w_i. As well,

$$h(\mathbf{w}^n(S), \mathscr{A}) \to 0 \qquad \text{as} \quad n \to \infty, \forall S \in \mathscr{H}(X). \tag{6}$$

2.2. *Selection of the Grey Level Maps*

For a general N-map IFS $\mathbf{w} = \{w_i: X \to X, \; i = 1, 2, ..., N\}$, it now remains to introduce and characterize the associated grey level maps $\Phi = \{\varphi_i: [0, 1] \to [0, 1], \; i = 1, 2, ..., N\}$ to define the IFZS $\{X, \mathbf{w}, \Phi\}$. Since our objective is to construct an operator on the class of fuzzy sets $\mathscr{F}^*(X)$, one condition to be satisfied by the functions φ_i is that they preserve uppersemicontinuity when composed with functions of $\mathscr{F}^*(X)$ (i.e., $\varphi_i \circ \mathbf{u}$ is u.s.c. for all $\mathbf{u}$ in $\mathscr{F}^*(X)$). If the base space X is finite, no conditions need to be imposed on the φ_i. For the infinite case, however, the φ_i will have to satisfy two conditions, together referred to as the n.d.r.c. condition.

DEFINITION. A function $\varphi: [0, 1] \to [0, 1]$ is said to be n.d.r.c. if and only if (i) φ is nondecreasing and (ii) φ is right continuous.

The following lemma justifies this definition.

LEMMA 2.2.1. *Let $\varphi: [0, 1] \to [0, 1]$ and X be an infinite compact metric space, then a necessary and sufficient condition for $\varphi \circ \mathbf{u}$ to be u.s.c. for all $\mathbf{u} \in \mathscr{F}^*(X)$ is that φ be n.d.r.c.*

Proof. (a) Proof of Sufficiency. Define $C_\alpha := \{x \in X: (\varphi \circ \mathbf{u})(x) \geqslant \alpha\}$. Then

$$C_\alpha = \begin{cases} \varnothing & \text{if} \quad \alpha > \varphi(1) \\ [\mathbf{u}]^\beta & \text{if} \quad 0 < \alpha \leqslant \varphi(1) \; (\beta = \inf\{t: \varphi(t) \geqslant \alpha\}) \\ X & \text{if} \quad \alpha = 0. \end{cases}$$

Thus for $\alpha \in [0, 1]$, C_α is closed, so $\varphi \circ \mathbf{u}$ is u.s.c.

(b) Proof of Necessity, in two steps. Firstly, for each $\alpha \in (0, 1)$ consider $A_\alpha = \varphi^{-1}([\alpha, 1])$. We now show that if $A_\alpha \neq \emptyset$, then $A_\alpha = [\beta, 1]$, where $\beta = \inf A_\alpha$. The inclusion $A_\alpha \subseteq [\beta, 1]$ is obvious, so we need only to prove the other inclusion. Indeed, assume the other inclusion is false: then there exists $t \in [\beta, 1]$, such that $t \notin A_\alpha$. If $a \in A_\alpha$, choose $x_0 \in X$ an accumulation point of X, and $x_1 \in X$, $x_1 \neq x_0$, and let $\mathbf{u}: X \to [0, 1]$ be defined as

$$\mathbf{u}(x) = \begin{cases} t & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \\ a & \text{otherwise.} \end{cases}$$

Then $\mathbf{u}$ is u.s.c., but $C_\alpha = [\varphi \circ \mathbf{u}]^\alpha$ is not closed, hence $\varphi \circ \mathbf{u}$ is not u.s.c., contradicting our hypothesis.

Secondly, we show that φ must be nondecreasing: Suppose that $t_1 < t_2$ and $\varphi(t_1) > \varphi(t_2)$. Let α be such that $\varphi(t_2) < \alpha \leqslant \varphi(t_1)$. Then $t_1 \in A_\alpha = [\beta, 1]$ implies that $\beta \leqslant t_1$ which, in turn, implies that $t_2 \in A_\alpha$ (since $t_1 < t_2$). Hence, $\varphi(t_2) \geqslant \alpha$ which is a contradiction. Thus $\varphi(t_1) \leqslant \varphi(t_2)$.

Using the fact φ is nondecreasing, a similar kind of argument can be applied to show right continuity of φ: First suppose that φ is not right continuous at $t_0 < 1$, i.e., $\varphi(t_0) < \varphi(t_0^+)$. Then let α be such that $\varphi(t_0) < \alpha \leqslant \varphi(t_0^+)$. Hence $t_0 \notin A_\alpha$ and therefore $A_\alpha = (t_0, 1]$ which is a contradiction. This completes the proof of the lemma. ∎

We now summarize the conditions which should be satisfied by a set of grey level maps $\Phi = \{\varphi_i, i = 1, 2, ..., N\}$ comprising an IFZS. For $i = 1, 2, ..., N$,

(1) φ_i: $[0, 1] \to [0, 1]$ is nondecreasing,

(2) φ_i is right continuous on $[0, 1)$,

(3) $\varphi_i(0) = 0$, and

(4) for at least one $j \in \{1, 2, ..., N\}$, $\varphi_j(1) = 1$.

A few remarks are necessary here. Properties 1 and 2 (n.d.r.c.), by Lemma 2.2.1 and Property 4, guarantee that the IFZS maps the class $\mathscr{F}^*(X)$ into itself. Property 3 is a natural assumption in the consideration of grey level functions: if the grey level of a point (pixel) $x \in X$ is zero, then it should remain zero after being acted upon by the φ_i maps.

It should be mentioned that in some practical treatments of the inverse problem, success has already been achieved by employing more general sets of grey level functions φ_i: for example, "place-dependent" grey level maps φ_i: $[0, 1] \to [0, 1]$. However, this will be the subject of future work. The analysis described here is concerned only with those functions φ_i satisfying the four properties mentioned above.

2.3. *Associative Operators on Fuzzy Sets*

This section is devoted to the introduction of a general class of operators mapping $\mathscr{F}(X)$ into itself, followed by a special class of operators which map the subclass $\mathscr{F}^*(X)$ into itself. The net result is the construction of an operator T_s which is contractive on the compact metric space $(\mathscr{F}^*(X), d_\infty)$ introduced in Section 2.1. The existence of a unique and attractive "fixed point" fuzzy set/grey level distribution $\mathbf{u} \in \mathscr{F}^*(X)$ will then be guaranteed.

Conforming to the *extension principle* for fuzzy sets [16, 11] and comforted by the same arguments that will justify our final choice for the operator $T: \mathscr{F}^*(X) \to \mathscr{F}^*(X)$ (see Eq. (22)), we define for each $\mathbf{u} \in \mathscr{F}(X)$ and each subset B of X,

$$\begin{aligned} \tilde{\mathbf{u}}(B) &:= \sup\{\mathbf{u}(y) : y \in B\}, \qquad \text{if} \quad B \neq \varnothing \\ \tilde{\mathbf{u}}(\varnothing) &:= 0, \end{aligned} \tag{7}$$

which implies, in particular, $\tilde{\mathbf{u}}(\{x\}) = \mathbf{u}(x)$ at each $x \in X$. For each w_i, $i = 1, 2, ..., N$, and each $x \in X$ we now define

$$\tilde{\mathbf{u}}_i(x) := \tilde{\mathbf{u}}(w_i^{-1}(x)), \tag{8}$$

where, of course, $w_i^{-1}(x) = \varnothing$ if $x \notin w(X)$. If $\mathbf{u} \in \mathscr{F}^*(X)$, then each of these functions $\tilde{\mathbf{u}}_i: X \to [0, 1]$ is a fuzzy set in $\mathscr{F}^*(X)$. (That $\tilde{\mathbf{u}}_i$ is u.s.c. will be proved in Lemma 2.4.1; the normality is straightforward.)

For a general IFZS $\{X, \mathbf{w}, \Phi\}$ consisting of N IFS maps and N grey level maps, consider the class of mappings $\mathbf{U}_N: [0, 1]^N \to [0, 1]$ and the operator $T: \mathscr{F}(X) \to \mathscr{F}(X)$ that associates to each fuzzy set $\mathbf{u}$ the fuzzy set $\mathbf{v} = T\mathbf{u}$ whose value at each $x \in X$ is given by

$$\mathbf{v}(x) = (T\mathbf{u})(x) := \mathbf{U}_N(q_1(x), q_2(x), ..., q_N(x)), \tag{9}$$

where

$$q_i(x) = \varphi_i(\tilde{\mathbf{u}}(w_i^{-1}(x))). \tag{10}$$

In other words, the function $\mathbf{U}_N$ operates on the modified grey levels of all possible pre-images of x under the IFS maps w_i, the grey levels having been transformed by the appropriate φ_i maps.

It appears totally natural to assume $\mathbf{U}_N$ symmetric in its arguments, i.e.,

$$\mathbf{U}_N(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, ..., \mathbf{v}_{i_N}) = \mathbf{U}_N(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N), \tag{11}$$

for every permutation $\{i_1, i_2, ..., i_N\}$ of $\{1, 2, ..., N\}$. However, it is just convenient for computational purposes to assume recursivity, i.e., the $\mathbf{U}_N$ are defined as

$$\mathbf{U}_N = \mathbf{U}_2[\mathbf{v}_1, \mathbf{U}_{N-1}(\mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_N)]. \tag{12}$$

In particular,

$$\mathbf{U}_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbf{U}_2[\mathbf{v}_1, \mathbf{U}_2(\mathbf{v}_2, \mathbf{v}_3)] = \mathbf{U}_3(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2)$$
$$= \mathbf{U}_2[\mathbf{v}_3, \mathbf{U}_2(\mathbf{v}_1, \mathbf{v}_2)] = \mathbf{U}_2[\mathbf{U}_2(\mathbf{v}_1, \mathbf{v}_2), \mathbf{v}_3],$$

which implies that the function $\mathbf{U}_2\colon [0, 1]^2 \to [0, 1]$ is an associative binary operation on the real numbers in $[0, 1]$. We shall let S denote such a binary operation. We shall assume the following set of additional properties to be satisfied by S:

(1) $S\colon [0, 1]^2 \to [0, 1]$ is continuous.

(2) For each $y \in [0, 1]$, the mapping $x \to S(x, y)$ is nondecreasing: the brighter a pixel, the brighter its combination with another pixel.

(3) 0 is an identity element, i.e., $S(0, y) = y$, $\forall y \in [0, 1]$: the combination of a pixel of brightness $y > 0$ with one of zero brightness yields a pixel with brightness y.

(4) For all $x \in [0, 1]$, $S(x, x) \geqslant x$: the combination of two pixels of equal brightness should not result in a darker pixel.

By recourse to a representation theorem for topological semi-groups on $\mathbb{R}$ [10], we have the following

THEOREM 2.3.1. *If $S\colon [0, 1]^2 \to [0, 1]$ satisfies (1)–(4) above, then there exists a sequence of disjoint open intervals $\{(a_r, b_r)\}$, with $a_1 = 0 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant 1$, and a sequence of increasing continuous functions $f_r\colon [a_r, b_r] \to [0, +\infty]$, with $f_r(a_r) = 0$, such that*

$$S(x, y) = g_r[f_r(x) + f_r(y)] \qquad \forall (x, y) \in [a_r, b_r]^2, \tag{13}$$

where g_r (pseudoinverse of f_r) is defined as

$$g_r(t) = \begin{cases} f_r^{-1}(t) & \text{if } t \in [0, f_r(b_r)] \\ b_r & \text{if } t \in [f_r(b_r), +\infty] \end{cases} \tag{14}$$

and finally,

$$S(x, y) = \sup\{x, y\} \qquad \text{if} \quad (x, y) \in [0, 1]^2 \setminus \bigcup_r [a_r, b_r]^2. \tag{15}$$

Clearly, $S(a_r, a_r) = a_r$ and $S(b_r, b_r) = b_r$ for all $r = 1, 2, \ldots$, that is, the a_r and b_r are idempotent for the operation S. Moreover, no element in the open intervals (a_r, b_r) is an idempotent for S. It is possible that the sequence $\{(a_r, b_r)\}$ may reduce to the single interval $(0, 1)$: indeed this is the case when S has 0 and 1 as its only idempotents, 0 being the identity, and 1 the annihilator. An example is given by the following operation, the p-norm, with p a positive integer,

$$S(x, y)=\begin{cases}[x^p+y^p]^{1/2} & \text{if } x^p+y^p \leqslant 1\\ 1 & \text{if } x^p+y^p \geqslant 1.\end{cases} \tag{16}$$

The functions f and g in Theorem 2.3.1 are given by

$$\begin{aligned} f(s) &= s^p \\ g(t) &= \begin{cases} t^{1/p} & \text{if } t \in [0, 1] \\ 1 & \text{if } t \in [1, +\infty].\end{cases}\end{aligned} \tag{17}$$

The other extreme case is when $S(x, x)=x$ for all $x \in [0, 1]$. In this case,

$$S(x, y)=\sup\{x, y\} \qquad \forall (x, y) \in [0, 1] \times [0, 1]. \tag{18}$$

In fact, from properties (2) and (4),

$$S(x, y) \geqslant S(x, 0)=x \qquad \text{and} \qquad S(x, y) \geqslant S(0, y)=y, \tag{19}$$

hence

$$S(x, y) \geqslant \sup\{x, y\}. \tag{20}$$

On the other hand, if $x<y$,

$$S(x, y) \leqslant S(y, y)=y=\sup\{x, y\}, \tag{21}$$

so that $S(x, y)=\sup\{x, y\}$. Even though this operation represents an extreme case, it appears to be the most natural one for our particular applications: the combination of two pixels with equal brightness t should result in a pixel with brightness t. As such, it will now be employed as the binary associative operation $\mathbf{U}_2$ introduced at the beginning of this section.

2.4. *Construction of the Contractive Operator T_s on the Class of Fuzzy Sets*

We now investigate the properties of the resulting operator $T: \mathscr{F}(X) \rightarrow \mathscr{F}(X)$ in Eq. (9), when $\mathbf{U}_2(\mathbf{v}_1, \mathbf{v}_2)=\sup\{\mathbf{v}_1, \mathbf{v}_2\}$, i.e., when

$$(T\mathbf{u})(x)=\sup\{\varphi_1(\tilde{\mathbf{u}}(w_1^{-1}(x))), ..., \varphi_N(\tilde{\mathbf{u}}(w_N^{-1}(x)))\}=:(T_s\mathbf{u})(x). \tag{22}$$

It will then be shown that T_s maps the class of fuzzy sets $\mathscr{F}^*(X)$ into itself.

LEMMA 2.4.1. *For all* $\mathbf{u} \in \mathscr{F}^*(X)$, *and* $\alpha \in [0, 1]$, *with* $q_i: X \rightarrow [0, 1]$ $(1 \leqslant i \leqslant N)$ *defined as in Eq.* (10), *we have*

(1) q_i *is u.s.c.*

(2) $[q_i]^\alpha = w_i([\varphi_i \circ \mathbf{u}]^\alpha)$

(3) $[T_s \mathbf{u}]^\alpha = \bigcup_{i=1}^N w_i([\varphi_i \circ \mathbf{u}]^\alpha)$.

(*Remark.* Note that q_i is not necessarily normal, therefore some of its level sets can be empty.)

Proof. (1) Because of Lemma 2.2.1, it is sufficient to show that $\tilde{\mathbf{u}}_i$ is u.s.c., or equivalently that, for each $0 \leqslant \alpha \leqslant 1$, the sets $F_\alpha = \{x \in X : \tilde{\mathbf{u}}_i \geqslant \alpha\}$ are closed. If $\alpha = 0$, $F_0 = X$ and is therefore closed. Let us analyze the case $\alpha > 0$: Because of the compactness of X and the uppersemicontinuity of $\mathbf{u}$, if $F_\alpha \neq \emptyset$, then $x \in F_\alpha$ implies that there exists $y \in X$ such that $w_i(y) = x$ and $\mathbf{u}(y) \geqslant \alpha$. Now assume $\bar{x} \in \bar{F}_\alpha$ and $\{x_n\}$ a convergent sequence in F_α, such that $\lim_{n \to \infty} x_n = \bar{x}$. Then there exists a sequence $\{y_n\}$ in X such that $w_i(y_n) = x_n$ and $\mathbf{u}(y_n) \geqslant \alpha$ for each n. By the compactness of X there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to a limit $\bar{y} \in X$. Then by the continuity of w_i,

$$\lim_{k \to \infty} w_i(y_{n_k}) = w_i(\bar{y}), \tag{23}$$

i.e.

$$\lim_{k \to \infty} x_{n_k} = w_i(\bar{y}) = \bar{x}. \tag{24}$$

By recourse to the u.s.c. of $\mathbf{u}$,

$$\mathbf{u}(\bar{y}) \geqslant \limsup_{k \to \infty} \mathbf{u}(y_{n_k}) \geqslant \alpha, \tag{25}$$

which means $\bar{x} \in F_\alpha$.

Remark. That $\tilde{\mathbf{u}}_i$ is u.s.c. can also be derived from Proposition 3.7 in [14].

(2) We divide the proof into two cases: (i) $\alpha > 0$, (ii) $\alpha = 0$.

(i) For $\alpha > 0$, the proof is straightforward.

(ii) For $\alpha = 0$, it is easy to show that

$$\{x \in X : \varphi_i(\tilde{\mathbf{u}}_i(x)) > 0\} = w_i(\{x \in X : (\varphi_i \circ \mathbf{u})(x) > 0\}). \tag{26}$$

Thus,

$$\begin{aligned} [q_i]^0 = \overline{\{x \in X : \varphi_i(\tilde{\mathbf{u}}_i(x)) > 0\}} &= \overline{w_i(\{x \in X : (\varphi_i \circ \mathbf{u})(x) > 0\})} \\ &= w_i(\overline{\{x \in X : (\varphi_i \circ \mathbf{u})(x) > 0\}}) \\ &= w_i([\varphi_i \circ \mathbf{u}]^0), \end{aligned}$$

where the penultimate equality follows from the continuity of w_i and the compactness of X.

(3) The proof follows immediately from (2). ∎

What can be considered as the main theorem of this paper will now be stated.

THEOREM 2.4.1. *The operator T_s is a contraction mapping on $(\mathscr{F}^*(X), d_\infty)$, i.e., T_s maps $\mathscr{F}^*(X)$ into $\mathscr{F}^*(X)$ and for $0 \leqslant s < 1$,*

$$d_\infty(T_s\mathbf{u}, T_s\mathbf{v}) \leqslant s d_\infty(\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{v} \in \mathscr{F}^*(X). \tag{27}$$

Proof. Clearly T_s maps $\mathscr{F}(X)$ into $\mathscr{F}(X)$. Moreover, if $\mathbf{u}$ is normal, the fact that $\varphi_k(1) = 1$ for at least one $k \in \{1, 2, ..., N\}$ guarantees that $T_s\mathbf{u}$ is normal. On the other hand, by Lemma 2.4.1, for each α, $[T_s\mathbf{u}]^\alpha$ is a finite union of closed sets, hence it is closed, implying that $T_s\mathbf{u}$ is u.s.c. This proves that T_s maps $\mathscr{F}^*(X)$ into itself.

In order to show the contractivity of the operator T_s let us consider for each $\alpha \in (0, 1]$ the set $P_\alpha = \{i : \alpha \leqslant \varphi_i(1)\}$ as well as the set $P_0 = \{i : 0 < \varphi_i(1)\}$. (Note that $i \in P_\alpha \Leftrightarrow \exists \mathbf{u} \in \mathscr{F}^*(X)$ such that $[\varphi_i \circ \mathbf{u}]^\alpha \Leftrightarrow \forall \mathbf{u} \in \mathscr{F}^*(X) [\varphi_i \circ \mathbf{u}]^\alpha \neq \varnothing$.)

Now

$$d_\infty(T_s\mathbf{u}, T_s\mathbf{v}) = \sup_{0 \leqslant \alpha \leqslant 1} \{h([T_s\mathbf{u}]^\alpha, [T_s\mathbf{v}]^\alpha)\}, \tag{28}$$

and

$$h([T_s\mathbf{u}]^\alpha, [T_s\mathbf{v}]^\alpha) = \max\{D([T_s\mathbf{u}]^\alpha, [T_s\mathbf{v}]^\alpha), D([T_s\mathbf{v}]^\alpha, [T_s\mathbf{u}]^\alpha)\}. \tag{29}$$

In addition,

$$\begin{aligned} D([T_s\mathbf{u}]^\alpha, [T_s\mathbf{v}]^\alpha) &= D\left(\bigcup_{i \in P_\alpha} w_i([\varphi_i \circ \mathbf{u}]^\alpha), \bigcup_{j \in P_\alpha} w_j([\varphi_j \circ \mathbf{v}]^\alpha)\right) \\ &= \max_{i \in P_\alpha} D\left(w_i([\varphi_i \circ \mathbf{u}]^\alpha), \bigcup_{j \in P_\alpha} w_j([\varphi_j \circ \mathbf{v}]^\alpha)\right) \\ &\leqslant \max_{i \in P_\alpha} D(w_i([\varphi_i \circ \mathbf{u}]^\alpha), w_i([\varphi_i \circ \mathbf{v}]^\alpha)) \\ &\leqslant s \max_{i \in P_\alpha} D([\varphi_i \circ \mathbf{u}]^\alpha, [\varphi_i \circ \mathbf{v}]^\alpha). \end{aligned}$$

An analogous result is obtained for the second argument in Eq. (29), so that

$$h([T_s\mathbf{u}]^\alpha, [T_s\mathbf{v}]^\alpha) \leqslant s \max_{i \in P_\alpha} h([\varphi_i \circ \mathbf{u}]^\alpha, [\varphi_i \circ \mathbf{v}]^\alpha). \tag{30}$$

But from the proof of Lemma 2.2.1, we have that for $i \in \{1, 2, ..., N\}$

$$[\varphi_i \circ \mathbf{u}]^\alpha = [\mathbf{u}]^{\beta(i)}, \qquad \text{where} \quad \beta(i) = \inf\{t : \varphi_i(t) \geqslant \alpha\}, \text{ with } \alpha \in (0, \varphi_i(1)]. \tag{31}$$

On the other hand, it is shown in Appendix A that for $i \in \{1, 2, ..., N\}$,

$$[\varphi_i \circ \mathbf{u}]^0 \neq \varnothing \Rightarrow [\varphi_i \circ \mathbf{u}]^0 = \lim_{n \to \infty} [\mathbf{u}]^{\beta_n}, \tag{32}$$

in Hausdorff distance, for some decreasing sequence β_n in (0, 1). Let us now consider:

• For $\alpha > 0$, $h([\varphi_i \circ \mathbf{u}]^\alpha, [\varphi_i \circ \mathbf{v}]^\alpha) = h([\mathbf{u}]^{\beta(i)}, [\mathbf{v}]^{\beta(i)}) \leqslant d_\infty(\mathbf{u}, \mathbf{v})$ for $i \in P_\alpha$.

• For $\alpha = 0$ and $i \in P_0$, $h([\varphi_i \circ \mathbf{u}]^0, [\varphi_i \circ \mathbf{v}]^0) = \lim_{n \to \infty} h([\varphi_i \circ \mathbf{u}]^{\alpha_n}, [\varphi_i \circ \mathbf{v}]^{\alpha_n}) = \lim_{n \to \infty} h([\mathbf{u}]^{\beta_n}, [\mathbf{v}]^{\beta_n}) \leqslant d_\infty(\mathbf{u}, \mathbf{v})$.

Hence,

$$h([T_s\mathbf{v}]^\alpha, [T_s\mathbf{u}]^\alpha) \leqslant s d_\infty(u, \mathbf{v}) \qquad \forall \alpha \in [0, 1]. \quad \blacksquare$$

By virtue of the Contraction Mapping Principle over the complete metric space $(\mathscr{F}^*(X), d_\infty)$ we have the following important

COROLLARY 2.4.1. *For each fixed IFZS* $\{X, \mathbf{w}, \Phi\}$ *there exists a unique fuzzy set* $\mathbf{u}^* \in \mathscr{F}^*(X)$, *such that*

$$T_s\mathbf{u}^* = \mathbf{u}^*. \tag{33}$$

This implies that there exists a unique solution to the functional equation in the unknown $\mathbf{u} \in \mathscr{F}^*(X)$,

$$\mathbf{u}(x) = \sup\{\varphi_1(\tilde{\mathbf{u}}(w_1^{-1}(x))), \varphi_2(\tilde{\mathbf{u}}(w_2^{-1}(x))), ..., \varphi_N(\tilde{\mathbf{u}}(w_N^{-1}(x)))\} \tag{34}$$

for all $x \in X$. The solution fuzzy set $\mathbf{u}^*$ will be called the *attractor* of the IFZS, since it follows from the Contraction Mapping Principle that

$$d_\infty((T_s)^n \mathbf{v}, \mathbf{u}^*) \to 0 \qquad \text{as} \quad n \to \infty, \forall \mathbf{v} \in \mathscr{F}^*(X). \tag{35}$$

This provides the rigorous justification of the iteration procedure outlined in the Introduction.

Another important consequence is the property

$$[\mathbf{u}^*]^\alpha = \bigcup_{i=1}^{N} w_i([\varphi_i \circ \mathbf{u}^*]^\alpha), \qquad 0 \leqslant \alpha \leqslant 1, \tag{36}$$

(cf. Lemma 2.4.1), which can be considered as a generalized self-tiling property of α-level sets of the fuzzy set attractor $\mathbf{u}^*$. Let us now show some properties of $\mathbf{u}^*$.

DEFINITION. For $\mathbf{u}, \mathbf{v} \in \mathscr{F}^*(X)$, $\mathbf{u} \leqslant \mathbf{v}$ iff $\mathbf{u}(x) \leqslant \mathbf{v}(x)\ \forall x \in X$.

It is easy to see that the operator $T_s: \mathscr{F}^*(X) \to \mathscr{F}^*(X)$ is monotone, namely $\mathbf{u}, \mathbf{v} \in \mathscr{F}^*(X)$, $\mathbf{u} \leqslant \mathbf{v}$ implies $T_s\mathbf{u} \leqslant T_s\mathbf{v}$. We then obtain the following

PROPOSITION 2.4.1. *Let $\mathscr{A} \in \mathscr{H}(X)$ be the attractor of the base space IFS $\{X, \mathbf{w}\}$ and let $\mathbf{u}^* \in \mathscr{F}^*(X)$ denote the fuzzy set attractor of the IFZS $\{X, \mathbf{w}, \Phi\}$ with corresponding operator T_s. Then, for $\mathbf{v} \in \mathscr{F}^*(X)$* and *$B \in \mathscr{H}(X)$,*

(1) (a) $T_s\mathbf{v} \leqslant \mathbf{v} \Rightarrow \mathbf{u}^* \leqslant \mathbf{v}$.

(b) $\mathbf{w}(B) \subseteq B \Rightarrow \mathscr{A} \subseteq B$.

(2) (a) $\mathbf{v} \leqslant T_s\mathbf{v} \Rightarrow \mathbf{v} \leqslant \mathbf{u}^*$.

(b) $B \subseteq \mathbf{w}(B) \Rightarrow B \subseteq \mathscr{A}$.

Proof. All these properties are proved in the same way. For the sake of brevity, we prove only (1)(a). From $T_s\mathbf{v} \leqslant \mathbf{v}$, the monotonicity of T_s implies

$$(T_s)^n\, \mathbf{v} \leqslant \mathbf{v}, \qquad \forall n \in \mathrm{N}. \tag{37}$$

Since the set $I_{\mathbf{v}} = \{\mathbf{u} \in \mathscr{F}^*(X), \mathbf{u} \leqslant \mathbf{v}\}$ is closed in $(\mathscr{F}^*(X), d_\infty)$ (see Appendix B), it follows that

$$\mathbf{u}^* = \lim_{n \to \infty} (T_s)^n\, \mathbf{v} \leqslant \mathbf{v}, \tag{38}$$

where the limit is taken in the metric space $(\mathscr{F}^*(X), d_\infty)$. ∎

We are now able to prove the following theorem which demonstrates the connection between the fuzzy set attractor of an IFZS and the attractor of the corresponding base space IFS:

THEOREM 2.4.2. *Let $\mathscr{A} \in \mathscr{H}(X)$ be the attractor of the IFS $\{X, \mathbf{w}\}$, and let $\mathbf{u}^* \in \mathscr{F}^*(X)$ denote the fuzzy set attractor of the IFZS $\{X, \mathbf{w}, \Phi\}$. Then* $\operatorname{supp}(\mathbf{u}^*) \subseteq \mathscr{A}$, *that is,*

$$[\mathbf{u}^*]^0 \subseteq \mathscr{A}. \tag{39}$$

Proof. Let χ_A be the characteristic function of $\mathscr{A}$. Since $x \notin \mathscr{A}$ implies $w_i^{-1}(x) \subset X \backslash \mathscr{A}$, for some $i \in \{1, 2, ..., N\}$,

$$T_s \chi_A \leqslant \chi_A, \tag{40}$$

and therefore, by Proposition 2.4.1,

$$\mathbf{u}^* = \lim_{n \to \infty} (T_s)^n \chi_A \leqslant \chi_A, \tag{41}$$

which implies $[\mathbf{u}^*]^0 \subseteq \mathscr{A}$. ∎

Note that equality holds in (39) for the following two cases:

- For all $i \in \{1, 2, ..., N\}$, $\varphi_i(1) = 1$, then $\mathbf{u}^* = \chi_A$.
- For all $i \in \{1, 2, ..., N\}$, φ_i are increasing at 0 (i.e., $\varphi_i^{-1}(0) = \{0\}$). Indeed, in this case $[\mathbf{u}^*]^0 = \bigcup_{i=1}^N w_i([\varphi_i \circ \mathbf{u}^*]^0) = \bigcup_{i=1}^N w_i(\mathscr{A}) = \mathbf{w}(\mathscr{A}) = \mathscr{A}$.

We should also point out that in the case $\varphi_j(0) > 0$ for one $j \in \{1, 2, ..., N\}$, the inequality (39) is not true.

Another noteworthy consequence of the contractivity of the T_s operator is the following.

THEOREM 2.4.3. (IFZS Collage Theorem). *Let $\mathbf{v} \in \mathscr{F}^*(X)$ and suppose that there exists an IFZS $\{X, \mathbf{w}, \Phi\}$ with contractivity factor s so that*

$$d_\infty(\mathbf{v}, T_s\mathbf{v}) < \varepsilon, \tag{42}$$

where the operator T_s is defined by Eq. (22). Then

$$d_\infty(\mathbf{v}, \mathbf{u}^*) < \frac{\varepsilon}{1-s}, \tag{43}$$

where $\mathbf{u}^* = T_s\mathbf{u}^*$ is the invariant fuzzy set of the IFZS.

The proof of this theorem proceeds in the same fashion as for the usual IFS [2].

3. EXAMPLES

This section is devoted to some examples which illustrate the main features of the IFZS. In particular, the generality afforded by the grey level maps is shown.

3.1. *Example* 1

$X = [0, 1]$, $N = 4$; $w_i(x) = 0.25x + 0.25(i-1)$, $i = 1, 2, 3, 4$. Here $\mathscr{A} = [0, 1]$. The following grey level maps were selected:

$$\varphi_1(t) = \begin{cases} 0.25t & \text{if} \quad 0 \leqslant t < 0.25 \\ t - 0.18, & \text{if} \quad 0.25 \leqslant t \leqslant 1 \end{cases}$$

$$\varphi_2(t) = t, \qquad t \in [0, 1].$$

$$\varphi_3(t) = 0.33t$$

$$\varphi_4(t) = \sin t.$$

The picture shown in Fig. 1 is a representation of the graph of the IFZS attractor $\mathbf{u}^*$ on $[0, 1]$.

In the remaining examples, $X = [0, 1]^2$ is the base space. Photographs of the computer approximations to the IFZS invariant sets are shown as normalized grey level distributions: the brightness value t_{ij} of a pixel p_{ij} representing a point $x \in X$ obeys $0 \leqslant t_{ij} \leqslant 1$, with $t_{ij} = \mathbf{u}^*(x)$: $t_{ij} = 0$ if x is in the background.

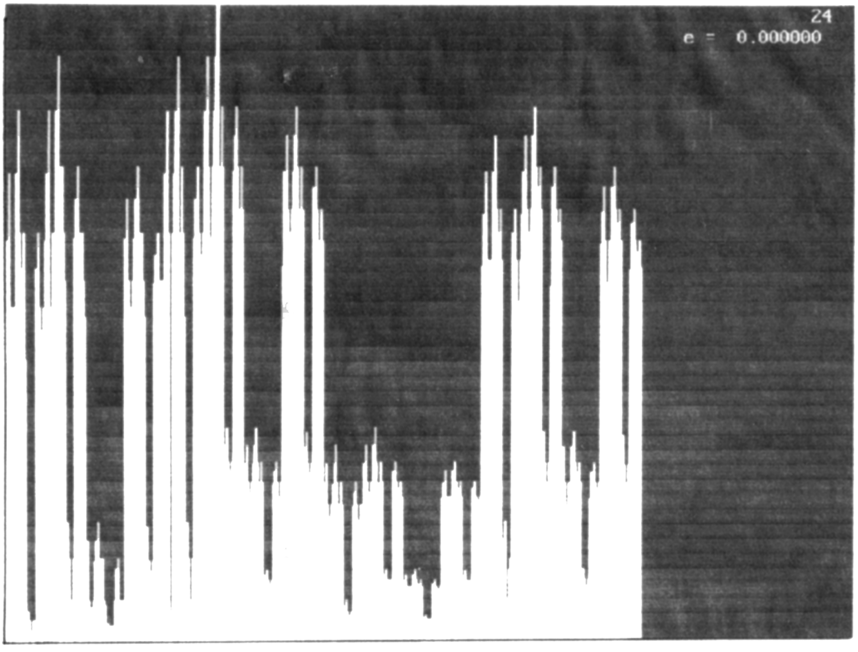
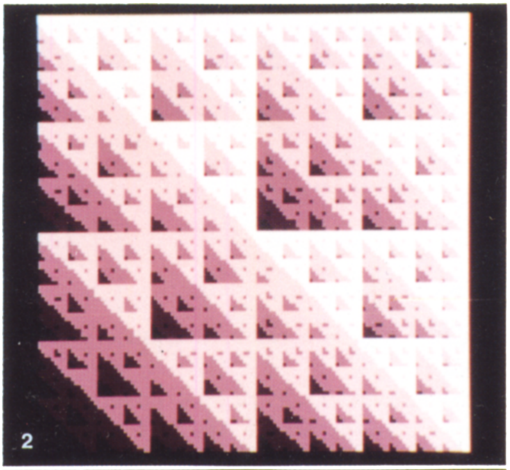
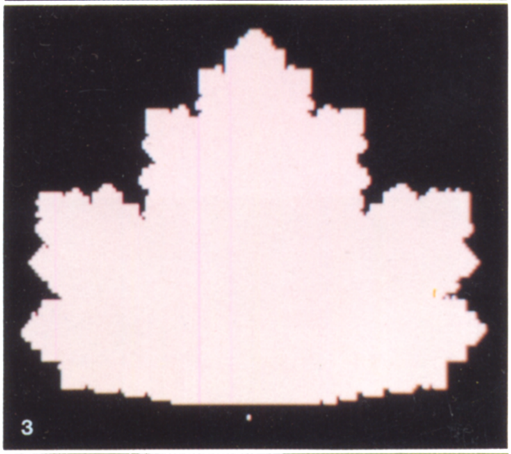
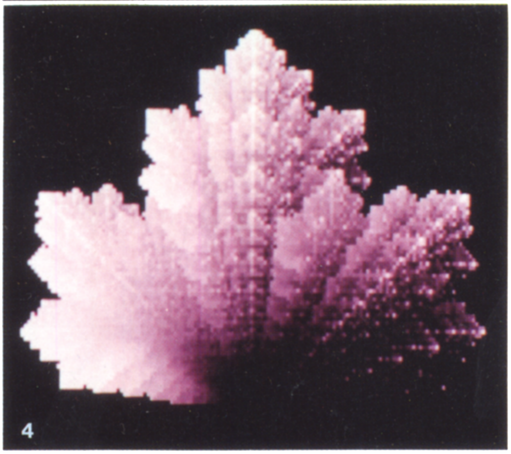


FIG. 1. Graph of IFZS attractor $\mathbf{u}^*$: $[0, 1] \to [0, 1]$ for Example 1.


2


3


4

3.2. *Example* 2

$N=4: w_1(x,y)=(0.5x, 0.5y)$, $w_2(x,y)=(0.5x+0.5,\ 0.5y)$, $w_3(x,y)=(0.5x,\ 0.5y+0.5)$, $w_4(x,y)=(0.5x+0.5,\ 0.5y+0.5)$. Here, $\mathscr{A}=X$. The following grey level maps were chosen:

$$\varphi_1(t)=\begin{cases} 0 & \text{if } 0 \leqslant t < 0.2505 \\ 0.25 & \text{if } 0.2505 \leqslant t < 0.505 \\ 0.5 & \text{if } 0.505 \leqslant t < 0.7505 \\ 0.75, & \text{if } 0.7505 \leqslant t \leqslant 1 \end{cases}$$

$$\varphi_2(t)=\varphi_3(t)=\varphi_4(t)=t, \qquad t \in [0, 1].$$

Note that $\mathbf{u}^* \subset \mathscr{A}$ (proper inclusion). This is due to the fact $\varphi_1(0)$ is not strictly increasing (cf. Theorem 2.4.2).

Some comments on the properties of $\mathbf{u}^*$, as evident from Fig. 2, would be instructive here. The grey level distribution exhibits the generalized self-tiling property of level sets, as given by Eq. (36). Let $\mathscr{A}_i=w_i(\mathscr{A})$, $i=1,2,3,4$. One can see the effect of the transformation φ_1 which is different from the other φ_i. Given that φ_2, φ_3, φ_4 are identity maps, the values of $\mathbf{u}^*$ on $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$ are the same. The presence of exactly four grey levels is due to the step function nature of φ_1.

The flexibility afforded by the grey level maps φ_i should now be apparent to the reader. The dynamics of maps on the unit interval (satisfying the n.d.r.c. condition) may be exploited to affect the pointwise shading of the image/fuzzy set in a rather controlled manner.

3.3. *Example* 3

$N=4$ and the transformations w_i taken from [4], define a base space IFS whose attractor $\mathscr{A}$ is a "leaf." We first consider identity grey level maps, i.e., $\varphi_i(t)=t$, $i=1,2,3,4$.

Since $\varphi_i(1)=1$, $i=1, ..., 4$, $\mathbf{u}^*=\chi_A$, i.e., $\mathbf{u}^*(x)=1$ if $x \in \mathscr{A}$, $\mathbf{u}^*(x)=0$ if $x \notin \mathscr{A}$. The attractor $\mathbf{u}^*$ is shown in Fig. 3.

3.4. *Example* 4

The base space IFS is as in Example 3, but with the following grey level maps:

$$\varphi_1(t)=0.85t, \quad \varphi_2(t)=t, \quad \varphi_3(t)=0.8\sqrt{t}, \quad \varphi_4(t)=0.4(t^2+t), t \in [0, 1].$$

The fuzzy set attractor $\mathbf{u}^*$ is shown in Fig. 4.

FIG. 2. IFZS attractor $\mathbf{u}^*: [0,1]^2 \to [0,1]$ for Example 2.

FIG. 3. IFZS attractor $\mathbf{u}^*: [0,1]^2 \to [0,1]$ for Example 3.

FIG. 4. IFZS attractor $\mathbf{u}^*: [0,1]^2 \to [0,1]$ for Example 4.

4. Concluding Remarks

Before getting involved into further investigations on IFZS it is useful and instructive to see how far we could actually proceed in setting up a fuzzy set approach by a direct translation of the IFS approach into fuzzy set language. For this purpose we reformulate the IFS approach step by step as follows:

(1) With (X, d) a complete metric space, and $\mathscr{P}(X)$ the set of parts of X, use the contractive point-to-point maps w_i that characterize the IFS to define set-to-set maps $\tilde{w}_i: \mathscr{P}(X) \to \mathscr{P}(X)$ as

$$\tilde{w}_i(B) := \{w_i(x) : x \in B\}, \qquad \forall B \in \mathscr{P}(X). \tag{44}$$

(2) Then, in order to be able to use the Hausdorff metric, restrict the domain of $\tilde{w}_i$, $i = 1, 2, ..., N$, to the set $\mathscr{H}(X)$ of all nonempty closed sets in X.

(3) At this point it can be shown that all $\tilde{w}_i$ are contractive maps on $(\mathscr{H}(X), h)$.

(4) Using the $\tilde{w}_i$, define an operator $\tilde{\mathbf{w}}: \mathscr{H}(X) \to \mathscr{H}(X)$ as

$$\tilde{\mathbf{w}}(B) := \bigcup_{i=1}^{N} \tilde{w}_i(B), \qquad \forall B \in \mathscr{H}(X). \tag{45}$$

In order to set up an IFZS approach, thereby extending characteristic functions to general fuzzy sets $\mathbf{u}$, we proceed as follows:

(1) With the given metric space (X, d) and point-to-point maps w_i, $i = 1, 2, ..., N$, use the extension principle for fuzzy sets [16, 11] and define the maps $\tilde{\tilde{w}}_i: \mathscr{F}(X) \to \mathscr{F}(X)$ as

$$\tilde{\tilde{w}}_i(\mathbf{u}) = \mathbf{v}, \qquad \text{where} \quad \mathbf{v}(x) := \sup\{\mathbf{u}(y) : y \in w_i^{-1}(x)\}. \tag{46}$$

(2) Restrict the domain of the maps $\tilde{\tilde{w}}_i$ to what corresponds (by the induced fuzzy set topology (see Weiss [14])) to $\mathscr{H}(X)$, namely, to the class $\mathscr{F}^*(X)$ of all normal ("nonempty") uppersemicontinuous ("closed") fuzzy sets.

(3) As a result, for each i, $\tilde{\tilde{w}}_i$ is contractive on the metric space $(\mathscr{F}^*(X), d_\infty)$.

(4) Then define the operator $\tilde{\tilde{\mathbf{w}}}: \mathscr{F}^*(X) \to \mathscr{F}^*(X)$ as

$$\tilde{\tilde{\mathbf{w}}} := \sup_{1 \leq i \leq N} \tilde{\tilde{w}}_i(\mathbf{u}) \qquad \forall\, \mathbf{u} \in \mathscr{F}^*(X). \tag{47}$$

This operator is contractive on $(\mathscr{F}^*(X), d_\infty)$, as is its IFS counterpart $\tilde{\mathbf{w}}$ on

$(\mathscr{H}(X), h)$, and its fuzzy set attractor is nothing but the characteristic function χ_A.

Notice that $\tilde{\mathbf{w}}$ of Eq. (47) coincides with our operator T_s in Eq. (22) when all φ_i are identity maps, i.e., $\varphi_i(t) = t$, $i = 1, 2, ..., N$.

The above comparison should indicate in which sense the IFZS approach is a generalization of IFS. As well, it should demonstrate why an additional flexibility to represent images is achieved with fuzzy sets $(\mathscr{F}^*(X))$ as compared to sets $(\mathscr{H}(X))$. The use of the functions φ_i instead of the numerical factors p_i provides the decisive advantage.

Moreover, Eq. (36) not only shows the self-tiling property of the operator T_s but it also provides a further motivation for the choice of such an operator, i.e., the use of the supremum as the associative operator on fuzzy sets. Indeed, it states that if an operator $T: \mathscr{F}^*(X) \to \mathscr{F}^*(X)$ ought to enjoy the self-tiling property, then $T = T_s$.

Appendix A

In connection with Theorem 2.4.1, we wish to analyze the 0-level set of the fuzzy set $\varphi \circ \mathbf{u}$, i.e., $[\varphi \circ \mathbf{u}]^0$, where $\mathbf{u} \in \mathscr{F}^*(X)$ and $\varphi: [0, 1] \to [0, 1]$ is n.d.r.c. First, let $\gamma = \inf\{t: \varphi(t) > 0\}$.

(1) if $(\gamma = 0)$ or $(\gamma > 0$ and $\varphi(\gamma) > 0)$ then $[\varphi \circ \mathbf{u}]^0 = [\mathbf{u}]^\gamma$:

if $\gamma = 0$ then $\varphi(\mathbf{u}(x)) > 0 \Leftrightarrow \mathbf{u}(x) > 0 \Rightarrow [\varphi \circ \mathbf{u}]^0 = [\mathbf{u}]^0$,

if $\gamma > 0$ and $\varphi(\gamma) > 0$ then

$$\varphi(\mathbf{u}(x)) > 0 \Leftrightarrow \mathbf{u}(x) \geqslant \gamma \Rightarrow \{x: \varphi(\mathbf{u}(x)) > 0\} = [\mathbf{u}]^\gamma \Rightarrow [\varphi \circ \mathbf{u}]^0 = [\mathbf{u}]^\gamma.$$

(2) if $(\gamma > 0$ and $\varphi(\gamma) = 0)$, then $[\varphi \circ \mathbf{u}]^0$ is not necessarily a level set of $\mathbf{u}$.

However, we have the following

Proposition A.1. *If φ is n.d.r.c. and $\mathbf{u}$ is u.s.c. then there exists a decreasing sequence $\beta_n \in (0, 1)$ such that $[\varphi \circ \mathbf{u}]^0 = \lim_{n \to \infty} [\mathbf{u}]^{\beta_n}$ in Hausdorff metric.*

Proof. $\{x: (\varphi \circ \mathbf{u})(x) > 0\} = \bigcup_{n \in \mathbb{N}} [\varphi \circ \mathbf{u}]^{\alpha_n}$, where $\alpha_n \searrow 0$, $\alpha_n > 0$. However, $[\varphi \circ \mathbf{u}]^{\alpha_n} = [\mathbf{u}]^{\beta_n}$ with β_n decreasing, which implies that $[u]^{\beta_n}$ is an increasing sequence of sets in X. Appealing to Lemma A.1 below, we have

$$h([\varphi \circ \mathbf{u}]^0, [\mathbf{u}]^{\beta_n}) \xrightarrow{n \to +\infty} 0,$$

which completes the proof. ▮

LEMMA A.1. *Let (X, d) be a compact metric space, and $(\mathscr{H}(X), h)$ be the space of nonempty closed sets in X, with the Hausdorff metric h. If $\{\mathscr{A}_n\}$ is a sequence of sets in $\mathscr{H}(X)$ such that $\mathscr{A}_n \subseteq \mathscr{A}_{n+1}$ $\forall n$, then*

$$\mathscr{A}_n \xrightarrow{h} \bar{\mathscr{A}}, \qquad \text{i.e., } h(\mathscr{A}_n, \bar{\mathscr{A}}) \to 0 \text{ as } n \to +\infty,$$

where $\bar{\mathscr{A}}$ indicates the closure of $\mathscr{A} = \bigcup_n A_n$.

Proof. We shall use the fact that for $A, B \in \mathscr{H}(X)$, $h(A, B) < \varepsilon$ iff $A \subseteq B^\varepsilon$ and $B \subseteq A^\varepsilon$, where $A^\varepsilon = \{x \in X : d(x, A) < \varepsilon\}$. Given an $\varepsilon > 0$, since $\mathscr{A}_n \subseteq \bar{\mathscr{A}}$ $\forall n$, then $\mathscr{A}_n \subseteq (\bar{A})^\varepsilon$ $\forall n$. We now show that for n sufficiently large, $\bar{\mathscr{A}} \subseteq \mathscr{A}_n^\varepsilon$.

Using the compactness of X we can find $m \in \mathbb{N}$ and $x_1, ..., x_m \in \bar{\mathscr{A}}$ such that

$$\bar{\mathscr{A}} \subseteq \bigcup_{i=1}^{m} \mathbb{B}(x_i, \varepsilon/2), \qquad \text{where} \quad \mathbb{B}(x_i, \varepsilon/2) = \{x \in X : d(x, x_i) < \varepsilon/2\}.$$

Now, for each x_i, $i = 1, 2, ..., m$, there exists $n_i \in \mathbb{N}$, such that $d(x_i, \mathscr{A}_n) < \varepsilon/2$, $\forall n \geqslant n_i$. Let $n_0 = \max\{n_i : 1 \leqslant i \leqslant m\}$. For $n \geqslant n_0$, if $y \in \bar{\mathscr{A}}$ then, for some i

$$d(y, \mathscr{A}_n) \leqslant d(y, x_i) + d(x_i, A_n) \leqslant \varepsilon,$$

which implies that $y \in \mathscr{A}_n^\varepsilon$. Therefore $\bar{\mathscr{A}} \subseteq \mathscr{A}_n^\varepsilon$ $\forall n \geqslant n_0$, completing the proof. ∎

APPENDIX B

Our main purpose in this appendix is to prove the following:

PROPOSITION B.1. *The sets*

$$I_v = \{\mathbf{u} \in \mathscr{F}^*(X) : \mathbf{u} \leqslant \mathbf{v}\}$$

and

$$O_{\mathbf{v}} = \{\mathbf{u} \in \mathscr{F}^*(X) : \mathbf{v} \leqslant \mathbf{u}\}$$

are closed in $(\mathscr{F}^*(X), d_\infty)$, $\forall \mathbf{v} \in \mathscr{F}^*(X)$.

This justifies the inequality (38) in the proof of Proposition 2.4.1. We need the following lemma, whose proof is immediate.

LEMMA B.1. *The following three properties hold:*

(i) *If $\mathbf{u} \in \mathscr{F}^*(X)$, then $\mathbf{u}(x) = \sup(\{0\} \cup \{\alpha \in (0, 1] : \mathbf{x} \in [\mathbf{u}]^\alpha\})$.*

(ii) *If* $\mathbf{u}, \mathbf{v} \in \mathscr{F}^*(X)$, *then* $\mathbf{u} \leqslant \mathbf{v} \Leftrightarrow [\mathbf{u}]^\alpha \subseteq [\mathbf{v}]^\alpha \ \forall \alpha \in [0, 1]$.

(iii) *If* $E \subseteq X$ *is closed, then* $I(E) = \{A \in \mathscr{H}(X) : A \subseteq E\}$ *and* $O(E) = \{A \in \mathscr{H}(X) : E \subseteq A\}$ *are closed in* $\mathscr{H}(X)$.

Proof of Proposition B.1. Consider $\{\mathbf{u}_n\} \subseteq I_\mathbf{v}$ and $\mathbf{u}_n \xrightarrow{d_\infty} \mathbf{u}$. Then $[\mathbf{u}_n]^\alpha \xrightarrow{h} [\mathbf{u}]^\alpha \ \forall \alpha \in [0, 1]$.

Since

$$\mathbf{u}_n \leqslant \mathbf{v},$$

Lemma B.1(ii) gives

$$[\mathbf{u}_n]^\alpha \subseteq [\mathbf{v}]^\alpha.$$

By (iii) from the same lemma,

$$[\mathbf{u}]^\alpha \subseteq [\mathbf{v}]^\alpha \qquad \forall \alpha \in [0, 1];$$

therefore $\mathbf{u} \leqslant \mathbf{v}$, which implies $\mathbf{u} \in I_\mathbf{v}$.

Hence $I_\mathbf{v}$ is closed. The closedness of $O_\mathbf{v}$ can be proved in a similar manner. ∎

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