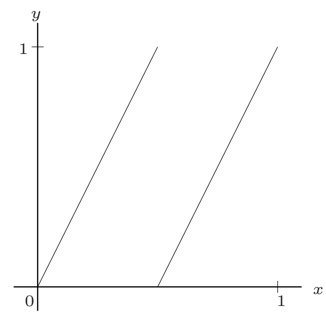
Lecture 21

Chaotic dynamics (cont'd)

Showing chaotic dynamics using "symbolic dynamics

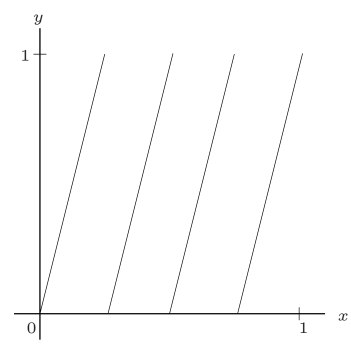
In the previous lecture, we showed, using symbolic dynamics, that the set of periodic points of the Baker map, shown below,

$$B(x) = 2x \bmod 1,\ 0 \le x \le 1\,,$$



is dense on [0,1]. This property could also have been shown from a knowledge of the above graph, as we did for the logistic map $f_4(x) = 4x(1-x)$. For example, the graph of $B^2(x) = B(B(x))$ is shown below.

$$y = B(B(x)),\ 0 \le x \le 1\,,$$



The function $B(B(x)$ is actually The graph of $B^2(x)$ intersects the line $y = x$ at three points – recall that we do not count the apparent fixed point at $x = 1$ since $B(1) = 0$ so that $B^2(1) = 0$. One of these three points is the fixed point $\bar{x} = 0$ of $B(x)$. The other two points, p_1 and p_2, make up the two-cycle $(\frac{1}{3}, \frac{2}{3})$:

$$\begin{aligned} B(\frac{1}{3}) &= 2 \cdot \frac{1}{3} \bmod 1 = \frac{2}{3} \\ B(\frac{2}{3}) &= 2 \cdot \frac{2}{3} \bmod 1 = \frac{4}{3} \bmod 1 = \frac{1}{3}\,. \end{aligned} \tag{1}$$

The reader may recognize that the graph of $B^2(x) = B(B(x)$ shown above may be expressed as follows,

$$B(B(x)) = 4x \bmod 1\,, \quad x \in [0,1]\,. \tag{2}$$

We can check this by showing that each of the points of the two-cycle given above are fixed points of

$B^2(x)$:

$$
\begin{aligned}
B^2(\frac{1}{3}) &= 4\cdot\frac{1}{3} \mod 1 = \frac{4}{3} \mod 1 = \frac{1}{3} \\
B^2(\frac{2}{3}) &= 4\cdot\frac{2}{3} \mod 1 = \frac{8}{3} \mod 1 = \frac{2}{3}\,.
\end{aligned} \tag{3}
$$

A pattern is developing here. The function $B^3(x)$ is given by the formula,

$$B^3(x) = 8x \mod 1\,, \quad x \in [0,1]\,. \tag{4}$$

The graph of $B^3(x)$ will consist of 8 lines with slope $2^3 = 8$, implying the existence of 7 period-3 points: The fixed point $\bar{x} = 0$ and two three-cycles. In general,

$$B^n(x) = 2^n x \mod 1\,, \quad x \in [0,1]\,. \tag{5}$$

The graph of $B^n(x)$ for $n > 1$ will consist of 2^n lines of slope 2^n. (Another way to see this is to consider the graph of $f(x) = 2^n x$ for $0 \leq x \leq 1$, which is a straight line of slope 2^n. The "mod 1" operation then takes all parts of this graph which extend from $y = 1$ upwards and brings them down into the interval $0 \leq y < 1$. This produces the 2^n "pieces" of the graph of $B^n(x)$.) The graph of $B^n(x)$ intersects with the line $y = x$ at $2^n - 1$ points, each of which lies in its own subinterval $I_{b_1 b_2 \cdots b_n}$ of length 2^{-n}. As $n \to \infty$, this set of $2^n - 1$ periodic points will get "thicker and thicker" over [0,1], as was the case for the logistic map $f_4(x)$. (The fact that there are $2^n - 1$ periodic points, implying that one subinterval is unoccupied, is not a problem since the lengths of the "missing" subintervals go to zero as $n \to \infty$.)

That all being said, it is not as easy – in fact, it will be difficult in general – to use the properties of the graphs of $B(x)$ and its iterates to show that $B(x)$ possesses the other two ingredients for chaotic behaviour, namely, sensitive dependence on initial conditions and transitivity. It is here where "symbolic dynamics" will show its strength.

Ingredient No. 2: Sensitive dependence to initial conditions (SDIC)

Let us first recall the definition of SDIC introduced in the previous lecture:

> **Definition:** A function $f : I \to I$ is said to have **sensitive dependence to initial conditions (SDIC)** at a point $x \in I$ if there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exists a $y \in N_\delta(x)$ and an $n > 0$ such that
>
> $$|f^n(x) - f^n(y)| > \epsilon \,. \tag{6}$$
>
> Recall that $y \in N_\delta(x)$ means that
>
> $$|x - y| < \delta \,. \tag{7}$$

Showing that the Baker map $B(x) = 2x \bmod 1$ is SDIC at any $x \in [0, 1]$ is quite straightforward using symbolic dynamics: We'll replace points in [0,1] with appropriate binary sequences in Σ_2 and show that the mapping S induced by the Baker map on Σ_2 is SDIC. Essentially, we do the following:

> Given any sequence $\mathbf{b} \in \Sigma_2$ (corresponding to a point $x \in [0, 1]$) show that we can find/construct a sequence $\mathbf{b}' \in \Sigma_2$ (corresponding to a point $y \in [0, 1]$ that is as close as desired to $\mathbf{b}$, i.e., lies within an arbitrary distance $\delta' > 0$ from $\mathbf{b}$ in d_{Σ_2} distance, but that after n iterations of the Bernoulli shift map, the sequences $S^n(\mathbf{b})$ and $S^n(\mathbf{b}')$ are separated by some distance $\epsilon' > 0$ which, in turn, implies that the points $B^n(x), B^n(y) \in [0, 1]$ corresponding to these two sequences are sufficiently separated.

Suppose that x and y have binary expansions $\mathbf{b}$ and $\mathbf{b}'$, respectively, that coincide for the first n digits, i.e. $b_k = b'_k$, $1 \leq k \leq n$. This implies that $x, y \in I_{b_1 b_2 \ldots b_n}$. Furthermore, suppose that $b_{n+1} \neq b'_{n+1}$. Since $S^n(\mathbf{b}) = (b_{n+1}, b_{n+2}, \ldots)$ and $S^n(\mathbf{b}') = (b'_{n+1}, b'_{n+2}, \ldots)$, it follows that $B^n(x) \in I_{b_{n+1}}$ and $B^n(y) \in I_{b'_{n+1}}$, i.e. $B^n(x)$ and $B^n(y)$ lie in different half-intervals of $[0, 1]$. How far they are from each other depends upon the remainders of their respective binary codes. The important point is that there exist pairs of points from the same subinterval $I_{b_1 b_2 \ldots b_n}$ that, after n applications of f, find themselves on opposite sides of the interval I. No matter how close these points are, i.e. how high we make n, the function f simply needs to be iterated n times. This is the essence of SDIC.

Ingredient No. 3: Transitivity

Let us recall the definition of transitivity, given in the previous lecture.

> A mapping $f : I \to I$ is transitive on I if, for any two points $x, y \in I$ and any two neighbourhoods $N_{\delta_1}(x)$ and $N_{\delta_2}(y)$ of these points, there exists a point $p \in N_{\delta_1}(x)$ and an $n > 0$ such that $f^n(p) \in N_{\delta_2}(y)$.

Once again, we want to translate the above statement into symbolic dynamics as follows:

> The mapping $S : \Sigma_2 \to \Sigma_2$ is transitive if, for any two sequences $\mathbf{b}, \mathbf{c} \in \Sigma_2$, and any two neighbourhoods $N_{\delta_1'}(\mathbf{b})$ (in Σ_2 distance) and $N_{\delta_2'}(\mathbf{c})$ of these two points, there exists a point $\mathbf{b}' \in N_{\delta_1'}(\mathbf{b})$ and an $n > 0$ such that $S^n(\mathbf{b}') \in N_{\delta_2'}(\mathbf{c})$.

As before, let

$$\begin{aligned} \mathbf{b} &= (b_1, b_2, b_3, \cdots) \\ \mathbf{c} &= (c_1, c_2, c_3, \cdots) \end{aligned} \tag{8}$$

be the two arbitrary sequences (which represent two arbitrary points $x, y \in [0, 1]$). The trick is to construct a sequence $\mathbf{b}'$ that will satisfy the conditions above.

1. **Step 1:** Given a $\delta_1' > 0$, there exists an n_1 which guarantees that if the first n_1 binary digits of the sequences $\mathbf{b}$ and $\mathbf{b}'$ agree, then $d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') < \delta_1'$, which implies that $\mathbf{b}' \in N_{\delta_1'}(\mathbf{b})$. (We don't have to worry about the details in finding n_1 – we only need to know that such an n_1 exists.) Therefore, we know the first n_1 digits of $\mathbf{b}'$, i.e.,

$$\mathbf{b}' = (b_1, b_2, \cdots b_{n_1} \, ?, ?, ?) \,, \tag{9}$$

where the question marks indicate that we still have some work to do.

2. **Step 2:** Given a $\delta_2' > 0$, there exists an n_2 which guarantees that if the first n_2 binary digits of the sequences $\mathbf{c}$ and $\mathbf{d}$ agree, then $d_{\Sigma_2}(\mathbf{c}, \mathbf{d}) < \delta_2'$. Therefore, if we simply add the first n_2 digits of the sequence $\mathbf{c}$ after the n_1st digit of $\mathbf{b}$, i.e.,

$$\mathbf{b}' = (b_1, b_2, \cdots, b_{n_1}, c_1, c_2, \cdots, c_{n_2}, \cdots) \,, \tag{10}$$

we've accomplished our goal – it doesn't matter what digits we put after $b'_{n_1+n_2} = c_{n_2}$.

3. **Step 3 (optional):** Let's just check the above result:

   - Since the first n_1 digits of $\mathbf{b}'$ agree with those of $\mathbf{b}$, it follows that $d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') < \delta_1'$, which implies that $\mathbf{b}' \in N_{\delta_1'}(\mathbf{b})$. So far, so good.

- Now apply the shift map n_1 times to the sequence $\mathbf{b}'$. The result is the sequence,

$$S^{n_1}(\mathbf{b}') = (c_1, c_2, \cdots, c_{n_2}, \cdots). \tag{11}$$

  Since the first n_2 digits of $S^n(\mathbf{b}')$ agree with those of $\mathbf{c}$, it follows that $d_{\Sigma_2}(S^{n_1}(\mathbf{b}'), \mathbf{c}) < \delta_2'$, which implies that $S^{n_1}(\mathbf{b}') \in N_{\delta_2'}(\mathbf{c})$. Mission accomplished!

Since the three ingredients for chaotic behaviour have been shown to hold, we can conclude that the shift map $S : \Sigma_2 \to \Sigma_2$ defines a chaotic dynamical system on the binary sequence space Σ_2 which, in turn, establishes that the map $B(x) = 2x \bmod 1$ is chaotic on $[0, 1]$.

The moral of the story: If you can establish a correspondence between a map $f : I \to I$ and a shift map S on a sequence space Σ, then the chances are good that you can show f to be chaotic on I. This is the essence of "symbolic dynamics".

Exercise: Sketch the graph of the mapping $f(x) = 3x \bmod 1$ on $[0, 1]$. What sequence space Σ would you use to analyze the action of f on $[0, 1]$?

Exercise: Same as above, but for the mapping $f(x) = 10x \bmod 1$ on $[0, 1]$.

An interesting consequence of transitivity – dense orbits

We shall now show, using symbolic dynamics, another remarkable property of chaotic mappings – the existence of **dense orbits**. This is made possible by the transitivity property of chaotic mappings. In fact, transitivity and the existence of dense orbits are equivalent properties – if you have one, you have the other. Showing this equivalence, however, is somewhat beyond the scope of this course (you can see a proof in the author's PMATH 370 notes).

First of all, let us recall the idea of the **orbit** of a point $x \in I$ under the action of a mapping $f : I \to I$. The orbit $O(x) \subset I$ is the set of points,

$$\begin{aligned} O(x) &= \{f^n(x)\}_{n=0}^{\infty} \\ &= \{x, f(x), f^2(x), \cdots\}. \end{aligned} \tag{12}$$

For the subset $O(x)$ to be dense in the set I, the following must hold, according to the definition of a dense subset of I: Given **any** point $y \in I$ and **any** $\delta > 0$, there exists a point $z \in O(x)$ such that $z \in N_\delta(y)$, i.e., z lies in the interval $(y - \delta, y + \delta)$. This implies that there exists an $n \geq 0$, such that the point $f^n(x) \in O(x)$ lies in the interval $N_\delta(y)$.

For an orbit $O(x)$ to be dense in I – often written as "for $O(x)$ to be a **dense orbit** in I" – is a rather special property of the set $O(x)$ which, in turn, depends on the point $x \in I$. Clearly, x cannot be a periodic point, e.g., a fixed point or an element of an N-cycle. In such cases, the orbit $O(x)$ is a finite set composed of the points in the N-cycle. There is no way that the elements of this finite set of points could approach an **arbitrary point** $y \in I$ to **arbitrary accuracy** $\delta > 0$.

We now show how a dense orbit can be constructed for the Baker map $B(x) = 2x \bmod 1$, using symbolic dynamics. In fact, we'll see that an infinite set of such orbits can be constructed.

The construction of a dense orbit using symbolic dynamics is quite clever. It relies on the following property that we have used on a number of occasions, but which is worthwhile to repeat:

Let $x, y \in [0, 1]$ with associated binary expansions

$$\begin{aligned} x: \quad & \mathbf{b} = (b_1, b_2, \ldots), \quad b_i \in \{0, 1\}, \\ y: \quad & \mathbf{b}' = (b'_1, b'_2, \ldots), \quad b'_i \in \{0, 1\}. \end{aligned}$$

Suppose that $b_k = b'_k$ for $1 \le k \le n$. Then $|x - y| \le 2^{-n}$.

Now consider $x \in (0, 1)$ with the the following binary representation,

$$x = \underbrace{.0\ 1}_{\text{length 1}} \quad \underbrace{00\ 01\ 10\ 11}_{\text{length 2}} \quad \underbrace{000\ 001\ 010\ 011\ 100\ 101\ 110\ 111}_{\text{length 3}} \quad \cdots \tag{13}$$

The binary representation of x is constructed by starting with the 2 possible strings of length 1, followed by all possible ($2^2 = 4$) strings of length 2, followed by all possible ($2^3 = 8$) strings of length 3, $\cdots$, all possible 2^n strings of length n, $\cdots$. In this way, the binary representation of x contains all possible strings of length n for $n = 1, 2 \cdots$.

A side note: We do not care about the actual "value" of x – only that such a point in $(0, 1)$ exists. We only need to know its binary representation. We can, in fact, construct many other such points in a similar fashion. How do we know this? By the fact that we employed a particular ordering of the strings of length 2,3,4, etc. in the above construction. We could employ another ordering, e.g.

$$y = \underbrace{.1\ 0}_{\text{length 1}} \quad \underbrace{11\ 10\ 01\ 00}_{\text{length 2}} \quad \underbrace{111\ 110\ 101\ 100\ 011\ 010\ 001\ 000}_{\text{length 3}} \quad \cdots \tag{14}$$

Or a different ordering for each string-length. It doesn't matter. The only important point is that for each $n \ge 1$, we produce **all** possible strings of length n **in some order**. Indeed, since the order of presenting the strings for two different values of n, say, $n = n_1$ and $n = n_2$, does not matter, the method described above can, in the limit, produce an infinity of points x with dense orbits.

It is now quite straightforward to show that the forward orbit of x under the action of the Baker map $B(x)$ is dense in $[0,1]$. We need to show that for any $y \in [0,1]$ and any $\delta > 0$, there exists an $n > 0$ such that

$$|B^n(x) - y| < \delta \,. \tag{15}$$

Let the binary representation of y be denoted as

$$y \,:\, \mathbf{b} = (b_1, b_2, \ldots)\,, \quad b_i \in \{0,1\}\,. \tag{16}$$

Now let $N > 0$ be such that $2^{-N} < \delta$.

For a point $B^n(x)$ in the orbit of x to satisfy (15), it is necessary that the first N digits of $B^n(x)$ agree with those of y. Since the binary expansion of x, by construction, contains **all** sequences of length $n \geq 1$, it will contain, by construction, the **particular** sequence,

$$b_1 b_2 \cdots b_N \,, \tag{17}$$

which is one of the 2^N possible sequences of length N. The first element of this string in the binary sequence of x will be situated somewhere after the first

$$2 + 2^2 + 2^3 + \cdots 2^{N-1} = 2^N - 2 \tag{18}$$

binary digits of x but before the next set of 2^{N+1} strings of length $N+1$, starting at

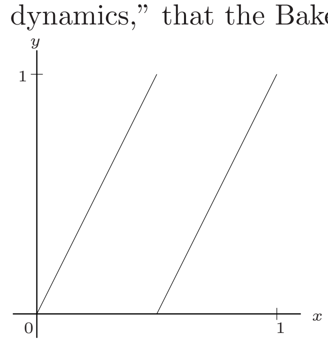
$$2 + 2^2 + 2^3 + \cdots 2^N = 2^{N+1} - 2\,. \tag{19}$$

From the fact that each application of the Baker map $B(x)$ to x produces a leftward shift (plus deletion of the first binary digit) of the binary expansion of x, the first digit of the N-string in (17) will become the first binary digit of $B^m(x)$ after m applications of the shift map S to the binary expansion of x in (13), where $2^N - 2 < m < 2^{N+1} - 2$. This implies that the inequality in (15) is satisfied for $n = m$. Since this inequality is satisfied for any $y \in (0,1)$ and any $\delta > 0$, we conclude that the orbit $\{B^n(x)\}_{n=0}^{\infty}$ is dense in $[0,1]$. Mission accomplished!

In summary, we have shown above, using the method of "symbolic dynamics," that the Baker map,

Baker map on $[0,1]$

$$B(x) = 2x \bmod 1,\; 0 \leq x \leq 1\,.$$



defines a chaotic dynamical system on the interval [0,1] which satisfies the three "ingredients" listed earlier, namely,

1. The set of periodic points of $B(x)$ is dense on [0,1],

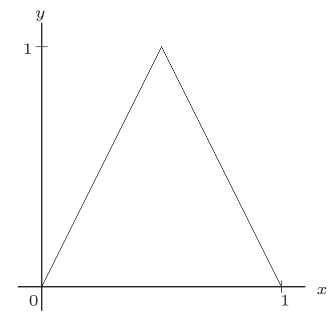
2. Sensitive dependence to initial conditions (SDIC),

3. Transitivity.

We now state, without proof, that the method of "symbolic dynamics" can be applied to the Tent Map introduced earlier,

Tent Map on $[0,1]$

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2-2x, & \frac{1}{2} < x \le 1 \,. \end{cases}$$

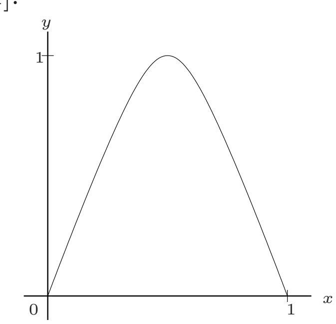


It just requires a bit of "rearranging" of the intervals used for the Baker Map because the part of $T(x)$ defined over $[1/2, 1]$ is "flipped". (We may return to this point in a little while.) As a result, we may conclude that the Tent Map defines a chaotic dynamical system on [0,1].

Finally, using an appropriate change of variable – details omitted here – there is a 1-1 mapping from the (chaotic) Tent Map to the Logistic map $f_4(x) = 4x(1-x)$. We may therefore conclude that the Logistic map $f_4(x)$ defines a chaotic dynamical system on [0,1].

Logistic map on $[0,1]$

$$f_4(x) = 4x(1-x)\,, \qquad 0 \le x \le 1\,.$$



At this point, one may wonder the following: OK, so the Baker Map, the Tent Map and the Logistic map $f_4(x)$ define chaotic dynamical systems on the interval [0,1]? Are they then, "the same", whatever that means? Well, since they are three different functions, the dynamical systems are not "the same," in terms of how points are being mapped over [0,1]. That being said, the Baker and Tent Maps, since they are composed of straight line segments, will share a property that is quite different from that of the Logistic map $f_4(x)$, the graph of which is composed of "curved" segments. We'll show this in the next section.

Lecture 22

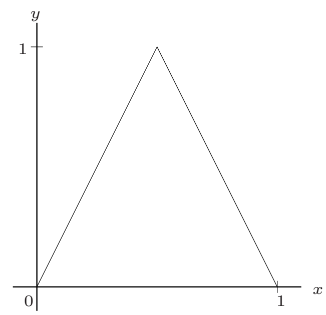
Chaotic dynamics (cont'd)

A return to the Tent Map

We would now like to use the symbolic dynamics approach shown above to show that the Tent Map, shown below, is chaotic.

Tent Map

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} < x \le 1\,, \end{cases}$$



There is a complication, however. The first portion of the tent map, i.e., the portion over $[0, 1/2]$, coincides with that of the Baker Map. As such, its action on a point $x \in [0, 1/2]$ induces a Bernoulli left-shift on the binary sequence of x. But the second portion of the tent map, the part over $[1/2, 1]$, is a "flipped over" version of the Baker Map. Its action on a point $x \in (1/2, 1]$ does not induce a Bernoulli left-shift on the binary sequence of x.

It turns out that we don't have to worry about this if we associate **another binary sequence** with an $x \in [0, 1]$, a **binary sequence that is defined by the action of the Tent Map, as opposed to the location of** x **in [0,1].** This binary sequence will, for reasons that will become clear, be called an **itinerary sequence**. In fact, as we'll show below, we can define itinerary sequences for other maps, e.g., the "curved" logistic map.

To do this, let's go back to the Baker Map for a moment. Recall that we considered the binary expansion of a point $x \in [0, 1]$ which had the form

$$x = \cdot b_1 b_2 b_3 \ldots = \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \cdots, b_i \in \{0, 1\}\,. \tag{20}$$

We considered the binary expansion of x as an element of the space of all infinite binary expansions, denoted as Σ_2, i.e.,

$$\Sigma_2 = \{\mathbf{b} = (b_1, b_2, b_3\ \ldots) \mid b_i \in \{0, 1\}\}. \tag{21}$$

We then observed that action of the Baker map $B(x)$ on a point x is equivalent to "left-shifting" the binary expansion of x and dropping the integer part, i.e.,

$$B(x) = 2x \bmod 1 = \cdot b_2 b_3 b_4 \ldots = \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \cdots, b_i \in \{0, 1\}\,. \tag{22}$$

We denoted this left-shift or "Bernoulli" operator on the space Σ_2 as "S". The action of $S \to \Sigma_2 \to \Sigma_2$ is given as follows,

$$S \ : \ b_1 b_2 b_3 \to b_2 b_2 b_4 \ldots \, . \tag{23}$$

We then used this equivalence of the action of the Baker map $B(x)$ on points $x \in [0,1]$ to that of the shift map S on binary sequences $\mathbf{b} \in \Sigma_2$ to show that the Baker map $B(x)$ is chaotic on $[0,1]$. By means of "symbolic dynamics," i.e., the action of the operator $S : \Sigma_2 \to \Sigma_2$, we were able to show that the Baker map $B(x)$ posseses the following three ingredients for chaotic dynamics:

1. The set of periodic points of $B(x)$ is dense on [0,1],
2. Sensitive dependence to initial conditions (SDIC),
3. Transitivity.

This was all fine, but let us now look at how we can generalize the use of binary sequences to allow them to be used for other maps. The first thing that we'll do – hopefully not causing much confusion! – is to slightly alter the indexing of sequence coefficients b_i for reasons that will become clear below. Our space of infinite binary sequences Σ_2 will be defined as follows,

$$\Sigma_2 = \{\mathbf{b} = b_0 b_1 b_2 \ldots \mid b_i \{0, 1\}\} \, . \tag{24}$$

In other words, the index of the first element in the sequence is now "0" instead of "1". The usual binary expansion of a number $x \in [0,1]$ will now have the form,

$$x = .b_0 b_1 b_2 \cdots = \frac{b_0}{2} + \frac{b_1}{4} + \frac{b_2}{8} + \cdots , b_i \in \{0, 1\} \, . \tag{25}$$

But we are now going to want to **forget** the above binary expansion since it works **only** for the Baker map. Instead, we focus on **binary sequence b** representing x, writing the representation as follows,

$$x \ : \ b_0 b_1 b_2 \cdots , \tag{26}$$

and observe the action of the Bernoulli shift map S on this sequence. Note that we no longer write "$x =$". We write "x :" to denote the correspondence. **There will be no equality – only a 1-1 correspondence – when we consider maps other than the Baker Map.**

Recall that a given number $n \geq 0$ of applications of the Baker Map $B(x)$ produces the point $y = B^n(x)$, with binary sequence,

$$y \ : \ b_n b_{n+1} b_{n+2} \ldots \, . \tag{27}$$

As we discussed many times in the past two lectures, the expansion in (27) is obtained by left-shifting the expansion of x in (26) n times. But now note how the location of the point y in the interval $[0,1]$ is determined by the first element b_n of its binary sequence,

- If $b_n = 0$, then $y = B^n(x) \in I_0 = [0, 1/2]$.
- If $b_n = 1$, then $y = B^n(x) \in I_1 = (1/2, 1]$.

Let's look at the first few cases.

1. $n = 0$:

   - If $B^0(x) = x \in [0, 1/2]$, then $b_0 = 0$.
   - If $B^0(x) = x \in (1/2, 1]$, then $b_0 = 1$.

   There is nothing new here. We know that the leading digit b_0 tells us the location of x in terms of the half-subintervals of the interval $[0, 1]$. The interval $[0, 1/2]$ will be denoted as I_0 since all of its elements have the binary digit $b_0 = 0$. Likewise, the interval $(1/2, 1]$ will be denoted as I_1 since all of its elements have the binary digit $b_0 = 1$.

2. $n = 1$:

   - If $B^1(x) \in [0, 1/2]$, then $b_1 = 0$.
   - If $B^1(x) \in (1/2, 1]$, then $b_1 = 1$.

   From the graph of $B(x)$ shown below, we see that there are two intervals of points which are mapped to $[0, 1/2]$ after one application of B: (i) $x \in [0, 1/4]$ and (ii) $x \in [1/2, 3/4]$. The **second** binary digit b_1 of points in either of these two intervals is therefore **0**.

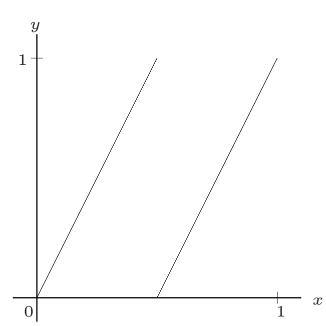
   - Since $[0, 1/4] \subset I_0$ (first binary digit is 0), we denote $[0, 1/4]$ as I_{00}.
   - Since $[1/2, 3/4] \subset I_1$ (first binary digit is 1), we denote $[1/2, 3/4]$ as I_{10}.

   Also from the graph of $B(x)$ shown below, we see that there are two intervals of points which are mapped to $(1/2, 1]$ after one application of B: (i) $x \in (1/4, 1/2]$ and (ii) $x \in (3/4, 1]$. The **second** binary digit b_1 of points in these two intervals is therefore **1**.

   - Since $(1/4, 1/2] \subset I_0$ (first binary digit is 0), we denote $(1/4, 1/2]$ as I_{01}.
   - Since $(3/4, 1] \subset I_1$ (first binary digit is 1), we denote $(3/4, 1]$ as I_{11}.

Baker map on $[0, 1]$

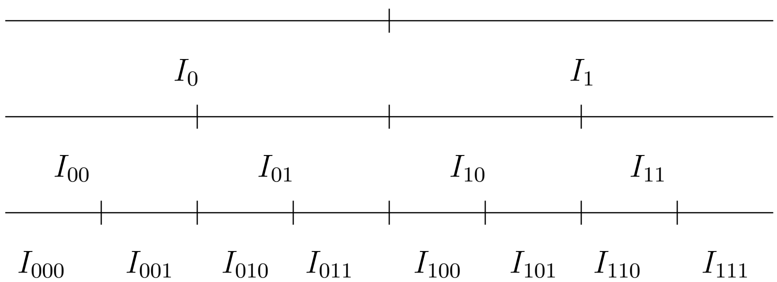
$$B(x) = 2x \bmod 1, \ 0 \le x \le 1 \,.$$



We have used the above observation to define the sequence representation of the point $x \in [0,1]$ on the basis of its trajectory under the action of the map B and forget/discard the fact that, in this particular case, the sequence in (26) coincides with the binary expansion of x.

With this idea in mind, the sequence in (26) is considered to characterize the **itinerary** of x under the action of the Baker Map B. Knowing the itinerary of x after n applications of B, hence n applications of the left-shift map, allows us to determine the location of $x \in [0,1]$ to an accuracy of 2^{-n}. If the first n digits of the sequence in (26) are known, then we know that x is an element of a subinterval that we shall designate as $I_{b_0 b_1 b_2 \dots b_{n-1}}$ and the length of this subinterval is 2^{-n}.

Recall that for the Baker Map, positions of these subintervals is ordered in a very regular way as shown in the figure below. And these subintervals are labelled in the same way as they were when we considered the binary expansions of points $x \in [0,1]$. But that will change when we study the Tent Map!



Itinerary sequences for Tent Map

Let us now move to the tent map $T(x)$ and construct sequences of the form,

$$x \,:\, \mathbf{b} = (b_0 b_1 b_2 \dots) \qquad b_i \in \{0,1\}\,, \tag{28}$$

for points $x \in [0,1]$ according to their itineraries under the tent map. (Note that we do not write "$x =$" in the above statement.) The binary digit representations of a point x will be assigned as follows,

- If $b_n = 0$, then $y = T^n(x) \in I_0 = [0, 1/2]$.
- If $b_n = 1$, then $y = T^n(x) \in I_1 = (1/2, 1]$.

First of all, the first element b_0 is the same as in the case of the Baker Map, since we apply the Tent Map zero times:

- If $T^0(x) = x \in [0, 1/2]$, then $b_0 = 0$.
- If $T^0(x) = x \in (1/2, 1]$, then $b_0 = 1$.

Let's now examine the case $n = 1$.

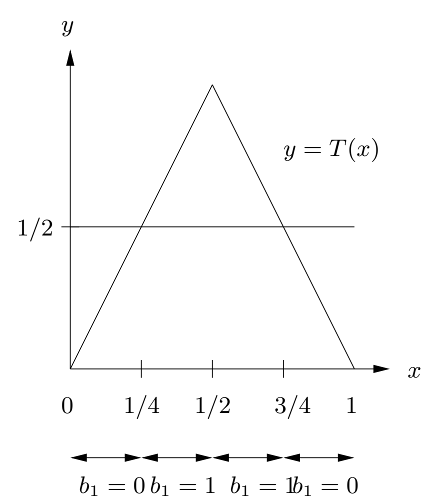
- If $T^1(x) \in [0, 1/2]$, then $b_1 = 0$.
- If $T^1(x) \in (1/2, 1]$, then $b_1 = 1$.

From the graph of $T(x)$ shown below, we see that there are two intervals of points which are mapped to $[0, 1/2]$ after one application of T: (i) $x \in [0, 1/4]$ and (ii) $x \in [3/4, 1]$, **not** $(1/2, 3/4]$ as was the case for the Baker Map. The **second** binary digit b_1 of points in these two intervals is therefore **0**.

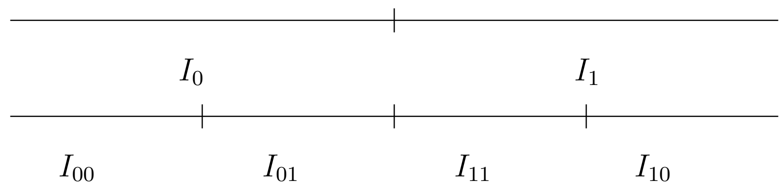
- Since $[0, 1/4] \subset I_0$ (first binary digit is 0), we denote $[0, 1/4]$ as I_{00}.
- Since $[3/4, 1] \subset I_1$ (first binary digit is 1), we denote $[3/4, 1]$ as I_{10}. (Note that this differs from the Baker Map case.

Also from the graph of $T(x)$ shown below, we see that there are two intervals of points which are mapped to $(1/2, 1]$ after one application of B: (i) $x \in (1/4, 1/2]$ and (ii) $x \in (1/2, 3/4]$. The **second** binary digit b_1 of points in these two intervals is therefore **1**.

- Since $(1/4, 1/2] \subset I_0$ (first binary digit is 0), we denote $(1/4, 1/2]$ as I_{01}.
- Since $(1/2, 3/4] \subset I_1$ (first binary digit is 1), we denote $(1/2, 3/4]$ as I_{11}.



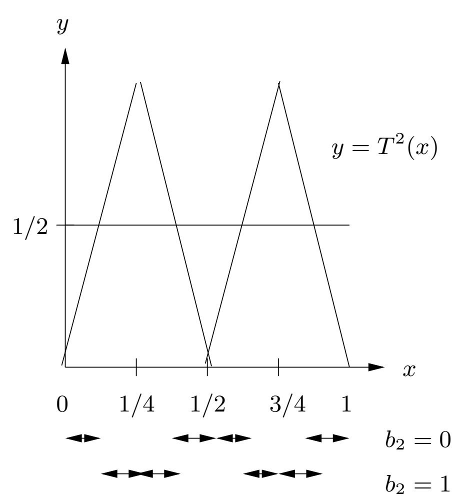
On the basis of these observations, we conclude that the first two sets of subintervals I_{ij} associated with the tent map are indexed as shown in the figure below.



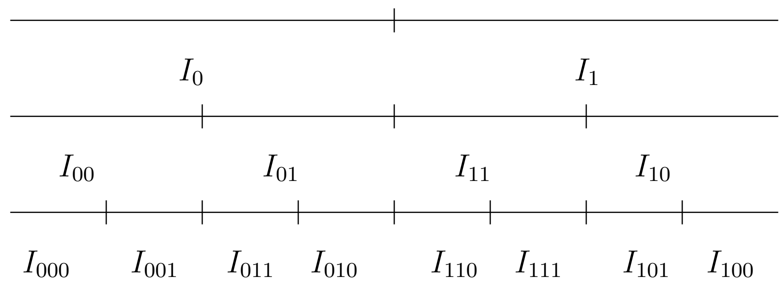
The first two levels of intervals associated with the itinerary scheme for the Tent Map.

It is important to note the difference between the labelling of the second level of the Tent Map scheme from that of the Baker Map scheme. In the Baker map scheme, the four intervals, from left to right, are indexed as I_{00}, I_{01}, I_{10} and I_{11}. There is a reversal in the final two indices of the Baker Map, which is produced by the "reversal" of the second function in the Baker Map to produce the second map of the Tent Map.

After a little work involving the graph of $T^2(x)$, we can find the the next set of subintervals. From the graph of $T^2(x)$ shown below, we see that there are four subintervals of points which are mapped to $[0, 1/2)$ after one application of $T^2(x)$ or, equivalently, two applications of $T(x)$. For these points, $b_2 = 0$. Likewise, there are four subintervals of points which are mapped to $[1/2, 1]$ after one application of $T^2(x)$ or, equivalently, two applications of $T(x)$. For these points, $b_2 = 1$.



From these results, we obtain third level of subintervals associated with the itinerary scheme for the Tent Map as shown in the figure below.



The first three levels of intervals associated with the itinerary scheme for the Tent Map.

We can now apply this method for all $n \geq 0$ to assign an itinerary sequence to each $x \in [0,1]$ under the action of the tent map, $T(x)$: For each $n \geq 0$,

- If $b_n = 0$, then $y = T^n(x) \in I_0 = [0, 1/2]$.
- If $b_n = 1$, then $y = T^n(x) \in I_1 = (1/2, 1]$.

Because of the piecewise linearity of the Tent Map, and the fact that the slopes of the "pieces" of the Tent Map have slopes ± 2, we can conclude that if the (itinerary) sequences of x and y agree to n digits, then

$$|x - y| \leq 2^{-n} \,. \tag{29}$$

Finally, the following can be established: Given an $x \in [0,1]$ with itinerary sequence,

$$b_0 b_1 b_2 \ldots \,, \quad b_i \in \{0, 1\} \,, \tag{30}$$

then for each $n \geq 0$, the point $T^n(x)$ has the associated sequence,

$$b_n b_{n+1} b_{n+2} \ldots \,, \tag{31}$$

obtained by left-shifting the sequence in (30) n times. In other words, **the Tent Map $T : [0,1] \to [0,1]$ induces the Bernoulli left-shift map S on the itinerary sequence (30) associated with x.**

The Bernoulli shift-map S on the itinerary sequence of the Tent Map will have the same properties as those of the shift-map on binary sequences of the Baker Map. On the basis of our earlier discussion of the Bernoulli shift-map on binary sequences, we can then conclude that

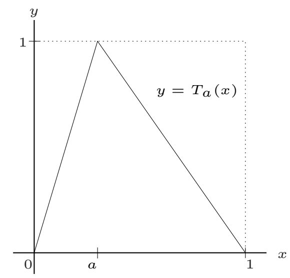
1. The periodic points of the Tent Map are dense on $[0,1]$,
2. The Tent Map has SDIC on $[0,1]$,
3. The Tent Map is transitive on $[0,1]$.

These, of course, are the three ingredients for chaotic behaviour on $[0,1]$. We therefore conclude that the **Tent Map is chaotic on** $[0,1]$..

Itinerary sequences for other maps

In closing this section, mention that the above definition of an itinerary sequence can be applied to other maps $f : [0,1] \to [0,1]$. In the case of piecewise linear maps, for example, the following family of "shifted" tent maps introduced in Problem Set. No. 3.:

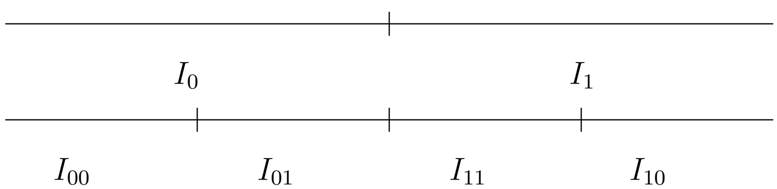
$$T_a(x) = \begin{cases} \frac{1}{a}x, & 0 \leq x \leq a, \\ \\ \frac{1}{1-a}(1-x), & a < x \leq 1\,. \end{cases} \tag{32}$$



In this case, it is natural to partition $[0,1]$ into the following two subintervals,

$$I_0 = [0,a]\,, \qquad I_1 = [a,1]\,. \tag{33}$$

(In order to ensure uniqueness of the sequences, the interval I_1 should be $(a,1)$, but we'll ignore this technicality here.) The next level of partitions will have the same indexing as that of the Tent Map, i.e.,



But the lengths of these intervals will be different than in the (unshifted) Tent Map case, because of the shifted nature of the map T_a. We leave it as an exercise for the reader to show that:

- Length of I_{00} is a^2.
- Length of I_{01} is $a(1-a)$.
- Length of I_{11} is $(1-a)^2$.
- Length of I_{10} is $a(1-a)$.

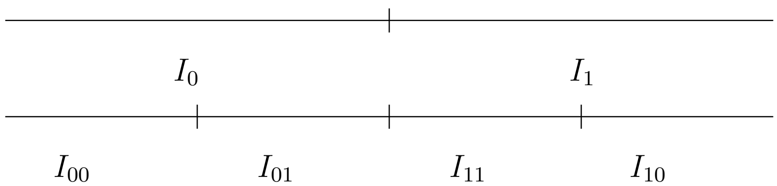
The total length of the four intervals is 1. The lengths of higher order subintervals $I_{b_0 b_1 \ldots b_n}$ can be determined in a recursive manner. (This will be a recurring theme during our study of fractals.)

The logistic map $f_4(x) = 4x(1-x)$

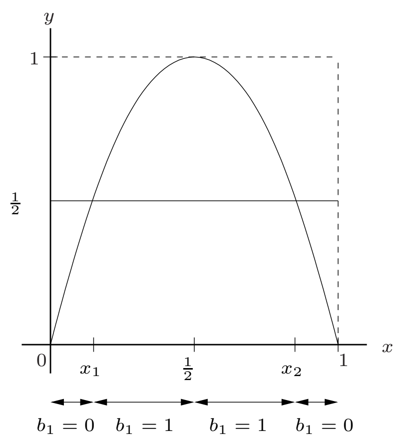
Since the chaotic logistic map, $f_4(x) = 4x(1-x)$, attains a maximum value at $x = 1/2$, we may, as in the case of the Baker and Tent Maps, use the two intervals,

$$I_0 = [0, 1/2]\,, \qquad I_1 = [1/2, 1]\,, \tag{34}$$

to define its associated itinerary sequences for points $x \in [0,1]$. Because the logistic map $f_4(x)$ has roughly the same shape as the Tent Map, the next level of partitions will have the same indexing as that of the Tent Map, i.e., The lengths of these intervals will be different, however, because of the



nonlinear nature of the map $f_4(x)$. Graphically, they can be viewed as follows,



The determination of the two additional points $x_1, x_2 \in [0,1]$ which will define the four intervals I_{00}, I_{01}, I_{11} and I_{10} is left as an exercise for the reader.

We conclude with the statement that for any $x \in [0,1]$, an itinerary sequence $\mathbf{b} \in \Sigma_2$, i.e.,

$$\mathbf{b} = (b_0, b_1, b_2, \cdots)\,, \qquad b_i \in \{0, 1\}\,, \tag{35}$$

can be defined using the above scheme. The result is that the action of the map f_4 on a point x induces a Bernoulli left-shift map S on its itinerary sequence $\mathbf{b}$. From the properties of the Bernoulli shift map on sequences in Σ_2, **we can conclude that the logistic map $f_4(x)$ is chaotic on [0,1]**.

Lecture 23

Chaotic dynamics (cont'd)

Spatial distribution of chaotic orbits, "visitation frequencies", invariant measures

Note: The material in this lecture is presented for information only, i.e., you will not be examined on this material. That being said, we may revisit these ideas in later sections of the course. Some portions of the notes below contain details which were not discussed in the lecture.

Suppose that we are told that a function $f : I \to I$ demonstrates chaotic behaviour on I, that is, for "almost all" seed points $x_0 \in I$, the orbit of x_0, given by the iteration procedure,

$$x_{n+1} = f(x_n)\,, \quad n \geq 0\,, \tag{36}$$

exhibits seemingly random behaviour when the x_n are plotted vs. n. Moreover, the orbit of x_0 is dense on I: Given any point $a \in I$, and any neighbourhood N_δ of a, i.e., the interval $(a - \delta, a + \delta)$, there is an element x_n to be found in N_δ.

If this were all the information that could be obtained from the iterates $\{x_n\}$ defined in (36), then there wouldn't seem to be a way to tell whether a chaotic sequence was generated from a function $f : I \to I$ or another function, say, $g : I \to I$. In other words, we wouldn't be able to tell the difference between a chaotic orbit generated by the iteration of the Tent Map and a chaotic orbit generated by iteration of the logistic map $f_4(x)$.

In this section, we show very briefly that there are differences between the two sets of chaotic orbits, even though they are both **dense** on I. **To see these differences, we look at how the iterates $\{x_n\}$ are distributed over the interval I.**

For example: Do the iterates $\{x_n\}$ tend to be spread out rather evenly over the interval I? Or are they somewhat "concentrated" at some parts of I and less concentrated at other parts. This question has been at the heart of an enormous amount of research in dynamical systems theory over the past half-century and more. Here we provide a small glimpse into the subject.

There is a relatively simple (apart from some possible problems due to finite precision) numerical procedure to visualize how the iterates x_n defined in (36) are distributed over the interval I. In what follows, we let $I = [a, b]$. (Typically, $I = [0, 1]$, but we'll keep the discussion general.)

Before we describe the algorithm in detail, let's mention very briefly the main idea. First, we divide the interval I into N nonoverlapping subintervals I_k of equal length. Then, we generate a chaotic sequence of iterates $\{x_n\}$, $1 \le n \le M$, according to Eq. (36) above. For each iterate x_n, we determine the subinterval I_k in which it lies. This allows us to count how many of the M iterates x_n lie in each subinterval I_k – call this number c_k. We then examine the numbers c_k, $1 \le k \le N$, first by plotting them vs. k.

Step No. 1: For an N relatively large (say 1000-10000), form a partition of the interval $[a, b]$ in the manner done in first-year Calculus, i.e., let

$$\Delta t = \frac{b-a}{N}\,, \tag{37}$$

and define

$$t_k = a + k\,\Delta t\,, \qquad 0 \le k \le N\,, \tag{38}$$

so that $t_0 = a$ and $t_N = b$.

Note: We are using the notations t_k and Δt instead of the usual x_k and Δx so that the **partition points** t_k will not be confused with the **iterates** x_k.

Step No. 2: This partition will define a set of N subintervals of I,

$$I_k = [t_{k-1}, t_k]\,, \quad 1 \le k \le N\,. \tag{39}$$

For reasons that will become clear below, define the following set of half-open intervals,

$$J_k = [t_{k-1}, t_k)\,, \quad 1 \le k \le N-1\,, \tag{40}$$

along with the final interval,

$$J_N = [t_{N-1}, t_N]\,. \tag{41}$$

In the special case that $I[0, 1]$, for which $a = 0$ and $b = 1$,

$$\Delta t = \frac{1}{N}\,, \tag{42}$$

and

$$t_k = k\Delta t = \frac{k}{N}\,, \tag{43}$$

with $t_0 = 0$ and $t_1 = 1$.

Step No. 3: Initialize a "counting vector," – call it $\mathbf{c}$, with N elements, so that

$$c_k = 0\,, \quad 1 \le k \le N\,. \tag{44}$$

The entries of $\mathbf{c}$ will be integers.

Step No. 4: Now choose a "good" seed point $x_0 \in I$, i.e., a point that is not preperiodic (or at least hopefully not preperiodic). Start computing the elements $\{x_n\}$ of the forward orbit of x_0 using (36), i.e.,

$$x_{n+1} = f(x_n)\,, \quad n \geq 0\,. \tag{45}$$

This will involve some kind of "loop" in your computer program. After you have computed each iterate x_n, determine the particular interval J_k, in which x_n lies. This can be done in the following way (or some slight modification of it):

$$k = \text{int}\left[\frac{1}{\Delta t}(x_n - a)\right] + 1\,, \tag{46}$$

where, for a $y \in \mathbb{R}$,

$$\text{int}[y] = \text{integer part of } y\,. \tag{47}$$

Rationale: If x_n lies in J_k, then

$$\begin{aligned} t_{k-1} \leq x_n < t_k \quad &\Longrightarrow \quad a + (k-1)\Delta t \leq x_n < a + k\Delta t \\ &\Longrightarrow \quad (k-1)\Delta t \leq x_n - a < k\Delta t \\ &\Longrightarrow \quad (k-1) \leq \frac{1}{\Delta t}(x_n - a) < k \end{aligned} \tag{48}$$

This implies that

$$k - 1 = \text{int}\left[\frac{1}{\Delta t}(x_n - a)\right]\,, \tag{49}$$

which yields (47).

After determining "k", the index of the interval I_k in which x_n lies, increase the appropriate entry of $\mathbf{c}$ by one, i.e.,

$$c_k = c_k + 1\,. \tag{50}$$

Step No. 5: Perform the iteration procedure in (45) for a sufficiently large number M of times, say $M = 50,000$, or, better yet, $M = 100,000$, or even $M = 10^6$. The larger the better: These computations do not take a lot of time.

At the end of the computation, you will have produced an N-vector, $\mathbf{c}$. Hopefully, some, if not all, of its entries c_k will be nonzero.

Question: What is this vector $\mathbf{c}$?

Answer: Each element c_k of this vector for $1 \leq k \leq N$, has recorded **the number of times that the interval** $I_k = [t_{k-1}, t_k)$ **has been visited** by an iterate x_n over the orbit $1 \leq n \leq M$.

If you now plot the values of the elements c_k vs. k, you will get an idea of how the iterates x_n are distributed over the interval I. Intervals I_k with higher numbers of visitation will have higher counts c_k.

In order to be able to compare the results of this counting experiment for different choices of M, the total number of iterates computed, it is convenient to **normalize** the count vector $\mathbf{c}$ by defining the following N-vector $\mathbf{p}$, the elements of which will not be integers, but fractions:

$$p_k = \frac{c_k}{M} \quad 1 \leq k \leq N \,. \tag{51}$$

Technically, we should write $p_k(M)$, since our values of p_k will depend on M, but we leave the M out for the moment. Note that $0 \leq p_k \leq 1$ for $1 \leq k \leq N$, and

$$\sum_{k=1}^{N} p_k = 1 \,. \tag{52}$$

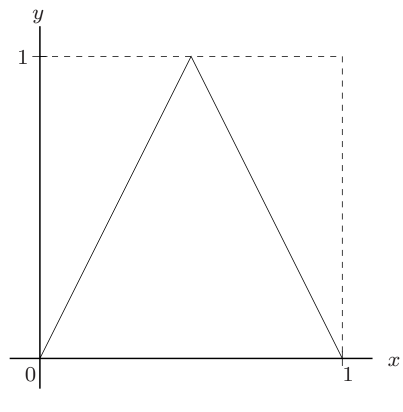
Each element p_k, $1 \leq k \leq N$, may be interpreted in at least two ways, which are not unrelated:

1. p_k is the **fraction of the iterates** $\{x_n\}_{n=1}^{M}$ that have visited interval J_k. If we view the set of iterates $\{x_n\}_{n=1}^{M}$ as a huge collection of numbers between 0 and 1, the p_k indicate how they are distributed in the N bins J_k, $1 \leq k \leq N$. This is, of course, a discrete approximation to how they are distributed over $[0, 1]$.

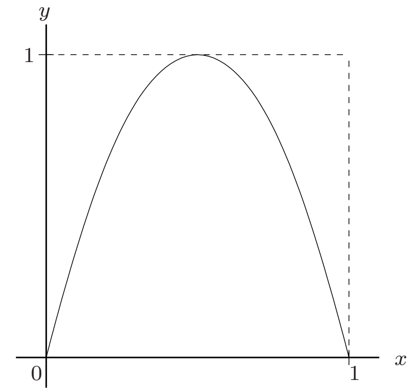
2. p_k is the **visitation frequency** of interval J_k – or at least an approximation of it – by the iterates $\{x_n\}_{n=1}^{M}$. This has a probabilistic interpretation: For each $n > 0$, we may view p_k as the **probability** that iterate x_n will be found in interval J_k.

The term "visitation frequency" sounds "statistical" and suggests that the results of our computations are approximations to a true "visitation frequency" that is obtained by letting the number of iterations $M \to \infty$.

Before going on, let us examine the results of a few computations for two chaotic maps on $[0, 1]$ that we have studied to date: (i) the Tent Map and (ii) the logistic map $f_4(x)$, the graphs of which are once again plotted below:



Tent Map $T(x)$.



Logistic map $f_4(x) = 4x(1-x)$.

In the plots on the next page are shown plots of the elements, p_k, $1 \leq k \leq M$, of the vector $\mathbf{p}$ obtained when the interval $[0,1]$ is divided into $N = 1000$ subintervals and $M = 2 \times 10^6$ iterates are used. The most noticeable difference between the two plots is that one (the tent map $T(x)$) is quite "flat" compared to the other (logistic-4 map $f_4(x)$).

With regard to the tent map case, note that the value of each p_k is roughly 0.001, i.e.,

$$p_k \approx 0.001 = \frac{1}{N}\,. \tag{53}$$

This is consistent with Eq. (52) and indicates that the iterates are visiting the entire interval [0,1] in a quite uniform manner.

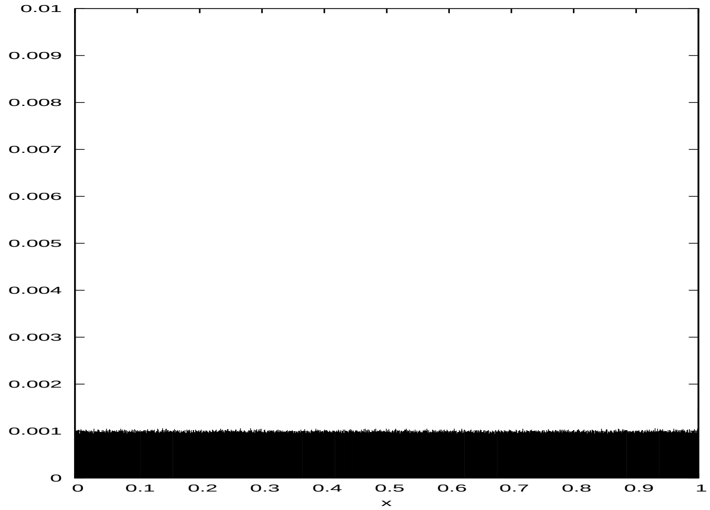
We might not be so surprised to see that the distribution associated with the tent tap is "flat". After all, the tent map is piecewise linear, i.e., the "pieces" are straight lines, as oppposed to the logistic function $f_4(x)$, which is "curved." This indeed has something to do with the flatness of the tent map case, and we'll "prove" that the distribution is flat, i.e., uniform.

The question then remains, "Why does the distribution for the logistic-4 map curl upwards near the ends of the interval, implying that these outer regions are visited more frequently than the inner region around $x = \frac{1}{2}$?" An explanation is now to be provided.

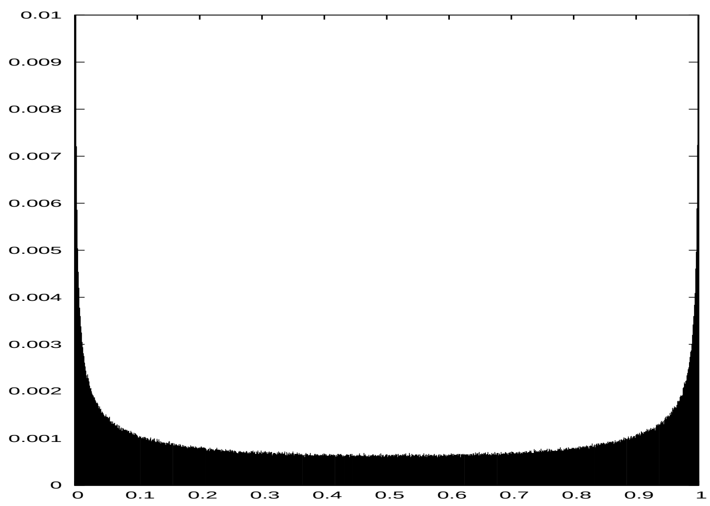
In order to understand the shapes of these visitation frequency plots, we shall have to make use of a kind of "conservation law", that acknowledges the 2-to-1 nature of the tent and logistic-4 maps. (The idea extends in a straightforward manner to other "many-to-one" maps, e.g., 3-to-1 maps.)

Consider an interval $[a,b] \subset [0,1]$. Instead of being concerned where points from $[a,b]$ are **going** under the action of a map f (tent or logistic-4), we are going to be concerned about what points from [0,1] are **coming** to $[a,b]$ under the action of f. It is therefore convenient to place the interval $[0,1]$ on the y-axis for each of the two plots of the graphs of $T(x)$ and $f_4(x)$, as shown below.

Approximations to visitation frequencies for two chaotic maps on [0,1]

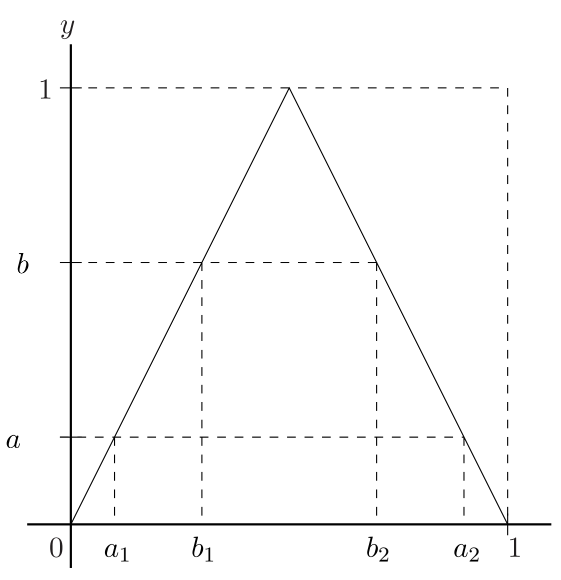


Tent Map $T(x)$

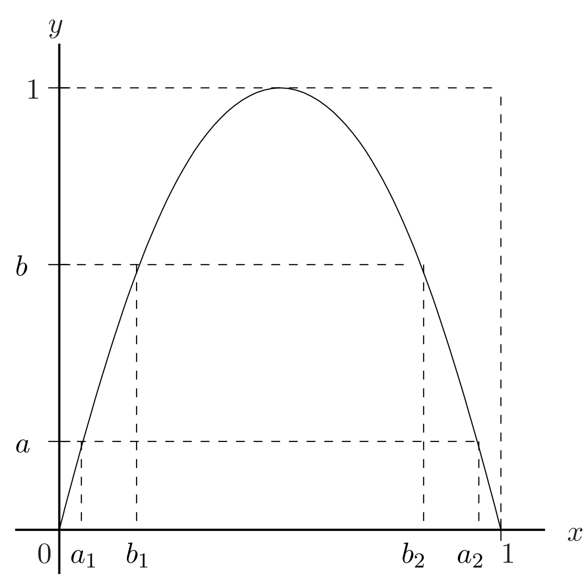


Logistic map $f_4(x) = 4x(1-x)$

In both cases, $M = 1000$ bins and $N = 2 \times 10^6$ iterates were used.



Tent Map $T(x)$.



Logistic map $f_4(x) = 4x(1-x)$.

In each figure are shown the two "preimages" of the interval $[a, b]$ under the action of the map concerned:

- Tent map: $T([a_1, b_1]) = [a, b]$ and $T([b_2, a_2]) = [a, b]$.
- Logistic-4 map: $f_4([a_1, b_1]) = [a, b]$ and $f_4([b_2, a_2]) = [a, b]$.

Note that for both maps, the interval $[b_2, a_2]$ is "flipped," due to the decreasing nature of the map for $\frac{1}{2} < x \leq 1$.

We now develop our "conservation law," which is essentially a law of probabilities. We'll be working in the discrete framework introduced earlier, that is, dividing the interval $[0, 1]$ into N subintervals J_k. Associated with each interval J_k is a probability p_k that the iterate will visit it during an orbit of length M. We'll also assume that N is sufficiently (enormously?) large so that the discrete approximations that we are employing are sufficiently accurate. We'll also assume that the length M of the orbit is large/enormous, essentially approaching the limit $M \to \infty$. Later, we shall actually let N, the number of subintervals, go to infinity in order to arrive at a continuous description of the visitation frequency which will then employ the differential dx instead of the bin width $\Delta x = \Delta t$.

In what follows, we shall, for the most part, adopt the first of the two interpretations of the quantities p_k, $1 \leq k \leq N$, defined in Eq. (51), that is, that each p_k is the **fraction of iterates** $\{x_n\}_{n=1}^{M}$ found in interval J_k. We use this interpretation to adopt the next set of assumptions:

1. Let $K \subseteq [0, 1]$ be an interval, and suppose that K is the union of a number of subintervals J_k, i.e.,

$$K = \bigcup_{k=k_1}^{k_2} J_k \,. \tag{54}$$

   This is equivalent to the statement,

$$K = [t_{n_1-1}, t_{n_2}] \,, \tag{55}$$

   where the t_k are the partition points defined earlier. Then the **fraction of iterates** $\{x_n\}_{n=1}^{M}$ that have visited interval K, to be denoted as $F(K)$ is given by

$$F(K) = \sum_{k=n_1}^{n_2} p_k \,. \tag{56}$$

   Special case: When $K = [0, 1]$, then $n_1 = 1$ and $n_2 = N$ so that the sum in (56) is 1, as it should be: The fraction of all iterates that lie in [0,1] is 1, since all iterates $x_n \in [0, 1]$.

   Note: Eq. (56) is often written in the more convenient form,

$$F(K) = \sum_{k}{}' p_k \,, \tag{57}$$

where the prime on the summation indicates that the summation is over only those $k \in \{1, 2, \cdots N\}$ for which $J_k \subseteq K$. Or, to remove any confusion, we could write,

$$F(K) = \sum_{\{k | J_k \subseteq K\}} p_k \,. \tag{58}$$

2. **This one is very important!** We shall consider the graphs of the Tent Map and the Logistic-4 Map on the previous page. **In each case, imagine that the iterates x_n that are generated by the repeated application of each map are distributed not only over [0,1] on x-axis but also [0,1] on the y-axis.** This will help in understanding the concepts that follow.

   We claim that the fraction of iterates that lie in the interval $[a, b]$ is equal to the sum of the fractions of iterates lying in $[a_1, b_1]$ and $[b_2, a_2]$. Mathematically,

$$F([a, b]) = F([a_1, b_1]) + F([b_2, a_2]) \,. \tag{59}$$

   This is a kind of **conservation of iterates** (which is essentially a conservation of mass).

   The iterates in $[a, b]$ come from the two intervals $[a_1, b_1]$ and $[b_2, a_2]$ under the action of the Tent or logistic-4 map. If we have arrived at a kind of stationary distribution that tells us how the iterates are distributed over the intervals, then the above conservation law has to hold. Later, we shall state this law mathematically.

   Note: We have relied on the assumption that Δt, the length of the subintervals J_k, is sufficiently small so that the intervals involved above, i.e., $[a, b]$, $[a_1, b_1]$ and $[b_2, a_1]$, can be expressed – or at least well approximated – as unions of the subintervals J_k, i.e., no subintervals J_k have been "split". In the limit $N \to \infty$, these approximations will become exact and Eq. (59) is valid without the use of the subintervals J_k.

We are now in a position to argue – not prove, but at least understand – why the distribution for the Tent Map is "uniform," i.e., all of the probabilites p_k are constant and equal $\frac{1}{N}$. It is, indeed, because of the fact that the two components of $T(x)$ are straight as well as having slopes of equal magnitude, namely, 2. By simple geometry, the lengths of the intervals $[a_1, b_1]$ and $[b_2, a_2]$ are equal and one-half the length of the interval $[a, b]$.

It should be fairly easy to see that if the conservation equation in (59) holds for **any** interval $[a, b] \subseteq [0, 1]$, then the p_k should all be equal, i.e.,

$$p_k = \frac{1}{N} \,, \quad 1 \le k \le N \,. \tag{60}$$

This is often referred to as a **uniform distribution**.

And what about the distribution associated with the Logistic-4 map? Why does it increase as we approach 0 and 1? Very loosely speaking, because the magnitude of the tangents to the graph of $f_4(x)$, i.e., $|f'(x)|$, increases as $x \to 0^+$ and $x \to 1^-$ and decreases as $x \to \frac{1}{2}$

Looking at the graph of $f_4(x)$ with the interval $[a, b]$ on the y-axis and its preimages, $[a_1, b_1]$ and $[b_2, a_1]$, note that the lengths of these intervals are shorter than one-half the length of $[a, b]$. This is due to the fact that the magnitudes of the tangents to the graph of $f_4(x)$, i.e., $|f'(x)|$ increase as $x \to 0^+$ and $x \to 1^{-1}$, approaching the value of 4 in the limits. As $[a, b]$ is moved downward toward the origin, the lengths of these two preimages gets even smaller. Loosely speaking (or writing), in order for the conservation equation in (59) to hold, the fractions of the iterates on these two intervals has to be greater than the uniform distribution in (60).

Admittedly, these are very "loose" or "heuristic" descriptions of why we expect the shape of the distributions of the p_k fractions to be what they are. Let us now go to a more mathematical description.

In order to do so, let us first consider the limit $M \to \infty$, where M is the number of iterates. As M increases, the number of iterates falling in the interval J_k, which we have called c_k, will generally increase with M. In fact, we should denote this number of iterates as $c_k(M)$. And we should once again acknowledge that the fractions p_k are also functions of M and write

$$p_k(M) = \frac{c_k(M)}{M}\,. \tag{61}$$

We now claim that the following limits exist,

$$\lim_{M\to\infty} p_k(M) = \lim_{M\to\infty} \frac{c_k(M)}{M} = p_k\,, \quad 1 \le k \le N\,. \tag{62}$$

Then p_k is the **limiting fraction of iterates** that are found in subinterval J_k as we let the number M of iterates go to infinity. Once again,

$$\sum_{k=1}^{N} p_k = 1\,. \tag{63}$$

We now consider the p_k to define a **piecewise constant** function $P(x)$ on $[0, 1]$:

$$P_N(x) = p_k \;\text{ if }\; x \in J_k\,, \quad 1 \le k \le N\,. \tag{64}$$

The subscript N reminds us that N subintervals J_k are used in the construction of this function.

Now we do something that will seem quite strange: We define a new function $\rho_N(x)$ as follows,

$$\rho_N(x) = \frac{1}{\Delta t} P_N(x) = \frac{p_k}{\Delta t} \qquad \text{if }\; x \in J_k\,, 1 \le k \le N\,. \tag{65}$$

This must seem very strange, indeed, since Δt, the size of the subintervals I_k, is very small, and since we eventually wish to take the limit $N \to \infty$, which implies that $\Delta t \to 0$. But as N, the number of intervals, increases, and Δt decreases, each p_k decreases – there are more intervals in which to find the iterates! The reason we define $\rho_N(x)$ in Eq. (65) is that the fraction $F(J_k)$ of iterates found in subinterval J_k is now given by

$$F(J_k) = p_k = \rho_N(t_k^*)\Delta\, x\,, \quad t_k^* \in J_k\,, \tag{66}$$

where we have now set $\Delta x = \Delta t$ (just to keep everything in terms of x). Note that the sample point t_k^* can be any point in J_k, since $F(x)$ is constant over each subinterval J_k.

Note: Is this starting to look like something from first-year Calculus, i.e., Riemann integration?

> **Another note:** We may view the piecewise constant function $\rho_N(x)$ in Eq. (65) as a **density function**, i.e., **the (normalized) number of iterates per unit length**. We write "normalized" since the total number of "normalized" iterates over the entire interval $[a, b]$ is 1, i.e.,
>
> $$\sum_{k=1}^{N} p_k = 1\,. \tag{67}$$
>
> In this way, one could think of the iterates as representing electric charges, in which case $\rho_N(x)$, $x \in J_k$, is the **lineal charge density** (charge per unit length) over the interval J_k.

We now perform the limiting operation $N \to \infty$. In this limit, Δx, the length of the subintervals J_k, will go to zero. The summation over these subintervals of length Δx will become an integration over the differential dx. We claim that in the limit $N \to \infty$, the piecewise constant functions $\rho_N(x)$ converge to a function $\rho(x)$, for $x \in [0, 1]$. For any subinterval $[a, b] \subseteq [0, 1]$, the limiting fraction of iterates in $[a, b]$ is **no longer a summation over all subintervals** J_k lying in $[a, b]$ as done in Eq. (56) **but rather an integration** over the interval $[a, b]$, i.e.,

$$F([a, b]) = \int_a^b \rho(x)\, dx\,. \tag{68}$$

We have arrived at a continuous description of the fractional distribution of iterates over the interval $[0, 1]$. Note that the function $\rho(x)$ is a normalized distribution since

$$F([0, 1]) = \int_0^1 \rho(x)\, dx = 1\,. \tag{69}$$

Eq. (68) leads to the following continuous version of the conservation equation in (59): For any $[a, b] \subseteq K$,

$$\int_a^b \rho(x)\, dx = \int_{a_1}^{b_1} \rho(x)\, dx + \int_{b_2}^{a_2} \rho(x)\, dx\,. \tag{70}$$

We are now going to state this conservation result more generally as well as mathematically. In what follows, we let I denote an interval on which a function $f : I \to I$ is defined. f may or may not be chaotic. For any subset $S \subset I$, we define the following set,

$$f^{-1}(S) = \{x \in I\,, f(x) \in S\}\,. \tag{71}$$

In other words, $f^{-1}(S)$ is the set of all points in I that are mapped by f to the set S. In the case of each of the Tent and Logistic-4 maps, S is the interval $[a,b] \subset [0,1]$ and

$$f^{-1}([a,b]) = [a_1, b_1] \cup [b_2, a_2] , \tag{72}$$

where the a_i and b_i depend on the maps. The above relation is true because

$$f([a_1, b_1]) = f([b_2, a_2]) = [a,b] . \tag{73}$$

Definition: Let I be an interval and $f : I \to I$. If there exists a function $\rho : I \to \mathbb{R}$ such that for all $S \subseteq I$,

$$\int_S \rho(x)\,dx = \int_{f^{-1}(S)} \rho(x)\,dx , \tag{74}$$

then ρ is said to the be **invariant (probability or normalized) density function** which defines the **invariant measure** associated with the mapping $f : I \to I$.

Note: The notation $\int_{f^{-1}(S)}$ implies an integration over the entire set $f^{-1}(S)$ defined earlier. Eq. (70) stated earlier,

$$\int_a^b \rho(x)\,dx = \int_{a_1}^{b_1} \rho(x)\,dx + \int_{b_2}^{a_2} \rho(x)\,dx , \tag{75}$$

is a special case of Eq. (74).

Note: The reason for the term "invariant measure," is that the density function ρ is considered to define a "measure" of subsets $S \subset I$ – a generalized notion of "length", a kind of "weighted length". Regions of I that have higher ρ-values, i.e., fractions of iterates, are weighted more heavily than those regions with lower ρ-values. Usually, the invariant measure associated with a dynamical system $f : I \to I$ is denoted as "μ". The invariant measure, or "μ-measure" of an interval $[a,b]$ is given by

$$\mu([a,b]) = \int_a^b \phi(x)\,dx , \tag{76}$$

which, as we saw earlier, is the fraction of iterates in the interval $[a,b]$. The conservation relation in (74) may be expressed as follows,

$$\mu(S) = \mu(f^{-1}(S)) \quad \text{for all} \quad S \in I . \tag{77}$$

The invariant density functions for the Tent and Logistic-4 maps

Tent Map $T(x)$

Here we simply state that, as expected, the invariant density function $\rho(x)$ for the Tent Map on $[0,1]$ is a constant function – no regions have a higher fraction of iterates than others. In the case that $I = [0,1]$,

$$\rho(x) = 1 , \quad x \in [0,1] , \tag{78}$$

Referring to the earlier figure which shows that graph of the Tent Map function $T(x)$ along with the interval $[a,b]$ and its two preimages, the conservation equation in (70) becomes

$$\int_a^b dx = \int_{a_1}^{b_1} dx + \int_{b_2}^{a_2} dx\,. \tag{79}$$

Let us finally state explicitly what the a_i and b_i are:

$$a_1 = \frac{1}{2}a\,, \quad b_1 = \frac{1}{2}b\,, \tag{80}$$

and

$$b_2 = 1 - \frac{1}{2}b\,, \quad a_2 = 1 - \frac{1}{2}a\,. \tag{81}$$

The integrals in (79) are, of course, simple to evaluate:

$$\begin{aligned} b - a &= (b_1 - a_1) + (a_2 - b_2) \\ &= \frac{1}{2}(b-a) + \frac{1}{2}(b-a) \\ &= b - a\,. \end{aligned} \tag{82}$$

which is satisfied for all $[a,b] \in [0,1]$.

The invariant measure μ defined by the density function ρ is

$$\begin{aligned} \mu([a,b]) &= \int_a^b \phi(x)\,dx \\ &= \int_a^b dx \\ &= b - a\,. \end{aligned} \tag{83}$$

In this case, the μ-measure of the interval $[a,b]$ is the length of the interval, the usual notion of the "size" of an interval. This is somewhat of a coincidence since the interval I on which the Tent Map $T(x)$ is defined is $[0,1]$. If we were to consider a tent map on $I = [0,2]$, then the constant density function ρ would be

$$\rho(x) = \frac{1}{2}\,, \tag{84}$$

so that

$$\int_I \rho(x)\,dx = \int_0^2 \frac{1}{2}\,dx = 1\,. \tag{85}$$

When the density function $\rho(x)$ is constant, the measure defined by ρ is often called **uniform measure**.

Logistic-4 map $f_4(x) = 4x(1-x)$

Referring to the earlier figure which shows the graph of the logistic-4 function $f_4(x)$ along with the interval $[a, b]$ and its two preimages, let us rewrite the conservation equation in (70) that would have to be solved by the invariant density function $\rho(x)$ associated with the $f_4(x)$ map.

$$\int_a^b \rho(x)\,dx = \int_{a_1}^{b_1} \rho(x)\,dx + \int_{b_2}^{a_2} \rho(x)\,dx\,. \tag{86}$$

Here we also state explicitly what the a_i and b_i are:

$$a_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1-a}\,, \quad b_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1-b}\,, \tag{87}$$

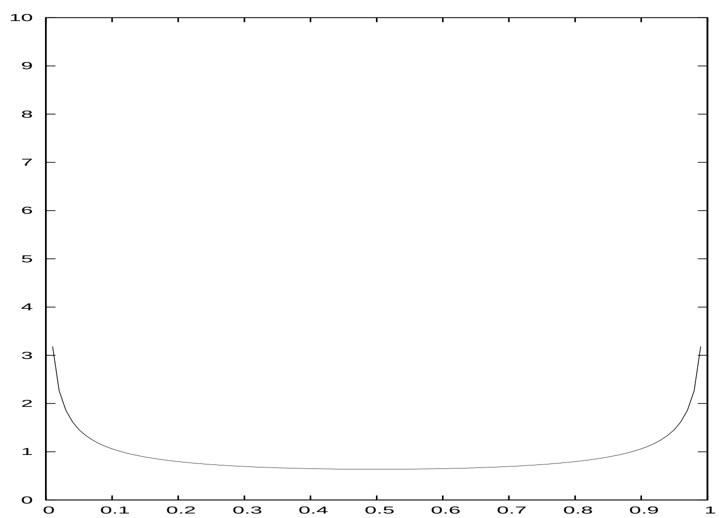
and

$$b_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-b}\,, \quad a_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-a}\,, \tag{88}$$

We now state a remarkable result – the density function $\rho(x)$ satisfying (86) with the a_i and b_i defined above, is known analytically:

$$\rho(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\,. \tag{89}$$

The graph of $\rho(x)$, presented in the figure below, demonstrates a strong similarity to the distribution of iterates of the logistic-4 map obtained numerically and presented earlier.



Plot of density function $\rho(x) = \dfrac{1}{\pi}\dfrac{1}{\sqrt{x(1-x)}}$, $x \in [0, 1]$, for logistic-4 function $f_a(x) = 4x(1-x)$.

Even though $\rho(x) \to \infty$ as $x \to 0^+$ and $x \to 1^-$, it is integrable:

$$\int_0^1 \rho(x)\,dx = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = 1\,. \tag{90}$$

The fact that the function $\rho(x)$ satisfies the conservation equation in (86) can be verified after a generous amount of Calculus, starting with the result (left as an exercise) that for $0 \leq a \leq b \leq 1$,

$$\begin{aligned} \int_a^b \rho(x)\,dx &= \frac{1}{\pi}\int_a^b \frac{1}{\sqrt{x(1-x)}}\,dx \\ &= \frac{1}{\pi}\left[\mathrm{Sin}^{-1}(2b-1) - \mathrm{Sin}^{-1}(2a-1)\right]. \end{aligned} \tag{91}$$

In the special case $a = 0$ and $b = 1$, the above result yields,

$$\begin{aligned} \int_0^1 \rho(x)\,dx &= \frac{1}{\pi}\left[\mathrm{Sin}^{-1}(1) - \mathrm{Sin}^{-1}(-1)\right] \\ &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right] \\ &= 1. \end{aligned} \tag{92}$$

This has been a very short introduction to the subject of dynamical systems and invariant measures – very little could be done which, of course, means that much has been omitted. But it was intended to be a starting point for anyone who is interested in pursuing the subject further.

One final note: The existence of the density function $\rho(x)$ in Eq. (74) is not always guaranteed. But an invariant measure μ generally exists. The complication is that the measure μ is a **measure**, and measures can be quite "irregular". They can include things such as "Dirac delta functions," i.e., "point masses", which cannot be modelled with "normal functions". This will certainly be the case when we study measures on fractal sets – at least lightly.