Lecture 18

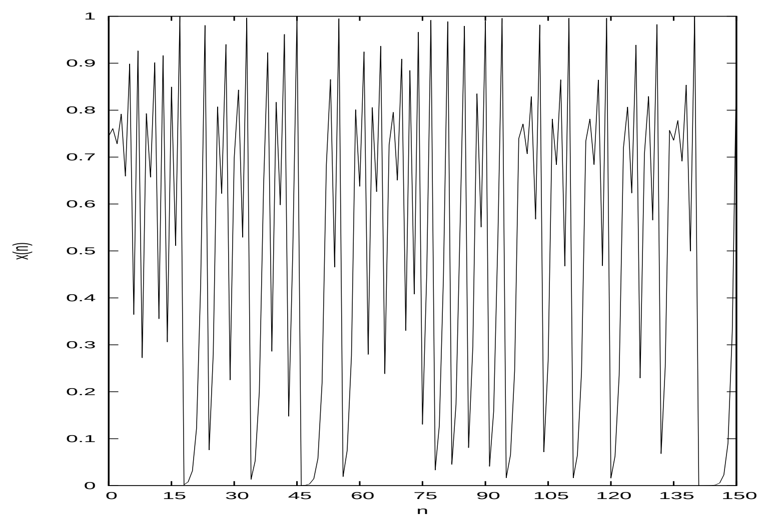
Dynamics of the Logistic Map $f_a(x) = ax(1-x)$ (cont'd)

The Case $a = 4$: "Chaotic" Behaviour on [0,1]

The case $a = 4$ represents a rather special dynamical system with very interesting behaviour. For "almost all" (we'll explain this term later) initial points $x_0 \in (0,1)$, the behaviour of the iterates given by

$$x_{n+1} = 4x_n(1-x_n) \quad n = 0, 1, 2, \cdots, \tag{1}$$

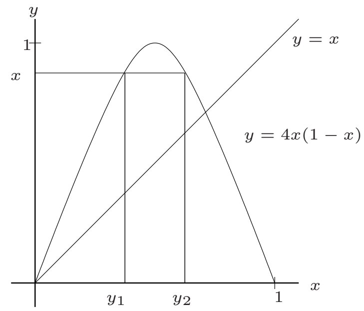
demonstrate seemingly random behaviour as shown in the figure below. (The initial condition employed in this computation was $x_0 = 0.333$.)



A plot of the iterates x_n for $0 \le n \le 150$ produced by Eq. (1) using the initial condition $x_0 = 0.333$.

The behaviour of these iterates "seems" to be random, but it is not, since the x_n were computed **deterministically** – no random numbers were involved in the iteration process. Such behaviour is said to be **chaotic**. In what follows, we shall provide some basic properties of the map $f_4(x)$ that are responsible for this chaotic behaviour.

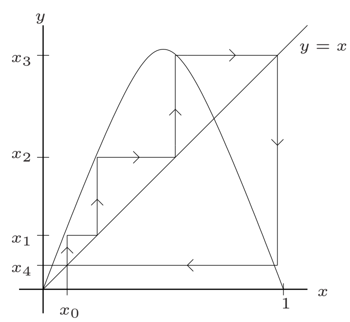
The map $f_4(x) = 4x(1-x)$ now maps $[0,1]$ **onto** itself and is **two-to-one**: Range (f_4) = Domain $(f_4) = [0,1]$. Every point $x \in [0,1]$ has two "preimages" y_1, y_2 such that $f(y_1) = f(y_2) = x$: See diagram below. In the special case $x = 1$, we consider the two preimages to be equal, i.e. $y_1 = y_2 = \frac{1}{2}$.



Graph of $f_4(x) = 4x(1-x)$ showing the two preimages y_1, y_2 of a point $x \in [0,1]$.

Recall that the fixed point $\bar{x}_2 = \frac{3}{4}$ is repulsive.

The fact that the graph of $f_4(x)$ begins at $(0,0)$, stretches up to $\left(\frac{1}{2}, 1\right)$ and comes down to $(1,0)$ accounts for some very rich behaviour of iteration sequences $x_{n+1} = 4x_n(1-x_n)$. It is possible to find points x close to the repulsive fixed point $\bar{x}_1 = 0$ that, after a finite number of iterations, come close again to $\bar{x}_1$ – a sample trajectory is sketched below.



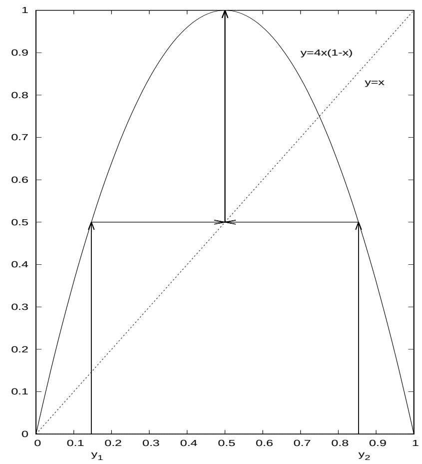
By "sliding" x_0 closer to $\bar{x}_1 = 0$, it is possible to bring the iterate x_k (it may take more than four iterations) closer to $\bar{x}_1 = 0$.

In fact, there is a great deal of "regularity" associated with this map, as we now show by examining higher order periodic orbits of f_4. First consider the map $g(x) = f_4(f_4(x)) = f_4^2(x)$. The fixed points of g will include the fixed points of f_4 as well as any possible two-cycles (p_1, p_2). We know that two-cycles exist since $a > 3$. The following features extracted from the graph of $f_4(x)$ will allow us to sketch the graph of $f_4^2(x)$:

1) $f_4(0) = 0 \Longrightarrow f_4^2(0) = 0$; $\quad f_4(1) = 0 \Longrightarrow f_4^2(1) = 0$

2) $f_4\left(\frac{1}{2}\right) = 1$ and $f_4(1) = 0 \Longrightarrow f_4^2\left(\frac{1}{2}\right) = 0$

3) Let y_1 and y_2 be the preimages of $x = \frac{1}{2}$, i.e. $f_4(y_1) = f_4(y_2) = \frac{1}{2}$, as shown in the figure below. (It is not difficult to show that $y_1 = \frac{1}{2} - \frac{\sqrt{2}}{4} \cong 0.146$ and $y_2 = \frac{1}{2} + \frac{\sqrt{2}}{4} \cong 0.854$.) Since $f_4\left(\frac{1}{2}\right) = 1$, it follows that $f_4^2(y_1) = f_4(f_4(y_1)) = f_4(\frac{1}{2}) = 1$ and $f_4^2(y_2) = f_4(f_4(y_2)) = f_4(\frac{1}{2}) = 1$.

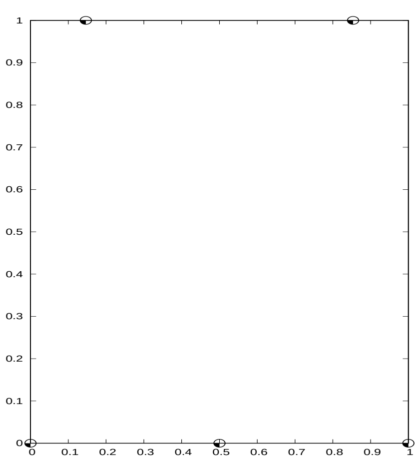


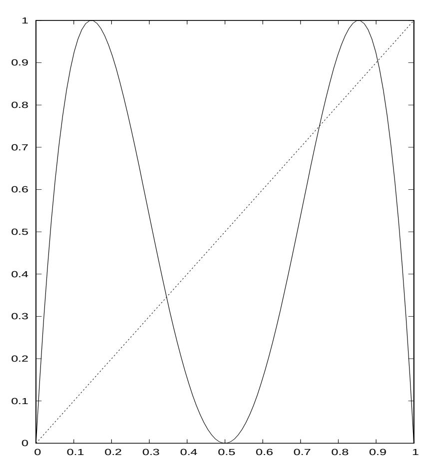
To summarize, we know five points on the graph of the function $f_4^2(x)$:

$$(0,0), \quad (y_1,1), \quad (\frac{1}{2},0), \quad (y_2,1), \quad (1,0)\,. \tag{2}$$

These five points are plotted in the left figure below.

Now we know that the function $f_4(x)$ is continuous on $[0,1]$. Moreover, it is monotonically increasing on $[0,\frac{1}{2}]$ and monotonically decreasing on $[\frac{1}{2},1]$. From the graph of $f_4(x)$ shown above, we see that as we move from $x=0$ to $x=y_1$, the value of $f_4^2(x)=f_4(f_4(x))$ increases continuously from the value 0 to the value 1. As we move from $x=y_1$ to $x=\frac{1}{2}$, the value of $f_4^2(x)=f_4(f_4(x))$ decreases continuously from the value 1 to the value 0. As such, we can connect the five points listed earlier – but using the actual values of $f_4^2(x)$ (as opposed to straight lines) to produce the following sketch of the graph of $f_2^2(x)$.





We see that the graph of $f_4^2(x)$ consists of two copies of $f_4(x)$ that are contracted horizontally in order to fit over the subintervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$.

The graph of $f_4^2(x)$ is shown again on the next page – top left. Note that f_4^2 has four fixed points: the two fixed points of $f_4(x)$, $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{3}{4}$, and the two cycle $(p_1, p_2) \cong (0.345, 0.904)$. Thus, there are **four** periodic points of period two. (Remember that a fixed point may be considered as a periodic point of any period $n \geq 1$.)

Let us now consider another iteration of $f_4(x)$: the graph of $g(x) = f_4^3(x)$. Once again, we use information from the graph of $f_4(x)$:

1) $f_4(0) = 0 \Longrightarrow f_4^3(0) = 0$; $f_4(1) = 0 \Longrightarrow f_4^2(1) = 0$

2) $f_4\left(\frac{1}{2}\right) = 1$ and $f_4(1) = 0 \Longrightarrow f_4^3\left(\frac{1}{2}\right) = 0$

3) For y_1, y_2 the preimages of $x = \frac{1}{2}$, we found that $f_4^2(y_1) = f_4^2(y_2) = 1$. Therefore $f_4^3(y_1) = f_4^3(y_2) = 0$.

4) Each of y_1 and y_2 has two preimages: $f_4(y_{11}) = f_4(y_{12}) = y_1$ and $f_4(y_{21}) = f_4(y_{22}) = y_2$. From 3),
$$f_4^3(y_{11}) = f_4^3(y_{12}) = f_4^3(y_{21}) = f_4^3(y_{22}) = 1.$$

We now have nine points of the graph of $f_4^3(x)$:

$$(0,0), \quad (1,0), \quad (y_1,0), \quad (y_2,0), \quad (y_{11},1), \quad (y_{12},1), \quad (y_{21},1), \quad (y_{22},1)\,. \tag{3}$$

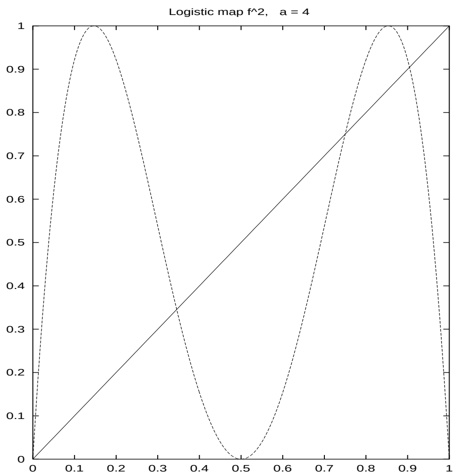
With a little more work, as in the case of $f_4^2(x)$ earlier, we are led to the conclusion that the graph of $f_4^3(x)$ consists of two copies of $f_4^2(x)$ that are contracted horizontally to fit over the subintervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. The graph of $f_4^3(x)$ is shown on the next page – top right. The function $f_4^3(x)$ has **eight** fixed points. Therefore there are eight periodic points of period 3: the two fixed points of $f_4(x)$, $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{3}{4}$, and two three-cycles (p_1, p_2, p_3) and (q_1, q_2, q_3). In the middle two graphs on the previous page, these two three-cycles are shown in relation to the graph of $f_4(x)$.

Note: Actually, the three-cycles are **not** shown. We'll leave it as an exercise for the reader to plot them. Just take one of the fixed points of $f_4^3(x)$ which is **not** a fixed point of $f_4(x)$ and determine where it travels under iteration of $f_4(x)$.

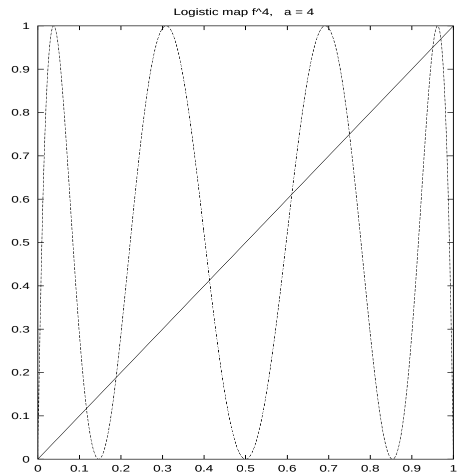
The bottom figures show the graphs of $f_4^4(x)$ and $f_4^5(x)$. Clearly, a pattern is emerging: The graph of $f_4^k(x)$ is obtained by (roughly) shrinking the graph of $f_4^{k-1}(x)$ to produce two copies that are placed on the subintervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. A little closer inspection shows that the shrunken copy over $[\frac{1}{2}, 1]$ is obtained by **reflecting** the shrunken copy over $[0, \frac{1}{2}]$ with respect to the line $x = \frac{1}{2}$.

To see why, let's write $f_4^k(x)$ as $f_4^{k-1}(f_4(x))$. As x moves from 0 to $\frac{1}{2}$. $f_4(x)$ moves from the value 0 to 1. This means that $f_4^{k-1}(f_4(x))$ must assume **all** values that the graph of $f_4^{k-1}(x)$ assumes over the interval $[0, 1]$. This produces the first, shrunken copy of the graph of $f_4^{k-1}(x)$ over the interval

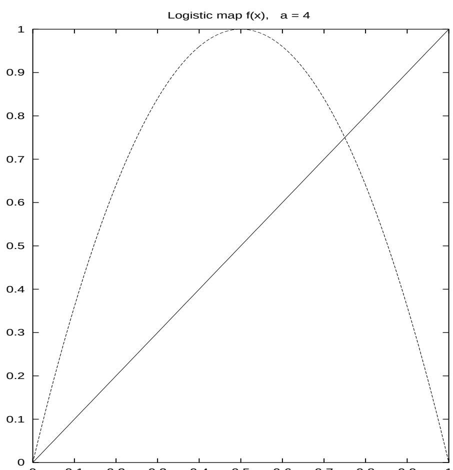
Some iterates $f_4^n(x)$ of the logistic map $f_4(x) = 4x(1-x)$

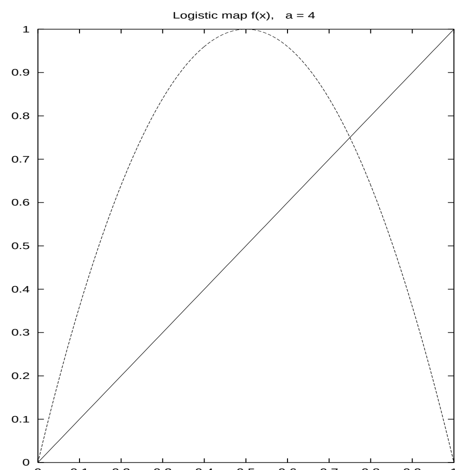


$f_4^2(x)$

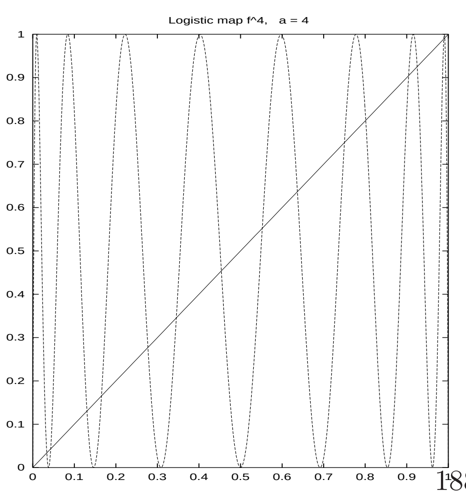


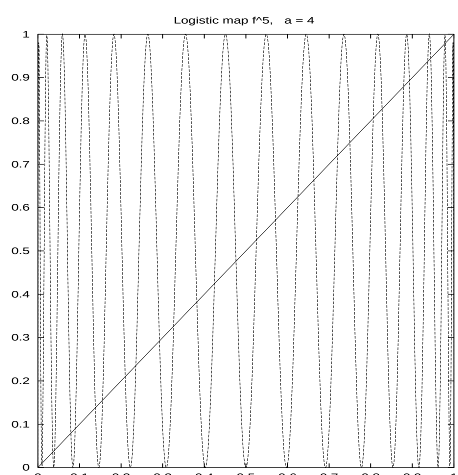
$f_4^3(x)$





The two three-cycles of $f_4(x)$.





$f_4^4(x)$

$f_4^5(x)$

$[0, \frac{1}{2}]$. Now as x moves from $\frac{1}{2}$ to 1, $f_4(x)$ moves from the value 1 to the value 0. This means that $f_4^{k-1}(f_4(x))$ must assume **all** values that the graph of $f_4^{k-1}(x)$ assumes over the interval $[0,1]$, but **in reverse order** since the inputs into this function are decreasing from 1 to 0. This produces the second, shrunken copy of the graph of $f_4^{k-1}(x)$ which is a **reflection** of the copy of $f_4^{k-1}(x)$ over $[0, \frac{1}{2}]$ that was produced earlier.

Let's summarize the above discussion. For a given $k \geq 2$, the graph of $f_4^k(x)$ is obtained from the graph of $f_4^{k-1}(x)$ as follows. The graph of $f_4^{k-1}(x)$ over [0,1] is "horizontally (and nonlinearly) shrunk" to fit over the interval $[0, \frac{1}{2}]$. This shrunken copy is then reflected about the line $x = \frac{1}{2}$ to produce a second shrunken, but inverted, copy of the graph of $f_4^{k-1}(x)$ which lies over the interval $[\frac{1}{2}, 1]$. The union of these two shrunked copies of the graph of $f_4^{k-1}(x)$ over [0,1] comprise the graph of $f_4^k(x)$ over $[0, 1]$.

This relationship between the graphs of $f_4^{k-1}(x)$ and $f_4^k(x)$ implies that the number of periodic points of $f_4^k(x)$ is twice the number of periodic points of $f_4^{k-1}(x)$. Since $f_4(x)$ has two fixed points, i.e., two periodic points of period 1, it follows that $f_4^2(x)$ has four fixed points, i.e., four periodic points of period 2, etc..

The net result: For an $n \geq 1$, the graph of $f_4^n(x)$ intersects the line $y = x$ at 2^n points. This implies that the function $f_4^n(x)$ has 2^n fixed points which, in turn, implies that

$$f_4(x) = 4x(1-x) \quad \text{has} \quad 2^n \quad \text{periodic points of period} \;\; n.$$

This "doubling" of the graph for each application of f_4 is a consequence of the two-to-one nature of f_4 on $[0, 1]$.

Lecture 19

Dynamics of the logistic map (cont'd)

Three ingredients for "chaotic behaviour"

We now describe three characteristics exhibited by the logistic map for $a = 4$ which will be responsible for the chaotic behaviour demonstrated by its iterates.

Ingredient No. 1: Density of repulsive periodic points

Note that all of the periodic points of the map $f_4(x)$ are **repulsive**. Moreover, as n increases, the graphs of $f_4^n(x)$ indicate that these repulsive periodic points are spread out over the entire interval $[0,1]$. In fact, a little more analysis shows that the set of all periodic points of $f_4(x)$ is **dense** on $[0,1]$. We'll formally define what "dense" means below. For the moment:

Translation: If you take **any** point $x \in [0,1]$ and **any** neighbourhood $N_\delta(x)$ of x, where

$$N_\delta(x) = \{y \mid y \in (x-\delta, x+\delta)\} \ , \tag{4}$$

no matter how small δ is, then you will find at least one periodic point of f_4 in it.

The existence of so many periodic points so densely packed on the interval suggests a very high degree of regularity associated with the map $f_4(x)$, since if x_0 is a periodic point, there is an $n > 0$ so that $f^n(x_0) = f^{2n}(x_0) = \cdots = f^{kn}(x_0) = x_0$. On the other hand, since these fixed points are repulsive, they treat points near to them as "hot potatoes" under iteration, wishing to throw them away from the periodic orbits of which they are members. This "regularity" of repulsive fixed points is one ingredient in the recipe for "chaotic behaviour" of the iteration dynamics $x_{n+1} = 4x_n(1-x_n)$.

Before moving on, let us discuss formally the idea of a **dense subset**. For the moment, we'll restrict our discussion to the real numbers. In the following discussion, we let X be a subset of the real numbers $\mathbb{R}$, for example an interval I or even the real line itself.

Definition: Let $X \subseteq \mathbb{R}$ and let $Y \subset X$. (Note that Y be a proper subset of X, i.e., Y cannot be X itself.) The set Y is **dense** in X if, given any $x \in X$, and any $\delta > 0$, we can find a point $y \in Y$ such that $y \in N_\delta(x)$. (The neighbourhood $N_\delta(x)$ was defined in Eq. (4) above.)

Translation: Given any point $x \in X$, you can find points in Y that are arbitrarily close to x. A standard example is

$$X = [0,1]\,, \quad Y = \{\text{ rational numbers in } X\,\}\,.$$

Another way of writing this is

$$X = [0,1]\,, \quad Y = X \cap \mathbb{Q}\,,$$

where $\mathbb{Q}$ denotes the set of rational numbers.

It is well known that the rational numbers form a dense subset of the real numbers. (People often simply say that "the rationals are dense in the reals.") If we let $x \in X$ be an irrational number, then given any $\delta > 0$, we can always find a rational number y in the interval $(x - \delta, x + \delta)$. We can consider δ as the error in approximating the irrational number x by the rational number y. From the density of the rationals, this implies that an irrational number can be approximated by a real number to any degree of accuracy.

Example: Let $x = \pi$. Now consider the following sequence of rational numbers,

$$x_1 = 3\,, \quad x_2 = \frac{31}{10}\,, \quad x_3 = \frac{314}{100}\,, \quad x_4 = \frac{3141}{1000}\,, \cdots . \tag{5}$$

The element x_n is obtained by taking the first n digits in the (infinite and nonrepeating) decimal expansion of π and dividing that number by 10^{n-1}. As $n \to \infty$, $x_n \to \pi$. This means that for any $\delta > 0$, we can find an element of the sequence $\{x_n\}$ such that $x_n \in N_\delta(\pi)$.

Ingredient No. 2: Sensitive dependence to initial conditions

The second ingredient, somewhat related to the repulsivity of the fixed points, is the property of **sensitive dependence to initial conditions**. This basically means that two distinct points, x and y, no matter how close they are to each other initially, may, under iteration of f, become separated significantly. Mathematically "SDIC" is expressed as follows:

> **Definition:** A function $f : I \to I$ is said to have **sensitive dependence to initial conditions (SDIC)** at a point $x \in I$ if there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exists a $y \in N_\delta(x)$ and an $n > 0$ such that
>
> $$|f^n(x) - f^n(y)| > \epsilon\,. \tag{6}$$
>
> Recall that $y \in N_\delta(x)$ means that
>
> $$|x - y| < \delta\,. \tag{7}$$

Note: From the above definition, a map f is SDIC at a point $x \in I$ if there exist points arbitrarily close to x which eventually separate from x by a distance of at least $\epsilon > 0$ under iteration of f. It should be emphasized that **not all points** near x need to separate in this way from x – only at least one point in each δ-neighbourhood of x.

Furthermore,

> **Definition:** A function $f : I \to I$ is said to have sensitive dependence to initial conditions (SDIC) on the interval I if it has SDIC at all $x \in I$.

We have already seen an example of an SDIC point for a function $f(x)$ – a repulsive fixed point $\bar{x} = f(\bar{x})$ at which $|f'(\bar{x})| > 1$. For x_0 near $\bar{x}$, iterates $x_n = f^n(x_0)$ are mapped away from $\bar{x}$. This indicates that the derivative $f'(x)$ – provided it exists – plays an important role.

Special example: Suppose that f is a C^1 function on an interval I and that there exists a $K > 1$ such that

$$|f'(x)| \geq K > 1 \quad \forall \quad x \in I \,. \tag{8}$$

Now consider any point $x \in I$. Then for any $y \in I$, by the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(c)| \, |x - y| \tag{9}$$

for some c between x and y. From the assumption in (8), it follows that

$$|f(x) - f(y)| \geq K|x - y| \,. \tag{10}$$

In other words, $f(x)$ and $f(y)$ are farther apart than x and y were. Now replace x with $f(x)$ and y with $f(y)$ so that Eq. (10) becomes

$$\begin{aligned} |f^2(x) - f^2(y)| &\geq K|f(x) - f(y)| \\ &\geq K^2|x - y| \quad \text{from Eq. (10)} \,. \end{aligned} \tag{11}$$

If we keep repeating this procedure, i.e., replacing x with $f(x)$ and y with $f(y)$, we obtain

$$|f^n(x) - f^n(y)| \geq K^n|x - y| \,, \tag{12}$$

for n as long as the points x and y remain in I. Given an $\epsilon > 0$, then for any nonzero initial separation $|x - y|$ as small as we wish, i.e., $|x - y| < \delta$ for any $\delta > 0$, there exists an $N > 0$ such that

$$K^N|x - y| > \epsilon \,. \tag{13}$$

As such, f is SDIC at every point $x \in I$. Technically, our choice of $\epsilon > 0$ must be such that the iterates $f^n(x)$ and $f^n(y)$ remain in the interval I.

There are some serious implications of SDIC regarding the numerical "analysis" of chaotic behaviour. In numerical computations, we rarely, if ever, compute iteration sequences using infinite precision (unless we compute them in exact rational form using a symbolic computation routine such as MAPLE). As a result, the finite precision approximation p^* of a periodic point p represents an error $\epsilon = |p - p^*|$. Thus, p^* may be viewed as the "y" in the previous definition of SDIC, and p as x. SDIC implies that $f^k(p)$ and $f^k(p^*)$ will become separated significantly. If p is a repulsive periodic point, period n, then there is little chance that $f^n(p^*)$ is even close to $f^n(p) = p$. In general, small errors in computation that are introduced by round-off (due to finite precision) may become magnified upon iteration. As R. Devaney points out in his excellent book, *An Introduction to Chaotic Dynamical Systems* (to be posted on LEARN),

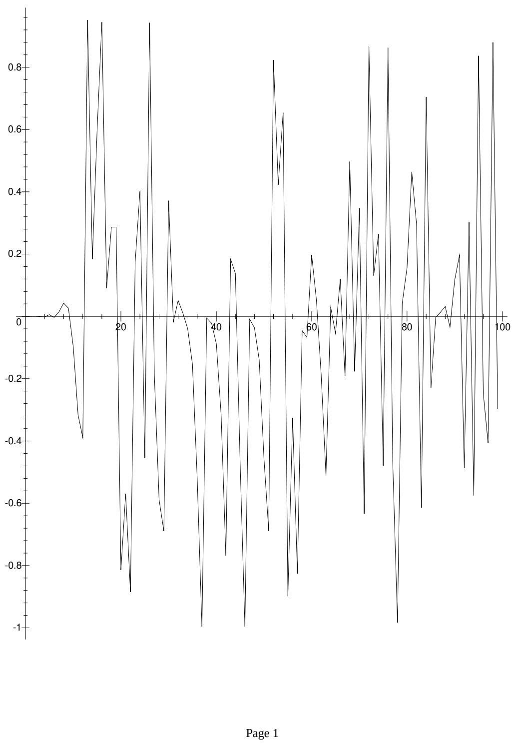
> "the results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever with the real orbit" (p. 49).

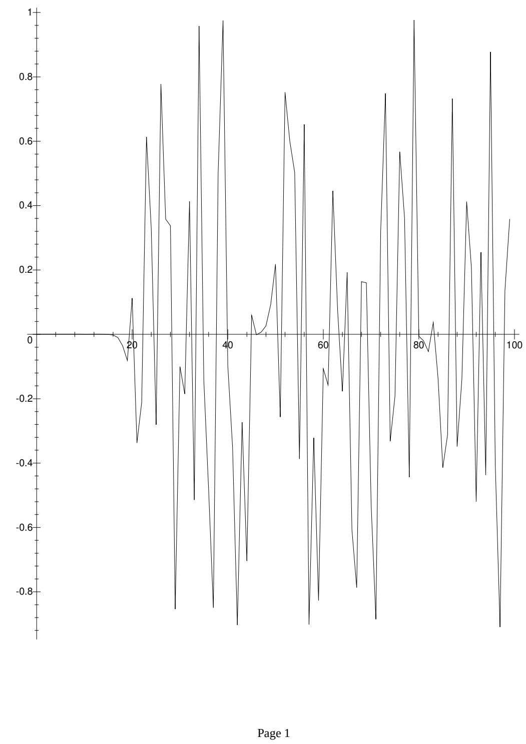
Note: SDIC alone is insufficient for chaotic or irregular behaviour. For example, the linear map $f(x) = 2x$ demonstrates SDIC: If $|x - y| = \delta$, then $|f^n(x) - f^n(y)| = 2^n\delta \to \infty$ as $n \to \infty$. However, the behaviour of $f^n(x) = 2^n x$ is rather straightforward – all points (except 0) go to infinity. What is necessary for chaotic behaviour is that points are eventually mapped back to a region so that the iteration process can continue. Linear maps don't do this, but nonlinear ones can. The nonlinear logistic map $f_4(x)$ is an example of a map that does this.

The SDIC property of chaotic dynamical systems is illustrated on the next page. The results of two numerical experiments are presented, in which the "exact" (at least to a few hundred digits of accuracy) orbits of two points x_0 and y_0, initially close, are followed. The graphs indicate that no matter how close x_0 and y_0 may be, their iterates x_n and y_n eventually travel far apart – almost to opposite ends of the interval $[0, 1]$ – only to come arbitrarily close again at another time. The process of separation and near-merging proceed in a seemingly random manner. Note, however, that this dynamical process is **not** a random one but rather a deterministic process since a point x_{n+1} is determined uniquely by its predecessor x_n.

Very shortly, we shall examine SDIC a little more closely, actually proving its existence for some simple maps.

Logistic map $f_4(x) = 4x(1-x)$:
Sensitive Dependence to Initial Conditions





The above graphs summarize the results of two numerical experiments that illustrate the SDIC property of the logistic map $f_4(x)$. In each experiment, two initial x_0 and y_0, with an initial separation of $e_0 = y_0 - x_0$, were chosen. The iteration sequences $x_n = 4x_{n-1}(1 - x_{n-1})$ and $y_n = 4y_{n-1}(1 - y_{n-1})$ were then computed to $n = 100$, using 1000 digits of floating-point precision. In the above graphs are plotted the quantities $e_n = y_n - x_n$ vs. n. In both cases, the separation e_n assumes values that are both arbitrarily small as well as arbitrarily close to 1 or -1. We may interpret y_0 as an initial approximation to x_0 with error e_0.

Left graph: $x_0 = 0.1$ and $y_0 = 0.1001$, so that $e_0 = 0.0001$. The iterates x_n and y_n remain close to about $n = 5$.

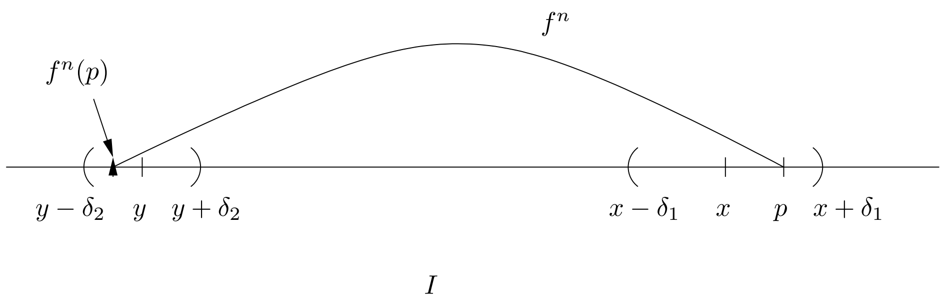
Right graph: $x_0 = 0.1$ and $y_0 = 0.1000001$, so that $e_0 = 0.0000001$. The iterates x_n and y_n remain close to about $n = 15$.

Ingredient No. 3: Transitivity

The final ingredient of chaotic behaviour, which is responsible for the great separation and near-merging of iterates mentioned above, is **transitivity**, which is a kind of **mixing** property of the mapping $f(x)$. One way of defining transitivity mathematically is as follows:

> $f : I \to I$ is transitive on I if, for any two points $x, y \in I$ and any two neighbourhoods $N_{\delta_1}(x)$ and $N_{\delta_2}(y)$ of these points, there exists a point $p \in N_{\delta_1}(x)$ and an $n > 0$ such that $f^n(p) \in N_{\delta_2}(y)$.

Transitivity implies that there are no special intervals in $I \subset \mathbb{R}$ or regions in $I \subset \mathbb{R}^n$ that can be completely avoided by orbits of points from other parts of I. For example, the map $f_2(x) = 2x(1-x)$ cannot be transitive on $[0,1]$, since Range $(f_2) = \left[0, \frac{1}{2}\right]$. Thus, no points are mapped into the interval $\left(\frac{1}{2}, 1\right]$. Even the map $f(x) = x^2$ cannot be transitive on $[0,1]$ since, if $x_n \in (0,1)$, $x_{n+1} = x_n^2 < x_n$. In other words a point $x_n \in (0,1)$ can never be mapped to a value $y > x_n$.



Summary: Ingredients for a chaotic dynamical system

In summary, the ingredients for a chaotic dynamical system $x_{n+1} = f(x_n)$, where $f : I \to I$ are:

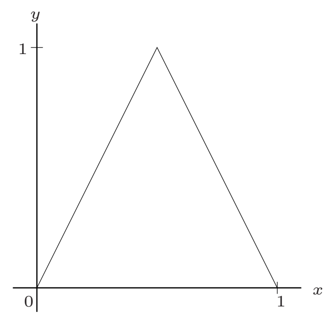
1. Regularity: The set of all periodic points of f is dense on I,
2. Sensitive dependence to initial conditions (SDIC),
3. Transitivity of f.

Here are some other examples of chaotic mappings on $[0, 1]$:

1. The "Tent Map"

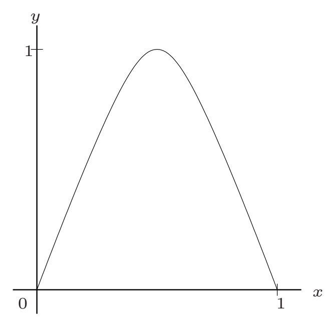
$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} < x \le 1\,, \end{cases}$$

a "straightened" version of the logistic map $f_4(x)$.



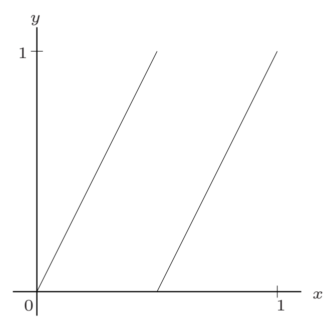
2. Another "unimodal" map"

$$f(x) = \sin \pi x, \qquad 0 \le x \le 1$$



3. The "Baker map" on $[0, 1]$

$$B(x) = 2x \bmod 1,\ 0 \le x \le 1\,.$$



In general, it may be rather complicated to show that a function $f(x)$ defined a chaotic dynamical system on an interval I, using the three ingredients listed earlier. However, if the action of f on points $x \in I$ can be put into one-to-one correspondence with another map $g : I \to I$ that is known to be chaotic, then f is chaotic. Historically, the logistic map f_4 was shown to be chaotic by means of a $1 - 1$ correspondence it has with the "tent map" in Example 1. The tent map, being piecewise linear, is easier to analyze.

Lecture 20

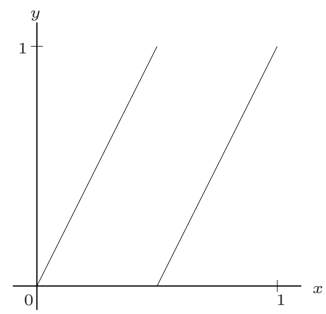
Chaotic dynamics (cont'd)

At the end of the previous lecture, we summarized the three ingredients for chaotic dynamics exhibited by the iteration of a map $f : I \to I$, where I is an interval on the real line $\mathbb{R}$. We claim that iteration of the logistic map

$$f_4(x) = 4x(1-x)\,, \tag{14}$$

produces chaotic behaviour on the interval [0,1]. We also gave three other examples of chaotic maps on [0,1]. The first two of these maps on Page 165 look quite similar to the logistic map, $f_4(x)$, in that they map 0 to 0, they map 1 to 0, and they increase from 0 to 1 on [0,1/2] and decrease from 1 to 0 on [1/2,1]. The third map, the so-called "Baker map",

$$\begin{aligned} B(x) &= 2x \bmod 1,\ 0 \le x \le 1\,, \\ &= \begin{cases} 2x\,, & 0 \le x < \frac{1}{2}\,, \\ 2x-1 & \frac{1}{2} \le x < 1\,, \\ 0\,, & x = 1\,. \end{cases} \end{aligned}$$



is also chaotic on [0,1]. We shall be using it below since it is easier to analyze.

Before we proceed, let us note that from the definition of $B(x)$ presented above, it has only one fixed point, namely $x = 0$, i.e., $B(0) = 1$. One might be tempted to say that $x = 1$ is a fixed point, but from the above definition, which includes the "mod 1" operation, $B(0) = 1$. Therefore, $x = 1$ is **not** a fixed point of $B(x)$.

One more point: Since $B'(0) = 2$, it follows that $x = 0$ is a repulsive fixed point.

Showing chaotic behaviour via "Symbolic Dynamics"

There is a method that can sometimes be used to show that a mapping f defines a chaotic dynamical system over an interval I. This method establishes a one-to-one correspondence between the mapping f and a mapping that operates on **sequences**. We'll illustrate below, using the Baker map given above. We'll also see that we don't even have to look at the actual iteration scheme using graphs of the function on [0,1] – everything will be accomplished with the use of sequences.

Firstly, let $x \in [0,1]$ and denote its **binary** expansion as

$$x = \cdot b_1 b_2 b_3 b_4 \ldots = \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \ldots\,, \qquad b_i \in \{0,1\}. \tag{15}$$

Every $x \in [0,1]$ has at least one binary expansion – some have two. For example, $x = \frac{1}{2}$ has binary expansions $.01111\ldots$ and $.1$. It is convenient to restrict the binary sequences to a particular class – for example choosing 0.1 over .0111 . . . – so that each $x \in [0, 1]$ will now have a unique binary sequence. However, in the brief discussion that follows, this point is not too important.

We may consider the binary expansion as a mapping ϕ from the interval $[0, 1]$ to a space Σ_2, the set of all binary sequences, i.e.

$$\Sigma_2 = \{\mathbf{b} = (b_1, b_2, b_3, \dots) | \quad b_i \in \{0, 1\}\} .$$

Then we may write $\phi : I \to \Sigma_2$ and $\phi(x) = \mathbf{b}$. (Once again, we shall not worry about ϕ being one-to-one or not.)

Let us now examine the action of the Baker map $B(x) = 2x \bmod 1$ on points $x \in [0, 1]$ and their associated binary expansions. From (15), the binary expansion of $2x$ is

$$2x = b_1 + \frac{b_2}{2} + \frac{b_3}{4} + \cdots = b_1 \cdot b_2 b_3 b_4 \dots , \quad b_i \in \{0, 1\}. \tag{16}$$

When we apply the "mod 1" operation to $y = 2x$, we simply delete the integer part of the binary expansion b_1. The net result is

$$2x \bmod 1 = \frac{b_2}{2} + \frac{b_3}{4} + \cdots = \cdot b_2 b_3 b_4 \dots , \quad b_i \in \{0, 1\}. \tag{17}$$

In other words, associated with the mapping $f : I \to I$, $B(x) = 2x \bmod 1$, is a mapping of a binary sequence to another binary sequence, i.e. $S : \Sigma_2 \to \Sigma_2$. The action of S on Σ_2 is given by

$$S : \ b_1 b_2 b_3 \dots \longrightarrow b_2 b_3 b_4 \dots . \tag{18}$$

The binary sequence $b = (b_1, b_2, b_3, \dots)$ has been "shifted" – more precisely "left-shifted" – to produce the binary sequence $S(b) = (b_2, b_3, b_4, \dots)$.

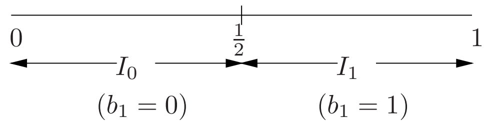
Note: The mapping S is **not** invertible since, given $S(\mathbf{b}) = (b_2, b_3, b_4, \dots)$, there is not a unique element, say $\mathbf{c} \in \Sigma_2$, such that $S(c) = (b_2, b_3, b_4, \dots)$. In fact, the element $(b_2, b_3, b_4 \dots)$ has **two** preimages:

$$\begin{aligned} \mathbf{c} &= (0, b_2, b_3, b_4, \dots) \\ \mathbf{d} &= (1, b_2, b_3, b_4, \dots) \end{aligned}$$

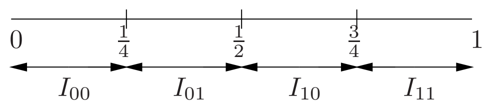
such that $S(\mathbf{c}) = S(\mathbf{d}) = (b_2, b_3, b_4, \dots)$. This is consistent with the fact that $B(x) = 2x \bmod 1$ is two-to-one $[0, 1]$.

The operator $S : \Sigma_2 \to \Sigma_2$ in (18) is often referred to as a "Bernoulli shift map" on Σ_2. You may well be asking, "So what? What is the significance of this mapping S and its correspondence with $B(x)$?" The answer lies in the interpretation of the binary expansion b of a point x as the "address" or "postal code" of x in the interval $[0, 1]$. Beginning with the first binary digit b_1, providing successive digits $b_2, b_3 \dots$ allows us to locate x more accurately in $[0, 1]$.

To illustrate: If $b_1 = 0$, then we know that $x \in \left[0, \frac{1}{2}\right]$. If $b_1 = 1$, then $x \in \left[\frac{1}{2}, 1\right]$. (We can delete the point of overlap, $x = \frac{1}{2}$, from either of the intervals by omitting various sequences from the space Σ_2, as discussed earlier.) Define $I_0 = \left[0, \frac{1}{2}\right]$ and $I_1 \left[\frac{1}{2}, 1\right]$. Then:



Now consider the second binary digit b_2 in the expansion of x. If $b_1 = 0$ and $b_2 = 0$, then $x \in I_{00} = \left[0, \frac{1}{4}\right]$. If $b_1 = 0$ and $b_2 = 1$, then $x \in I_{01} = \left[\frac{1}{4}, \frac{1}{2}\right]$. Likewise, if $b_1 = 1$ and $b_2 = 0$, then $x \in I_{10} = \left[\frac{1}{2}, \frac{3}{4}\right]$. If $b_1 = 1$ and $b_2 = 1$,then $x \in I_{11} = \left[\frac{3}{4}, 1\right]$:



The pattern should be clear by now. We construct a series of nested subintervals, by splitting each subinterval $I_{i_1 i_2 \ldots i_k}$ of length 2^{-k} into two smaller subintervals $I_{i_1 i_2 \ldots i_k 0}$ and $I_{i_1 i_2 \ldots i_k 1}$ of length $2^{-(k+1)}$. The net result: A knowledge of the first n digits $b_1, b_2, \ldots b_n$ of the binary expansion of $x \in [0,1]$ allows us to conclude that x lies in the subinterval $I_{b_1 b_2 \ldots b_n}$ of length 2^{-n}. Clearly, as we specify more binary digits, i.e. as n increases, then the accuracy to which we know the position of x increases. This idea leads to the following result, which is very important to our study of chaotic dynamical systems.

Proposition 1: Let $x, y \in [0,1]$ with associated binary expansions

$$\begin{aligned} x: \quad & \mathbf{b} = (b_1, b_2, b_3, \ldots), \quad b_i \in \{0,1\}, \\ y: \quad & \mathbf{b}' = (b'_1, b'_2, b'_3, \ldots), \quad b'_i \in \{0,1\}. \end{aligned}$$

Suppose that $b_k = b'_k$ for $1 \leq k \leq n$. Then $|x - y| \leq 2^{-n}$.

Proof: If the binary expansions of x and y agree to n places, then x and y must lie in the same subinterval $I_{b_1 b_2 \ldots b_n}$ of length 2^{-n}. As $n \to \infty$, $|x - y| \to 0$.

Aside: This simple fact allows us to define a "distance function" between sequences $\mathbf{b}$ in Σ_2. The more initial digits $b_1, b_2, \ldots, b_n$ that two sequences share, the closer the points, x and y, on the real line $\mathbb{R}$. Therefore, we expect these sequences to be "closer" as well. A suitable distance function, or **metric**, between two sequences $\mathbf{b}, \mathbf{b}' \in \Sigma_2$ is as follows,

$$d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') = \sum_{k=1}^{\infty} \frac{|b_k - b_{k'}|}{2^k}. \tag{19}$$

Let's see why this is a good distance function. Suppose that two sequences, $\mathbf{b}$ and $\mathbf{b}'$, agree to the first n binary digits, i.e.,

$$b_k = b'_k \,, 1 \leq k \leq n \,, \tag{20}$$

Then the d_{Σ_2} distance between these two sequences is

$$d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') = \sum_{k=n+1}^{\infty} \frac{|b_k - b_{k'}|}{2^k} \,. \tag{21}$$

Since $b_k, b'_k \in \{0,1\}$, the largest value that any difference between corresponding digits can be is 1, i.e.,

$$0 \le |b_k - b'_k| \le 1 \,. \tag{22}$$

This implies that

$$\begin{aligned} d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') &\le \sum_{k=n+1}^{\infty} \frac{1}{2^k} \\ &= \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots \\ &= \frac{1}{2^n} \,. \end{aligned} \tag{23}$$

Consequently, $d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') \to 0$ as the number of identical digits, $\to \infty$.

Important note: The distance bound $d_{\Sigma_2}(\mathbf{b}, \mathbf{b}') \le \frac{1}{2^n}$ in Eq. (23) is a **distance between two sequences $\mathbf{b}, \mathbf{b}' \in \Sigma_2$**. It just happens to agree with the maximum distance between the two points, $x, y \in [0,1]$, which the sequences represent. There is no need that the numerical values of the bounds be the same. We could have modified the formula for $d_{\Sigma_2}(\mathbf{b}, \mathbf{b}')$ to produce other bounds. The most important property that they must share, however, is that they both tend to zero as n, the number of identical digits at the fronts of the sequences, tends to infinity.

Now recall that the action of $B(x) = 2x \bmod 1$ on $[0,1]$ induced a shift map S on the associated binary sequence space Σ_2, cf. Eq. (17). **We now show how some important properties of the dynamical system $x_{n+1} = B(x_n)$ can be determined by examining the action of the associated map S on Σ_2.**

Ingredient No. 1: Density of periodic points

We first ask whether S, hence $B(x)$, has "fixed points", i.e. sequences $\mathbf{b} = (b_1, b_2, b_3, \ldots)$ such that $S(\mathbf{b}) = \mathbf{b}$. The answer is "Yes", since

$$S : (0, 0, 0, 0, \ldots) \to (0, 0, 0, 0, \ldots)$$

and

$$S : (1, 1, 1, 1, \ldots) \to (1, 1, 1, 1, \ldots).$$

Of course, $(0, 0, 0, \ldots)$ is the binary expansion of $x = 0$. And the point $x = 0$ is definitely a fixed point of the map $B(x) = 2x \bmod 1$, as we can see from the graph of $B(x)$ presented earlier. But what about the sequence $(1, 1, 1, \ldots)$? If we compute the number x associated with this binary expansion, we find that

$$0.11111\ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1\,. \tag{24}$$

But we said earlier that $x = 1$ is **not** a fixed point of $B(x)$. So what is happening here? Recall that the point $x = 1$ is actually "equivalent" to the point $x = 0$ under the operation "mod 1," i.e.,

$$0 = 1 \bmod 1\,. \tag{25}$$

How do we reconcile this fact with the binary expansion $0.111\ldots$? The answer is that we "round off" the binary number $0.111\cdots$ to the number $1.000\cdots \bmod 1 = 0.000\cdots$. Do we have to do this every time? The answer is "No" – we would simply declare that the binary sequence $0.111\cdots$ is inadmissible, as we discussed earlier. This suggests that we do not allow any sequences which end in an infinite string of 1's in our sequence space. For example, if $x = \frac{1}{2}$, then its one and only binary sequence representation is

$$\frac{1}{2} = 0.10000\cdots = 0.1\dot{0}\,. \tag{26}$$

We do **not** allow its other representation,

$$\frac{1}{2} = 0.01111\cdots = 0.0\dot{1}\,. \tag{27}$$

In summary, we have determined that $B(x)$ has one fixed point, $x = 0$, i.e., $B(0) = 0$.

Let us now continue by asking if the Bernoulli shift map S has any "two-cycles", i.e. a pair of sequences, $\mathbf{b}$ and $\mathbf{c}$, such that $S(\mathbf{b}) = \mathbf{c}$ and $S(\mathbf{c}) = \mathbf{b}$. The answer is again "Yes." Let

$$\mathbf{b} = (0, 1, 0, 1, 0, 1, \cdots)\,, \qquad \mathbf{c} = (1, 0, 1, 0, 1, 0, \cdots) \tag{28}$$

and note that

$$\begin{aligned} S :&(0, 1, 0, 1, 0, 1, \ldots) \to (1, 0, 1, 0, \ldots) \\ S :&(1, 0, 1, 0, 1, 0, \ldots) \to (0, 1, 0, 1, \ldots)\,. \end{aligned}$$

These sequences define the two points $p_1, p_2 \in [0,1]$ such that $B(p_1) = p_2$, $B(p_2) = p_1$.

Exercise: Determine p_1 and p_2 from a knowledge of their binary expansions above.

Solution:

$$p_1 = 0.010101\cdots = \frac{1}{4} + \frac{1}{16} + \frac{1}{64}\cdots . \tag{29}$$

This is a geometric series with initial term $\frac{1}{4}$ and ratio $\frac{1}{4}$. The sum of this series is

$$p_1 = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}. \tag{30}$$

Likewise,

$$p_2 = 0.101010\cdots = \frac{1}{2} + \frac{1}{8} + \frac{1}{32}\cdots . \tag{31}$$

This is a geometric series with initial term $\frac{1}{2}$ and ratio $\frac{1}{4}$. The sum of this series is

$$p_2 = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}. \tag{32}$$

Let's check this result using the formula $B(x) = 2x \bmod 1$:

$$B(\frac{1}{3}) = 2 \cdot \frac{1}{3} \bmod 1 = \frac{2}{3} \bmod 1 = \frac{2}{3}. \tag{33}$$

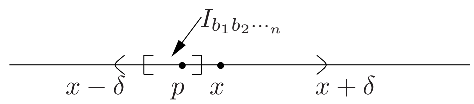
$$B(\frac{2}{3}) = 2 \cdot \frac{2}{3} \bmod 1 = \frac{4}{3} \bmod 1 = \frac{1}{3}. \tag{34}$$

We have verified that $(p_1, p_2) = (\frac{1}{3}, \frac{2}{3})$ is a two-cycle of $B(x)$.

We can continue this procedure to construct three-cycles, four-cycles, etc. In general, a periodic point of period n will have a binary expansion of the form,

$$\mathbf{b} = (b_1, b_2, b_3, \ldots b_n, b_1, b_2, b_3, \ldots b_n, b_1, b_2, \ldots b_n, \ldots).$$

Since each $b_i \in \{0,1\}$, $1 \le i \le n$, there are 2^n possible periodic points of period n. (Well, technically, we have to exclude the sequence $(1,1,1,1,\cdots)$ from each set for reasons explained earlier, so we really have only $2^n - 1$ periodic points/sequences. But recall that the sequence $(1,1,1,1,\cdots)$ represents the point $x = 1$ which, in fact, is equivalent to the point $x = 0$. As such, we are excluding the point $x = 1$ from our set of periodic points. This is not a problem at all since the point $x = 1$ lies at the farthest end of the interval [0,1].) We may also conclude that each periodic point of period n will lie in a distinct subinterval $I_{b_1 b_2 \ldots b_n}$ of length 2^{-n}. In other words, these periodic points are distributed equally over all subintervals $I_{b_1 b_2 \ldots b_n}$, one per subinterval. As $n \to \infty$, these periodic points are distributed over $[0,1]$ equally over smaller subintervals. This guarantees that the set of all periodic points is **dense** on the interval $[0,1]$. The reason: Given **any** point $x \in [0,1]$, and **any** neighbourhood $N(x)$ of x, say $(x-\delta, x+\delta)$, with δ as small we want (or $[0,\delta)$, $(1-\delta, 1]$ at the endpoints), we shall always be able to find a subinterval $I_{b_1 b_2 \ldots b_n}$, where n is sufficiently large, so that $I_{b_1, b_2 \ldots b_n}$ is contained in the neighbourhood $N(x)$ as shown in the figure below. Since $I_{b_1, b_2 \ldots b_n}$ will contain a periodic point of period n, we have guaranteed that at least one periodic point – it doesn't matter what the period of that point is – lies in the neighbourhood $N(x)$ of x. This is the definition of "denseness".



p periodic point, period n

It looks as if we have shown, using "symbolic dynamics", that the map S, hence the Baker map $B(x) = 2x \bmod 1$, has the property of regularity, one of the "ingredients" of chaotic systems. Can we establish the other two properties? The answer is "Yes".