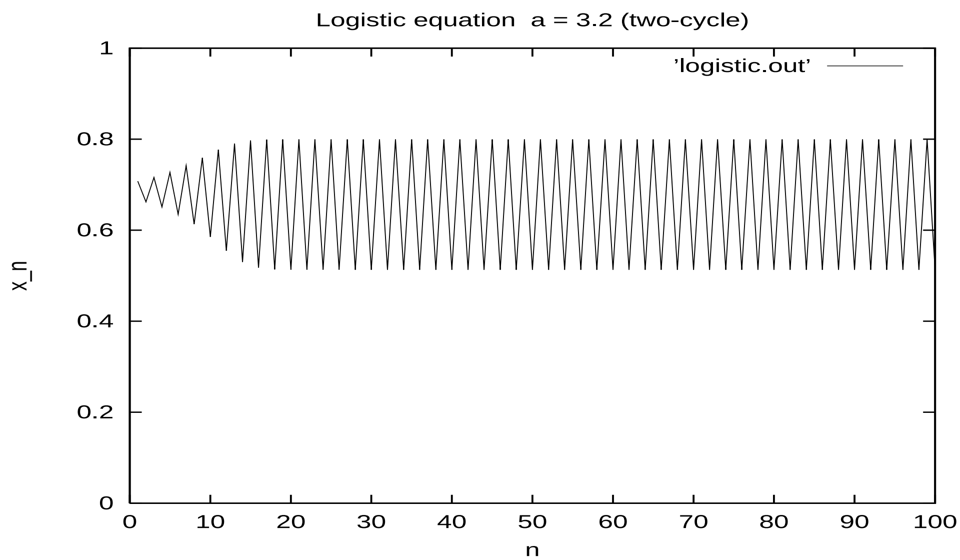
Lecture 15

Dynamics of the logistic map (cont'd)

The appearance of "two-cycles"

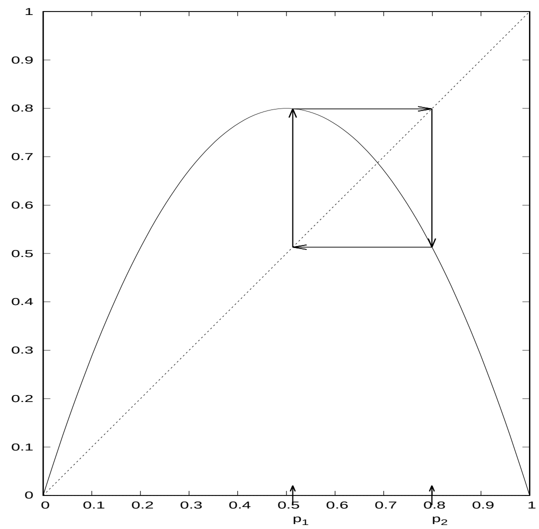
As stated earlier, for values $a > 3$, the fixed point $\bar{x}_2 = \frac{a-1}{a}$ is repulsive. If we begin with a seed value $x_0 \neq 0$, then, for values of a slightly greater than 3, e.g. $a = 3.1$, sequences produced by the iteration scheme $x_{n+1} = f_a(x_n)$ are observed to approach an oscillation between two numbers $\{p_1, p_2\}$. When $a = 3.2$, $p_1 \cong 0.799$, $p_2 \cong 0.513$. A plot of the iterates $\{x_n\}$ vs. n resulting from the initial seed $x_0 = \frac{1}{3}$ is shown below.



This pair of numbers $(p_1, p_2) \cong (0.799, 513)$ is known as a "two-cycle." It is a "two-cycle" since $f_a(p_1) = p_2$ and $f_a(p_2) = p_1$. From the behaviour of the iterates x_n, it is an **attractive two-cycle** since iterates x_n produced by the process $x_{n+1} = f_a(x_n)$ are attracted to it. A graphical depiction of this two-cycle is shown in the figure on the next page. The point p_1 is mapped to p_2 which is, in turn, mapped to p_1, etc.. Note that the square that is formed by this two-cycle encloses the repulsive fixed point $\bar{x} = \frac{3.2-1}{3.2} = 0.6875$.

If the parameter a decreases from 3.2 to 3, the box formed by the two-cycle decreases in size as p_1 and p_2 approach the value 3 at $a = 3$. Recall that the fixed point $\bar{x} = 3$ is neutral. For $a > 3$, the fixed point $\bar{x} = \frac{a-1}{a}$ is repulsive. It is as if the fixed point $\bar{x} = 3$ "gives birth" to a two-cycle as a increases from the value 3.

As the parameter a increases from the value 3, the elements of the two-cycle, p_1 and p_2, are observed to move farther and farther apart. At $a \cong 3.45$, the iterates no longer approach an oscillation between two numbers, but rather four values, as was observed in our initial numerical experiments



Graphical illustration of two-cycle $f(p_1) = p_2$, $f(p_2) = p_1$ for logistic map $f_a(x)$ with $a = 3.2$.

with the nonlinear population model.

In order to understand this behaviour, recall that the forward orbit of a point x under iteration by a function $f(x)$ can be written as

$$x_0 = x, \quad x_1 = f(x_0), \quad x_2 = f^2(x_0), \ldots\ x_n = f^n(x_0), \tag{1}$$

where $f^n(x)$ denotes the n-fold composition of the function f with itself. The "two-cycle" $\{p_1, p_2\}$ mentioned above is an example of a **periodic orbit**: The period of this cycle is two: If $f^n(x) = p_1$ for some $n \geq 0$, then $f^{n+1}(x) = p_2$, $f^{n+2}(x) = p_1$ so that $x_{n+2} = x_n$ for all n. (In this way, a fixed point $p = f(p)$ of f is trivially a periodic orbit with period 1.)

Note that if $f(p_1) = p_2$ and $f(p_2) = p_1$, then $f(f(p_1)) = f(p_2) = p_1$. In other words, p_1 **is a fixed point of the map** $g(x) = f^2(x)$. In the same way, p_2 is also a fixed point for $g(x) = f^2(x)$ since $g(p_2) = f(f(p_2)) = f(p_1) = p_2$. (The function g represents a two-fold application of f on a point.)

We may generalize this idea as follows: Let $\{p_1, p_2, \ldots, p_N\}$ be distinct points such that

$$f(p_1) = p_2, \quad f(p_2) = p_3, \ldots, f(p_{N-1}) = p_N, \quad f(p_N) = p_1. \tag{2}$$

Then the set $\{p_1, p_2, \ldots, p_N\}$ comprises **a periodic orbit of period N, or an "N-cycle".** (We are assuming that this cycle is "indecomposable", i.e. that "N" is the smallest integer for which such a periodic orbit exists. For example, if $N = 6$ and $p_1 = p_4$, $p_2 = p_5$, $p_3 = p_6$, then we really don't have a 6-cycle but a 3-cycle.) Each of the points $p_1, p_2, \ldots p_N$ is a fixed point of the map $g(x) = f^N(x)$, i.e. $g(p_1) = p_1, \ldots, g(p_N) = p_N$.

We are now in a position to be able to understand how periodic orbits $\{p_1, p_2, \ldots p_N\}$ can be attractive or repulsive, as is the case for fixed points (which are periodic orbits of period 1). For a two-cycle, $\{p_1, p_2\}$, we need to consider the mapping $g(x) = f^2(x) = f(f(x))$. Since $g(p_1) = p_1$ and

$g(p_2) = p_2$, we need to examine the multipliers $g'(p_1)$ and $g'(p_2)$. If $|g'(p_1)| < 1$, then we expect iterates $g^n(x)$ to be attracted to p_1. Of course, as these points get closer to p_1, we expect the points "in between" (since $g = f^2$) to get closer to p_2. Therefore, if $|g'(p_1)| < 1$, we expect $|g'(p_2)| < 1$ as well. Likewise, if p_1 is a repulsive fixed point of g, then we expect p_2 to be repulsive as well. In fact, we may compute the multipliers of g in terms of multipliers of f as follows: Since $g(x) = f(f(x))$, we have, by the Chain Rule

$$g'(x) = f'(f(x))f'(x). \tag{3}$$

For $x = p_1$, $f(x) = p_2$ so that $g'(p_1) = f'(p_2)f'(p_1)$. For $x = p_2$, $f(x) = p_1$ so that $g'(p_2) = f'(p_1)f'(p_2)$. In summary, for a two-cycle $\{p_1, p_2\}$ of f, we have

$$\boxed{g'(p_1) = g'(p_2) = f'(p_1)f'(p_2).} \tag{4}$$

In general, for an N-cycle $\{p_1, p_2, \ldots, p_N\}$ of f, we have, for $g = f^N$,

$$g'(p_i) = f'(p_1)f'(p_2)\ldots f'(p_N), \quad 1 \le i \le N. \tag{5}$$

Let us now return to the logistic map $f_a(x) = ax(1-x)$. We first look for values of a for which two-cycles $\{p_1, p_2\}$, $p_1 \neq p_2$, can exist:

$$\begin{aligned} ap_1(1-p_1) &= p_2 \qquad (f(p_1) = p_2), \\ ap_2(1-p_2) &= p_1 \qquad (f(p_2) = p_1). \end{aligned} \tag{6}$$

There are a number of ways to achieve our goal. Here is one method. First we subtract the second equation from the first to obtain,

$$a(p_1 - p_2) - a(p_1^2 - p_2^2) = p_2 - p_1\,. \tag{7}$$

We see that $p_1 = p_2$ satisfies this equation. But that would imply that our two-cycle is, in fact, a one-cycle, i.e., a fixed point. So we assume that $p_1 \neq p_2$ and divide both sides of this equation by $p_1 - p_2$ to obtain

$$a - a(p_1 + p_2) = -1\,. \tag{8}$$

A rearrangement of this equation produces the result,

$$p_1 + p_2 = \frac{1+a}{a}. \tag{9}$$

If we write p_2 in terms of p_1 (or vice versa) and substitute into either of the above equations, we obtain quadratic equations in the p_i with roots

$$\boxed{p_{1,2} = \frac{a+1}{2a} \pm \frac{1}{2a}\sqrt{(a-1)^2 - 4}.} \tag{10}$$

For real roots, we must have $(a-1)^2 - 4 \geq 0$, which implies that $a \geq 3$. Note that at $a = 3$, $p_1 = p_2 = \frac{2}{3}$. **This is precisely the value of a beyond which the fixed point $\bar{x}_2(a) = \frac{a-1}{a}$ becomes repulsive.**

Now we know that two-cycles exist for $a > 3$. It remains to determine the a-values, if any, for which such two-cycles are attractive or repulsive. Since $f_a'(x) = a - 2ax$, we have, after a little work,

$$\begin{aligned} f'(p_1)f'(p_2) &= a^2[1-2p_1][1-2p_2] \\ &= a^2\left[1 - \frac{2(1+a)}{a} + \frac{4(1+a)}{a^2}\right] \\ &= -a^2 + 2a + 4. \end{aligned} \tag{11}$$

Thus, the two-cycle $\{p_1, p_2\}$ will be attractive for a-values such that $|g'(p_1)| = |g'(p_2)| < 1$,

$$|-a^2 + 2a + 4| < 1 \quad \Longrightarrow \quad -1 < -a^2 + 2a + 4 < 1\,. \tag{12}$$

We may consider this inequality as being satisfied by those positive a-values (recall that we do not consider $a < 0$) for which the graph of the function,

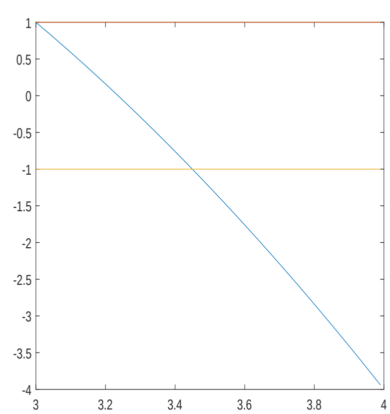
$$h(a) = -a^2 + 2a + 4\,, \tag{13}$$

an "upside down" parabola, lies between the lines $y = 1$ and $y = -1$. First of all, since

$$h'(a) = -2a + 2\,, \tag{14}$$

we see that $h'(1) = 0$ and $h(a)$ is increasing for $a < 1$ and decreasing for $a > 1$, i.e., $h(a)$ achieves a global maximum value of 5 at $a = 1$. But let's also recall our earlier result that two-cycles exist for $a > 3$. So we only need to consider the behaviour of $h(a)$ for $a \geq 3$.

A plot of the graph of $h(a)$ vs. a for $3 \leq a \leq 4$ is shown in the figure below along with the lines $y = 1$ and $y = -1$.



Plot of $h(a) = -a^2 + 2a + 4$ for $3 \leq a \leq 4$.

We are clearly interested in the values of a for which $h(a) = 1$ and $h(a) = -1$. From the graph, $h(3) = 1$. We now solve for a such that $h(a) = -1$. From the graph, it looks like this value of a will lie between 3.4 and 3.5:

$$h(a) = -a^2 + 2a + 4 = -1 \quad \Longrightarrow \quad a^2 - 2a - 5 = 0\,. \tag{15}$$

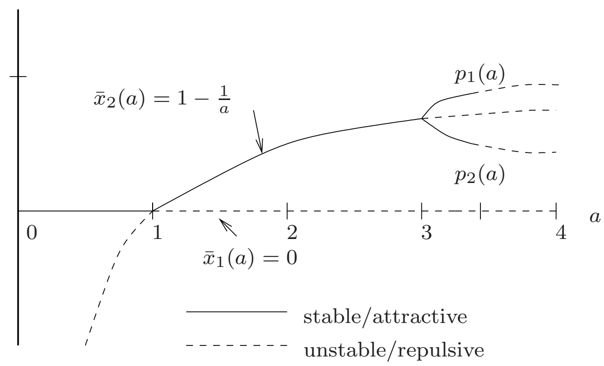
The roots of this quadratic are $a = 1 \pm \sqrt{6}$. Only the positive root is acceptable, namely,

$$a = 1 + \sqrt{6} \cong 3.44948\,. \tag{16}$$

We therefore conclude that the two-cycle $\{p_1, p_2\}$ for the logistic map $f_a(x) = ax(1-x)$ is attractive, or stable, for

$$3 < a < 1 + \sqrt{6} \cong 3.44948.$$

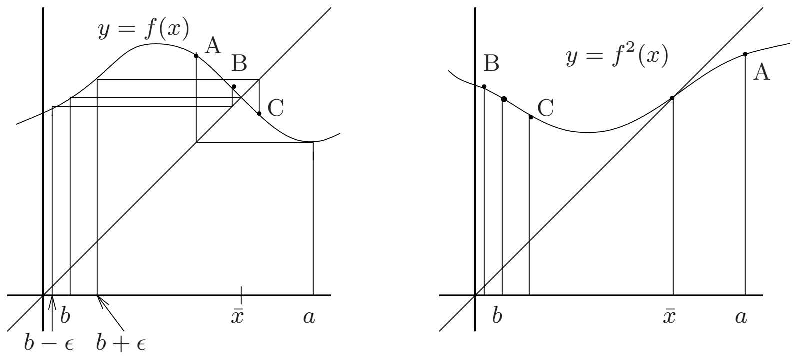
At $a = 1 + \sqrt{6}$, the two-cycle is neutral. For $a > 1 + \sqrt{6}$, the two-cycle is repulsive. We may now update the diagram from the previous lecture with this new information:



At $a = 1 + \sqrt{6}$, iterates of the logistic equation (assuming $x_0 \neq 0$) will still approach the neutral two-cycle $\{p_1, p_2\}$.

Naturally, the next question is, "What happens to iterates for $a > 1 + \sqrt{6}$?" We have already mentioned that numerical experiments indicate that an attractive four-cycle or periodic orbit of period 4 appears. One could try to perform an analysis similar to above to find a-values for which 4-cycles could occur, and then to determine their stability characteristics. However, this is quite complicated since we are working with **four** equations in the unknowns p_1, p_2, p_3, p_4. No closed-form expressions exist for the p_i. It turns out that a deeper analysis of the transition, or **bifurcation**, that occurs at $a = 3$, from a stable 1-cycle to a stable 2-cycle, will give us an idea of what goes on at $a = 1 + \sqrt{6}$. We perform such an analysis, in terms of the graphs of $f_a(x)$ and $g_a(x) = f_a^2(x)$, below.

Given a function $f(x)$, we can obtain an idea of what the graph of its iterate $g(x) = f^2(x)$ looks like from the graph of f: For every point x, we find $y = f(x)$, then travel horizontally to the line $y = x$ and then input $f(x)$ into f to obtain $f^2(x)$. This is illustrated in the figure below. Clearly, fixed points of f, $\bar{x} = f(\bar{x})$, are fixed points of $g(x)$. The point $x = a$ gets mapped, after two applications of f, to $g(a) = f^2(a)$. The point $x = b$ has been chosen so that $f(b) = \bar{x}$, i.e. b is a preperiodic point of $\bar{x}$. A point just to the right of $x = b$, say $x = b + \varepsilon$, will be mapped to a value $f(b+\varepsilon) > f(b) = \bar{x}$. However, from the graph, $f(f(b+\varepsilon)) < \bar{x}$. The reader is encouraged to examine other x values in this example.



Lecture 16

Dynamics of the logistic map (cont'd)

The appearance of two-cycles (cont'd)

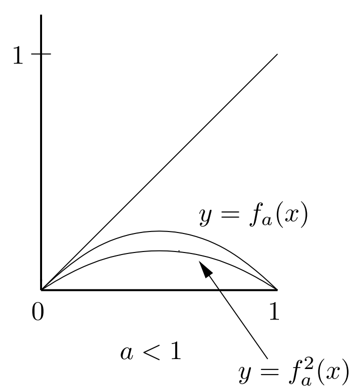
We continue our discussion from the previous lecture. In this lecture, we shall analyze the logistic map $f_a(x) = ax(1-a)$ with the help of the graphs of f_a and various iterates plotted on Pages 171 and 172.

Reader alert: Some parts of this discussion will be more detailed and technical than usual in order to provide a reasonable analysis of what is going on. There is no need to worry! The purpose of this course is not to analyze everything in great detail, but only to get an idea of what is going on "behind the scenes" when the fixed point $\bar{x}_2$ ceases to be attractive and "gives birth" to an attractive two cycle (p_1, p_2).

At the end of the last lecture, we mentioned that it will be helpful to understand the structure of the graph of $f_a^2(x) = f(f(x))$ in order to explain the appearance of two-cycles (p_1, p_2). Recall that each of the components of a two-cycle is a fixed point of $f^2(x)$, i.e., if $f(p_1) = p_2$ and $f(p_2) = p_1$, it follows that $f^2(p_1) = p_1$ and $f^2(p_2) = p_2$.

First of all, let us comment that for $0 < a < 2$, the graph of $f_a^2(x)$ is not fundamentally different from the graph of $f_a(x)$. Let's examine this in more detail for particular ranges of the parameter a.

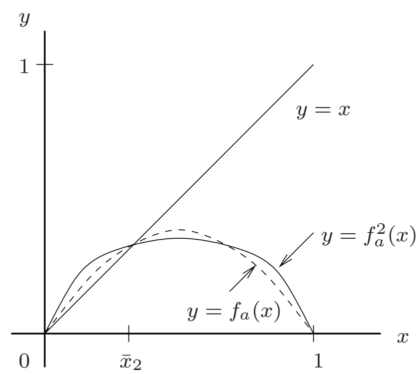
Case 1: $0 < a \leq 1$. In this case, $f_a(x) < x$, so that $\bar{x}_1 = 0$ is the only fixed point. Furthermore, $f_a^2(x) < f_a(x)$ for all $x \in (0, 1]$, implying that the graph of f_a^2 is obtained by somewhat "squashing down" the graph of of f_a, as shown below.



Sketch of graphs of $f_a(x)$ and $f_a^2(x)$ for $0 < a < 1$.

Case 2: $1 < a < 2$. In this case, the second fixed point $\bar{x}_2(a) = \frac{a-1}{a}$ appears in the interval $\left(0, \frac{1}{2}\right)$. It will, of course, be a fixed point of f_a^2 as well. Since $f_a(x)$ is increasing on $(0, \bar{x}_2(a))$ and $f_a(x) > x$, it follows that $f_a^2(x) > f_a(x)$ for $x \in (0, \bar{x}_2(a))$. Thus, the graph of f_a^2 is obtained by "pulling up" the graph of f_a between $x = 0$ and $x = \bar{x}_2(a)$.

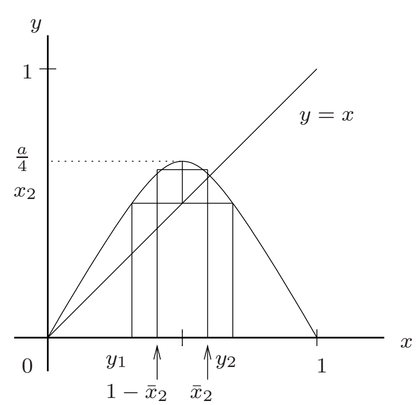
For $x \in \left(\bar{x}_2(a), \frac{1}{2}\right)$, $f_a(x) < x$ so that $f_a^2(x) < f_a(x)$, i.e. we once again "push down" on the graph of f_a to obtain the graph of f_a^2. All of the above behaviour is mirrored on the other half of the interval, i.e. $x \in \left(\frac{1}{2}, 1\right)$ since $f_a(x) = f_a(1-x)$. The net result is shown schematically in the figure below.



Sketch of graph of $f_a^2(x)$ for $1 < a < 2$.

Case 3: $a = 2$. In this case, the fixed point $\bar{x}_2 = \frac{1}{2}$ coincides with the maximum value of $f_2(x)$ on $[0, 1]$. At this point, the graph of f_a^2 – shown on Page 171, middle right – is quite flat at $x = \frac{1}{2}$.

Case 4: $2 < a < 3$. In this case, $\bar{x}_2 > \frac{1}{2}$ so that $f_a\left(\frac{1}{2}\right) > f_a(\bar{x}_2) = \bar{x}_2$. In other words, there is a piece of the graph of $f_a(x)$ that lies above the fixed point value $\bar{x}_2$. More precisely, $f_a(x) > \bar{x}_2$ for all $x \in \left(\bar{x}_2 - \frac{1}{2}, \bar{x}_2\right)$: see the figure immediately below.

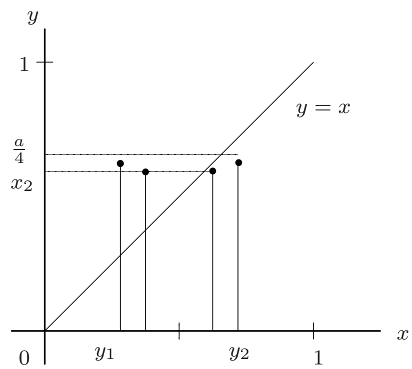


Sketch of graph of $f_a(x)$ for $2 < a < 3$.

There are some serious consequences for the graph of $f_a^2(x)$. Firstly, note that $f_a\left(\bar{x}_2 - \frac{1}{2}\right) = f_a(\bar{x}_2) = \bar{x}_2$. It follows that $f_a^2\left(\bar{x}_2 - \frac{1}{2}\right) = f_a^2(\bar{x}_2) = \bar{x}_2$. Therefore, we know two points on the graph of $f_a^2(x)$.

(We also know that $f_a^2(0) = f_a^2(1) = 0$.)

Now, we know that $f_a\left(\frac{1}{2}\right) = \frac{a}{4}$, the absolute maximum value of $f_a(x)$ on $[0,1]$. We ask the question: For what value(s) of x does $f_a^2(x) = \frac{a}{4}$? Answer: The values of x such that $f_a(x) = \frac{1}{2}$, i.e. the "preimages" of $x = \frac{1}{2}$. There are two such points, y_1 and y_2, which we have shown on the above figure. So far, we have the following information:

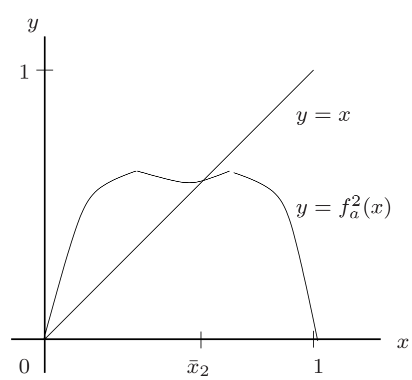


It now remains to join these points somehow in order to obtain the graph of $f_a^2(x)$. A few observations, using graphical methods, will help:

1) $f_a\left(\frac{1}{2}\right) = \frac{a}{4}$ and $f_a\left(\frac{a}{4}\right) < \bar{x}_2 \Rightarrow f_a^2\left(\frac{1}{2}\right) < \bar{x}_2$. Moreover, if we move slightly away from $x = \frac{1}{2}$, $f_a^2\left(\frac{1}{2} \pm \varepsilon\right) > f_a^2\left(\frac{1}{2}\right)$, implying that $x = \frac{1}{2}$ is a local minimum of $f_a^2(x)$.

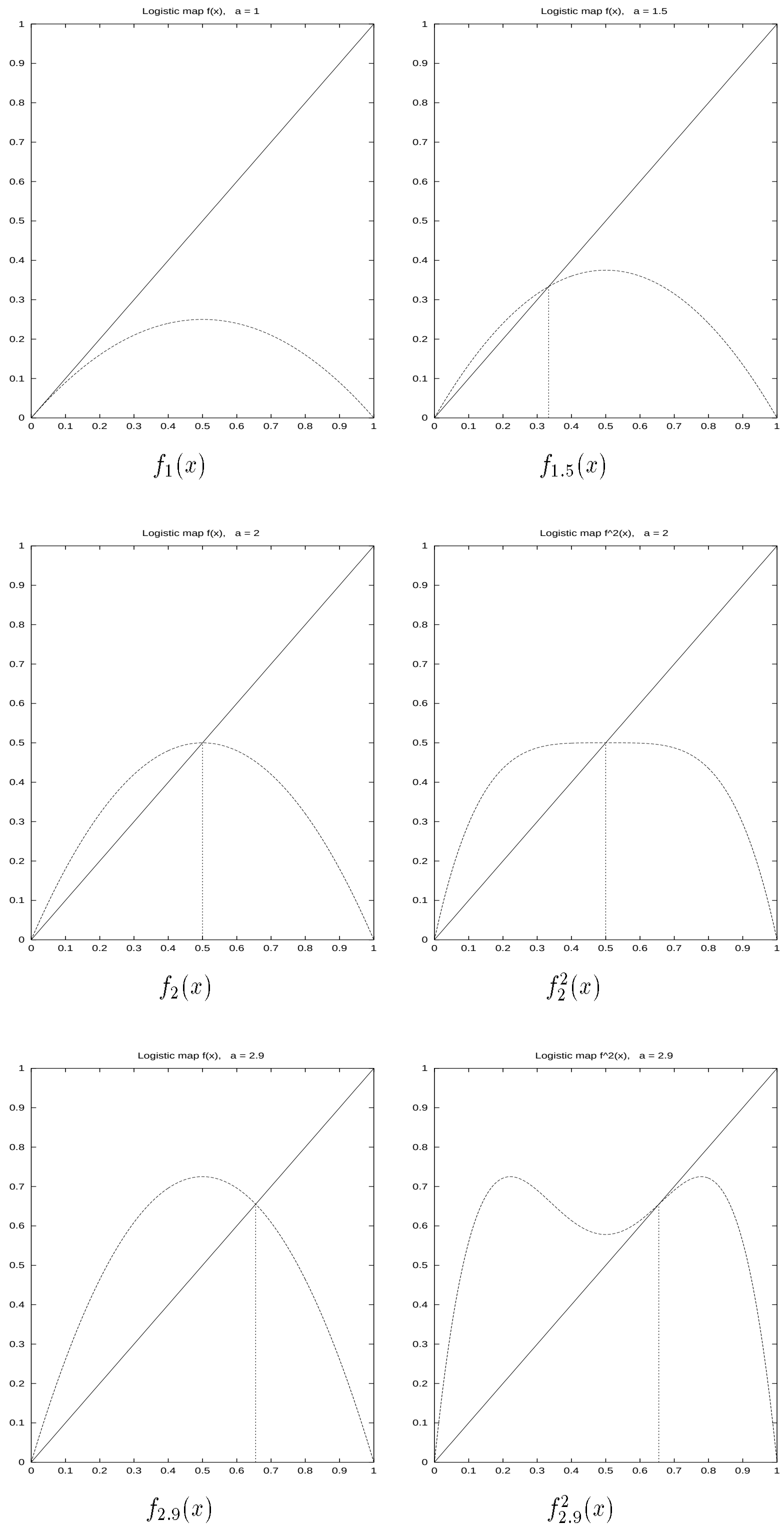
2) Consider y_1, where $f_a(y_1) = \frac{1}{2}$. Since $x = \frac{1}{2}$ is a local maximum of $f_a(x)$, moving slightly away from y_1, i.e. $x = y_1 \pm \varepsilon$ will cause us to move slightly away from the max value $\frac{1}{2}$. Hence, $f_a^2(y_1 \pm \varepsilon) < f_a^2(y_1) = \frac{a}{4}$. Thus, $x = y_1$ and $x = y_2$ are local maxima of $f_a^2(x)$.

We now have enough information to fill in the details to obtain a rough idea of the graph of $f_a^2(x)$:

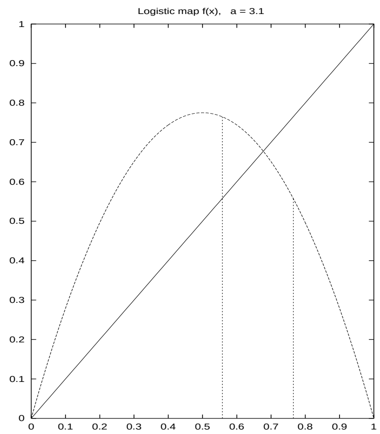


Note that $|f_a^{2\prime}(\bar{x}_2)| < 1$ for $1 < a < 3$.

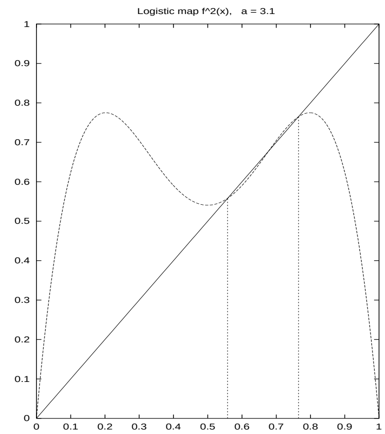
As a increases from 2 to 3, the minimum of $f_a^2(x)$ at $x = \frac{1}{2}$ moves further downward and the maxima $f_a(f_a(y_1)) = f_a(f_a(y_2)) = \frac{a}{4}$ move upward. The result: the slope of $f_a^2(\bar{x}_2)$ approaches the value 1, as can be seen in the case $a = 2.9$ on the next page, bottom right.



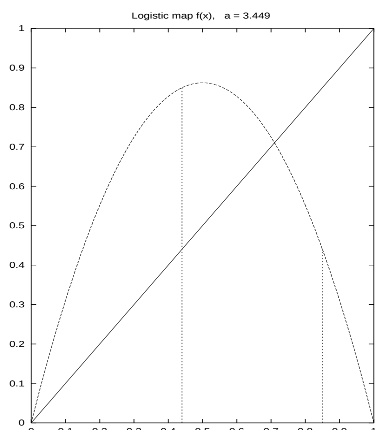
Graphs of $f_a(x)$ and $f_a^2(x)$, showing fixed points.



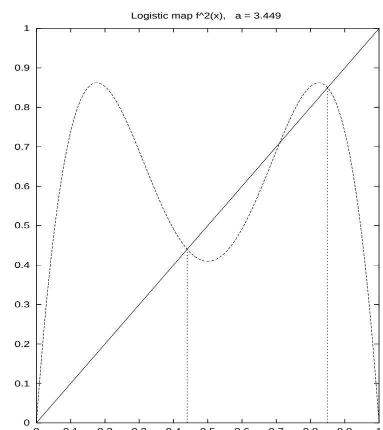
$f_{3.1}(x)$



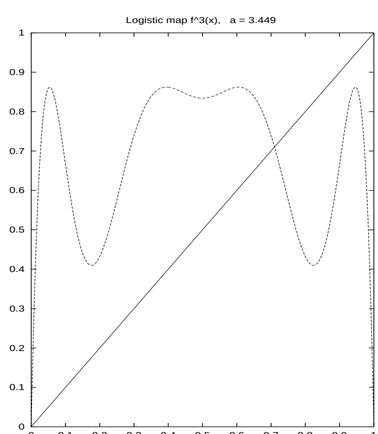
$f_{3.1}^2(x)$



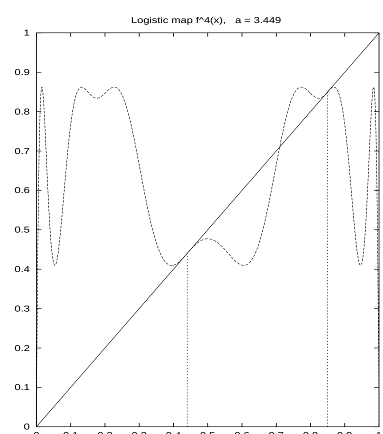
$f_{3.449}(x)$



$f_{3.449}^2(x)$



$f_{3.449}^3(x)$



$f_{3.449}^4(x)$

Graphs of $f_a(x)$ and various iterates, showing two-cycles.

Case 5: $a = 3$. In this special case, the second fixed point $\bar{x}_2 = \frac{2}{3}$ and its multiplier is $f_a'(\bar{x}_2) = -1$. This implies that $(f_a^2)'(\bar{x}_2) = 1$. (Why?) In other words, **the fixed point $\bar{x}_2(a) = \frac{2}{3}$ is neutral, with respect to iteration of f_a as well as f_a^2.**

Case 6: $a = 3 + \varepsilon$, where $0 < \varepsilon \ll 1$. In this case, $f_a'(\bar{x}_2) > 1$ so that $(f_a^2)'(\bar{x}_2) > 1$. The result, as can be seen in the case $a = 3.1$, is that the graph of $f_a^2(x)$ must intersect the line $y = x$ at two additional points $p_1 < \bar{x}_2$ and $p_2 > \bar{x}_2$. **Note that these two points, p_1 and p_2 are not fixed points of $f_a(x)$. They must be components of a two-cycle (p_1, p_2): $f_a(p_1) = p_2$, $f_a(p_2) = p_1$.** As we computed earlier, this two-cycle is attractive for $3 < a < 1 + \sqrt{6} \cong 3.449$.

In summary, the behaviour of the graph of $f_a^2(x)$ for values of a running from $2 < a < 3$ to $a = 3$ and on to $a = 1 + \sqrt{6}$ explains the appearance of a two-cycle (p_1, p_2) that is attractive for $a \in (3, 1 + \sqrt{6})$. For these a-values, **most** iteration sequences $x_{n+1} = f_a(x_n)$ will approach these attractive two-cycles. The word "most" is emphasized, since **not all sequences will approach the two-cycle.** For example, if $x_0 = 0$, then $x_n = 0$. As well, if $x_0 = 1$, then $x_n = 0$ for $n \geq 1$. There are also the **preperiodic** points that map, in a finite number of iterations, to the repulsive fixed point $\bar{x}_2 = \frac{a-1}{a}$.

A natural question to ask is, "What happens for $a \geq 1 + \sqrt{6}$?" At $a = 1 + \sqrt{6}$, the two-cycle is neutral but it still attracts neighbouring iterates. What has happened is that the values of the derivatives $(f^2)'(p_1)$ and $(f^2)'(p_2)$ have changed in value from $+1$ at $p_1 = p_2 = \frac{2}{3}$ when $a = 3$ to -1 at p_1 and p_2 when $a = 1 + \sqrt{6}$. Remarkably, as shown in the plots on the next two pages, **it appears as if $f_a^2(x)$ is behaving at p_1 and p_2 as $a \to 1 + \sqrt{6}$ in the same way as $f_a(x)$ behaved at $\bar{x}_2$ as a approached the value 3.** A look at the graph of $h(x) = f_a^2(f_a^2(x)) = f_a^4(x)$ shows that this is, indeed, the case. At $x = p_1$ and $x = p_2$ we now have scaled-down copies of the same "curling up" behaviour that f_a^2 exhibited at $a = 3$. The result is a birth of two new points at each of p_1 and p_2, implying the creation of a four-cycle of points (q_1, q_2, q_3, q_4) such that $f(q_1) = q_2$, $f(q_2) = q_3$, $f(q_3) = f(q_4)$, $f(q_4) = q_1$. Numerical experiments have shown that this four-cycle is stable until $a \cong 3.544$. The process repeats itself and at $a \cong 3.544$, an eight-cycle is born.

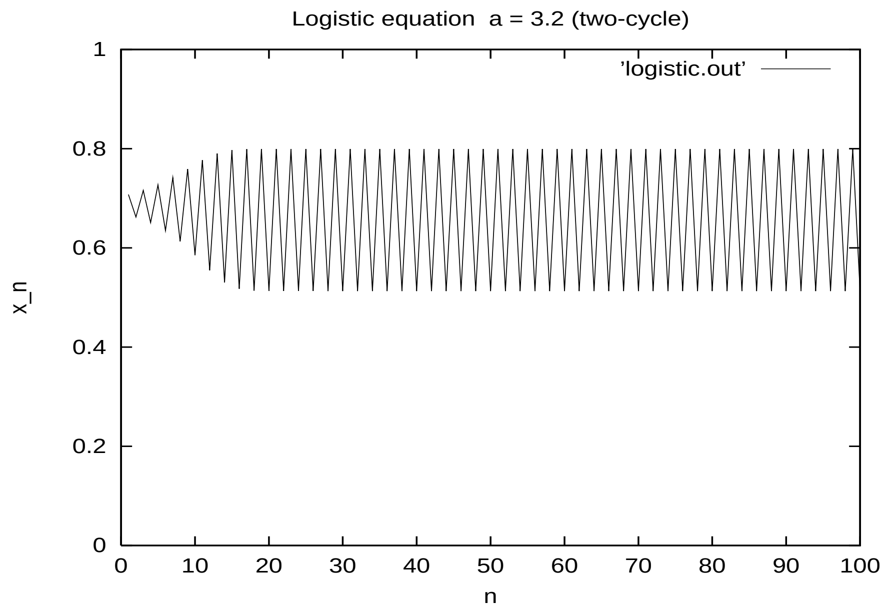
On the next two pages are presented plots of iteration sequences $x_{n+1} = ax_n(1 - x_n)$, for several values of a to illustrate the period-doubling phenomenon. In all cases the starting point of the iteration sequence was $x_0 = \frac{1}{3}$. The values of a are:

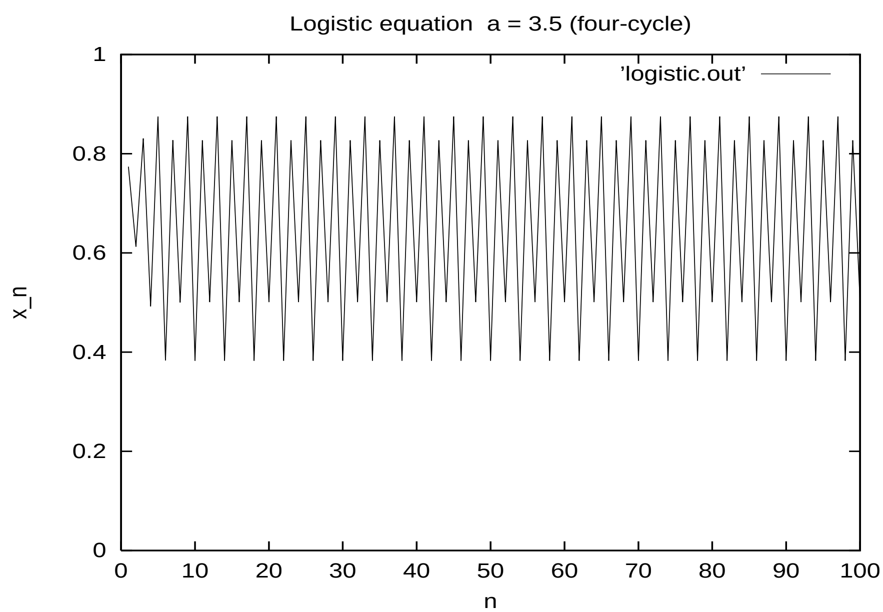
i) $a = 3.2, \quad x_n \to$ 2-cycle,

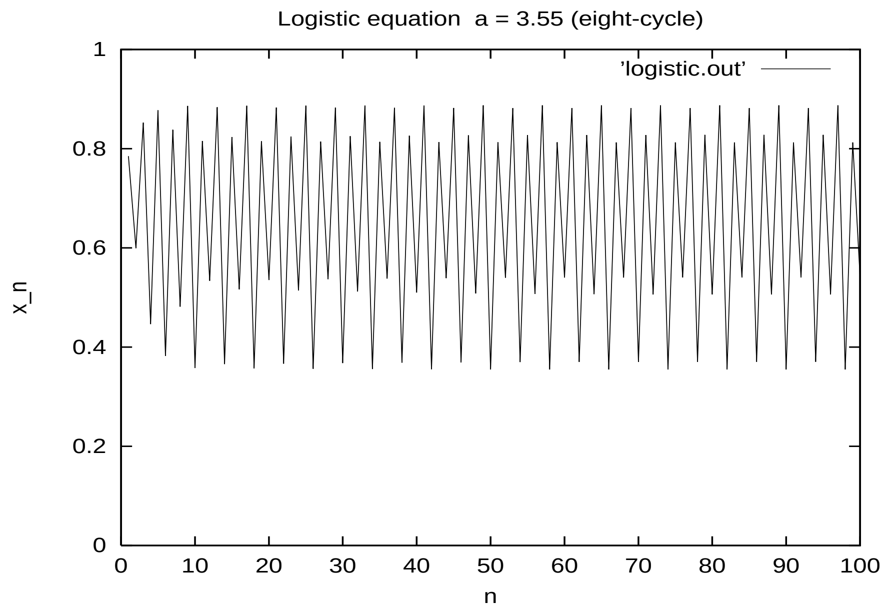
ii) $a = 3.5, \quad x_n \to$ 4-cycle,

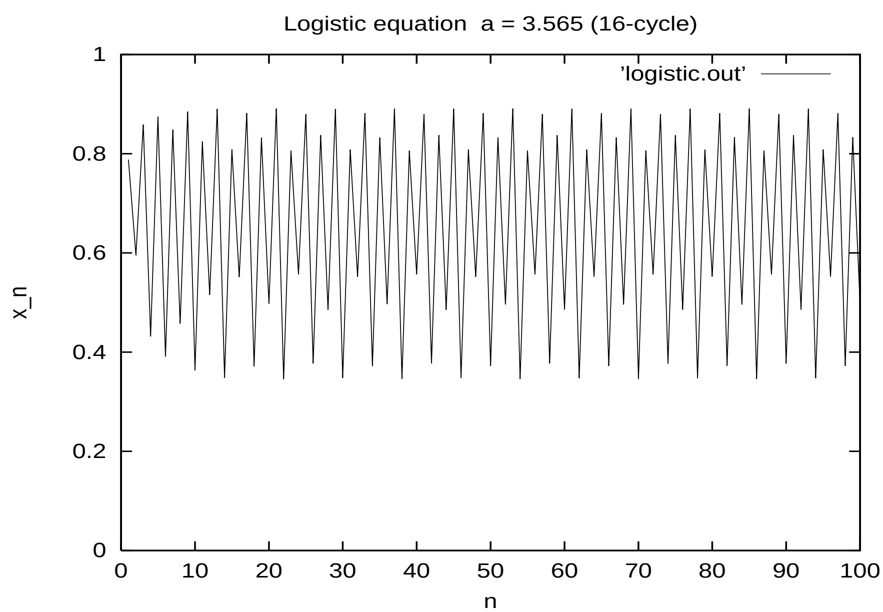
iii) $a = 3.55, \quad x_n \to$ 8-cycle,

iv) $a = 3.565, \quad x_n \to$ 16-cycle.


Logistic equation a = 3.2 (two-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100


Logistic equation a = 3.5 (four-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100


Logistic equation a = 3.55 (eight-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100


Logistic equation a = 3.565 (16-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100

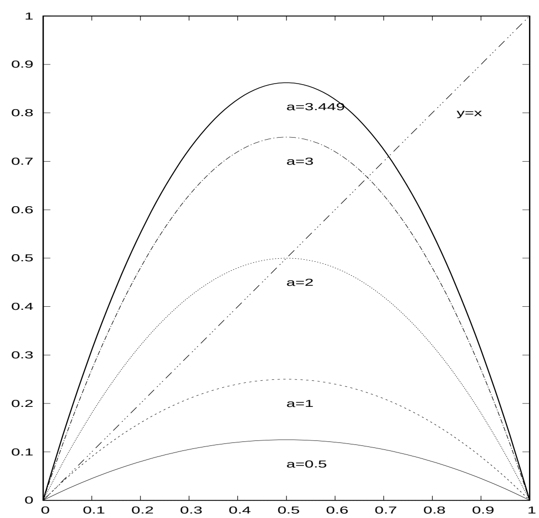
Lecture 17

Dynamics of the logistic map (cont'd)

Bifurcations and the "period-doubling route to chaos"

(Note: Portions of the material below were presented in the previous lecture but then reviewed in this lecture. As such, they have been placed here in one coherent form.)

Let us now quickly summarize what we have found so far for the logistic map $f_a(x) = ax(1-x)$ and the associated iteration procedure $x_{n+1} = f_a(x_n)$ with reference to the following figure showing the graphs of $f_a(x)$ for relevant values of a:



1) For $0 < a < 1$, the fixed point $\bar{x}_1 = 0$ is attractive or stable, and all iterates $x_n \to \bar{x}_1$ as $n \to \infty$.

2) At $a = 1$, $\bar{x}_1 = 0$ is neutral. Nevertheless, all iterates $x_n \to 0$ as $n \to \infty$.

3) For $1 < a < 3$, the fixed point $\bar{x}_2(a) = \frac{a-1}{a}$ is attractive, with $x_n \to \bar{x}_2(a)$ as $n \to \infty$.

4) At $a = 3$, $\bar{x}_2 = \frac{2}{3}$ is neutral. Nevertheless, all iterates $x_n \to \bar{x}_2$ as $n \to \infty$.

5) For $3 < a < 1 + \sqrt{6} \cong 3.449$, the two-cycle $(p_1(a), p_2(a))$ computed in the previous section is attractive.

The a-values $0, 1, 3$ and $1 + \sqrt{6}$ represent **bifurcation points** of the discrete dynamical system $x_{n+1} = f_a(x_n)$. **Bifurcations** are changes in the dynamical behaviour of a system as one or more parameters are varied. These changes are normally due to the changes in stability properties of fixed points or periodic orbits, e.g. from attractive/stable to repulsive/unstable. Such changes in stability may be accompanied by the appearance of new periodic orbits which may "take over" in the attraction of iterates x_n.

Let us examine the bifurcations that take place with the logistic map $f_a(x)$ as the parameter "a" is allowed to vary from $a = 0$. We can think of "a" as a tunable parameter – once we lock in on a desired a-value, we than "turn on" the iteration process $x_{n+1} = f_a(x_n)$ and observe the long-term behaviour of the iterates x_n.

For $0 < a < 1$, the fixed point $\bar{x}_2(a) = \frac{a-1}{a} < 0$ and plays no role in the dynamics of the iteration process on $[0, 1]$. The fixed point $\bar{x}_1 = 0$ is stable and attracts all iteration sequences $\{x_n\}$ as $n \to \infty$. At $a = 1$, $\bar{x}_2(1) = 0$ "collides" with $\bar{x}_1 = 0$. The single fixed point $\bar{x} = 0$ is neutral. As "a" increases past 1, the fixed point $\bar{x}_2 = \frac{a-1}{a}$ is now attractive and $\bar{x}_1 = 0$ is unstable. It is as if the "collision" caused the two fixed points to exchange their stability characteristics. The point $a = 1$ is a bifurcation point. (In technical jargon, it is a "transverse bifurcation point".) At $a = 3$, the fixed point $x_2 = \frac{a-1}{a}$ ceases to be attractive. As "a" increases past 3, an attractive two-cycle (p_1, p_2) is created; $\bar{x}_2(a) = \frac{a-1}{a}$ is now repulsive. The point $a = 3$ is a "period-doubling bifurcation point" (since we go from a period one orbit – a fixed point – to a period two orbit – a two-cycle). This phenomenon is also called a "pitchfork bifurcation" because of the curves traced out by the fixed point $\bar{x}_2(a)$ and the two-cycle as we saw in the plot in Lecture 15 on Page 166. At $a = 1 + \sqrt{6}$, the two-cycle ceases to be stable and bifurcates (period-doubling again) into a four-cycle.

What is now remarkable is that this period-doubling behaviour continues: from 2- to 4-cycle, 4- to 8-cycle, 8- to 16-cycle,etc. A great deal of theoretical and computational work has been done for this apparently simple iteration process. On the next page, the asymptotic behaviour of iterates $x_{n+1} = f_a(x_n)$ is plotted as a function of a. For a fixed value of a, say $a = K$, the intersection of the line $a = K$ and the points will give the points in $[0, 1]$ to which the iterates $\{x_n\}$ are attracted. This is a completion of the figure on Page 166, with the exception that the repulsive periodic points are omitted. The figure on the right is a blow-up of the region $3.4 \le a \le 4$ to give the reader an idea of the complexity - and beauty - of this process.

Below the figures are listed numerical values of the bifurcation points a_k at which 2^{k-1}-cycles give birth to 2^k-cycles. Early numerical experiments suggested that the bifurcation points a_k approach a limiting value "a_∞" as $k \to \infty$. In fact, an examination of the first differences $\Delta_k = a_{k+1} - a_k$ and ratios of first differences $R_k = \Delta_{k+1}/\Delta_k$ (some values are shown in the table) suggested that the a_k were approaching a_∞ geometrically: If

$$\frac{\Delta_{k+1}}{\Delta_k} = \frac{a_{k+1} - a_k}{a_k - a_{k-1}} \to r < 1 \quad \text{as} \quad k \to \infty$$

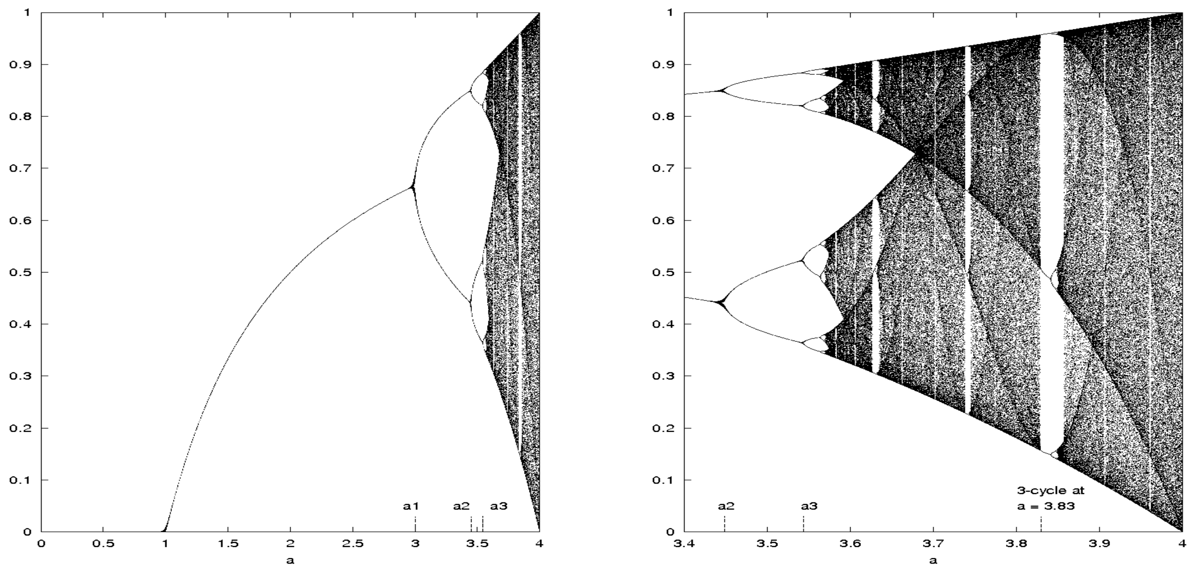
then

$$a_{k+n} - a_{k+n-1} \simeq (a_{k+1} - a_k) r^n.$$

The ratio $\delta = \frac{1}{r} \cong 4.6692$ is referred to as "Feigenbaum's constant". It is now known that

$$\lim_{k \to \infty} a_k = a_\infty \cong 3.5699456.$$

Asymptotic Behaviour of Iteration Sequences $x_{n+1} = ax_n(1 - x_n)$ vs. a

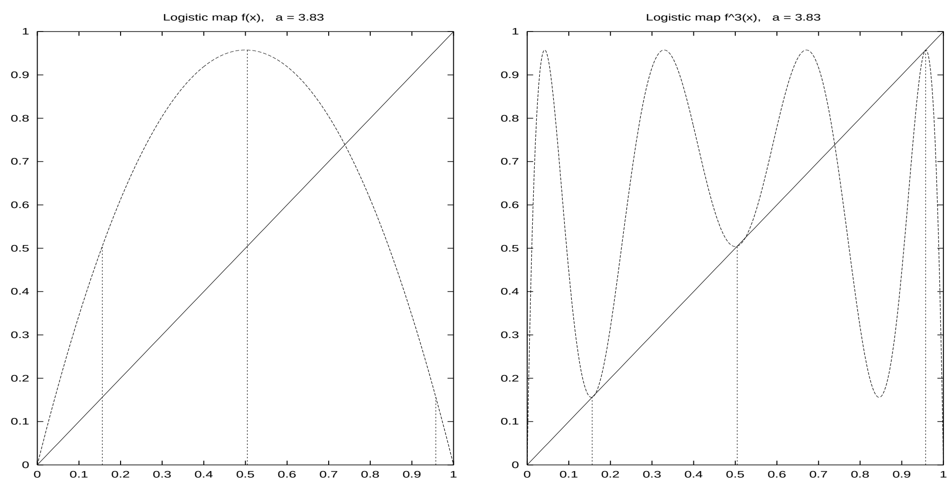


Finite-Difference Analysis of Bifurcation Points a_k

k	a_k	$\Delta_k = a_{k+1} - a_k$	$r_k = \Delta_k / \Delta_{k+1}$
1	3		
2	3.449499	0.449499	4.752
3	3.544090	0.094591	4.656
4	3.564407	0.020317	4.668
5	3.568759	0.004352	4.449
6	3.569692	0.000932	4.669
7	3.569891	0.000200	4.669
8	3.569934	0.000043	

A natural question is, “What happens at $a = a_\infty$?” The answer is that the iterates x_n are attracted to a “fractal, Cantor-like” subset of $[0, 1]$. Very briefly, such a subset is “full of holes” in the sense that it has no intervals of the form $[a_k, b_k]$. Nevertheless, it still has a lot of points – an **uncountable infinity** of points, to be precise. (A set such as $S = \left\{ \frac{1}{n} \,|\, n \geq 1 \right\}$ has a countable infinity of points since they can all be put into one-to-one correspondence with the integers. The real line, and even the real interval $[0, 1]$, are uncountable – points in these sets cannot be put into one-to-one correspondence with the integers.) Thus, in an “intuitive” sense, a fractal set is a very “thick” set. We shall return to this idea in the next section.

Before closing we simply mention that some rather exotic dynamical behaviour is demonstrated by iteration sequences $x_{n+1} = f_a(x_n)$ for $a > a_\infty \cong 3.5699456$. For example, at $a \cong 3.83$, there is a quite prominent three-cycle that eventually undergoes period-doubling bifurcations to 6, then 12, then 24, $\cdots 3 \cdot 2^n$ cycles, up to a “$3 \cdot 2^\infty$” cycle. At various a-values, 5-cycles appear, only to undergo the same type of period doubling, i.e. $5 \to 10 \to 20 \to 40$ etc. A graph of $f_a(x)$ for $a = 3.83$ along with the 3-cycle is shown below.



Graphs of $f_a(x)$ and $f_a^3(x)$ for $a = 3.83$, showing the three-cycle (p_1, p_2, p_3). At each point p_i of the three-cycle, a vertical line is drawn from the x-axis to each graph. Note that each of the points p_i of the three-cycle are fixed points of the map $f_a^3(x)$.

As written earlier, volumes have been written on this dynamical system and related ones. In fact, some rather remarkable results show that there is a kind of “universality” property shared by one parameter families of mappings $g_c(x)$ of an interval, say $[0, 1]$ into itself. For $c_{\min} \leq c \leq c_{\max}$,

1) $g_c(0) = g_c(1) = 0$

2) $g_c(x)$ has a single (global) maximum at an interior point $p \in (0, 1)$

3) $g_c'(x) > 0$ for $x \in [0, p)$; $g_c'(x) < 0$ for $x \in (p, 1]$; $g_c'(p) = 0$, $g''(p)$ exists.

These maps are known as "unimodal maps". As the parameter c is increased, the maps exhibit period-doubling bifurcations, of 2^{k-1}- to 2^k- cycles for $k = 1, 2, 3, \ldots$. An example of such a family of non-quadratic unimodal maps is

$$g_c(x) = c \sin \pi x, \qquad x \in [0, 1],$$

for $c \in [0, 1]$. The reader may wish to explore the asymptotic behaviour of iteration sequences as the parameter c is varied from 0 to 1, in particular, locating the point c_1 where an attractive fixed point becomes neutral, giving rise to an attractive two-cycle. For such unimodal maps, the convergence of the bifurcation points $c_k \to c_\infty$ is geometric.

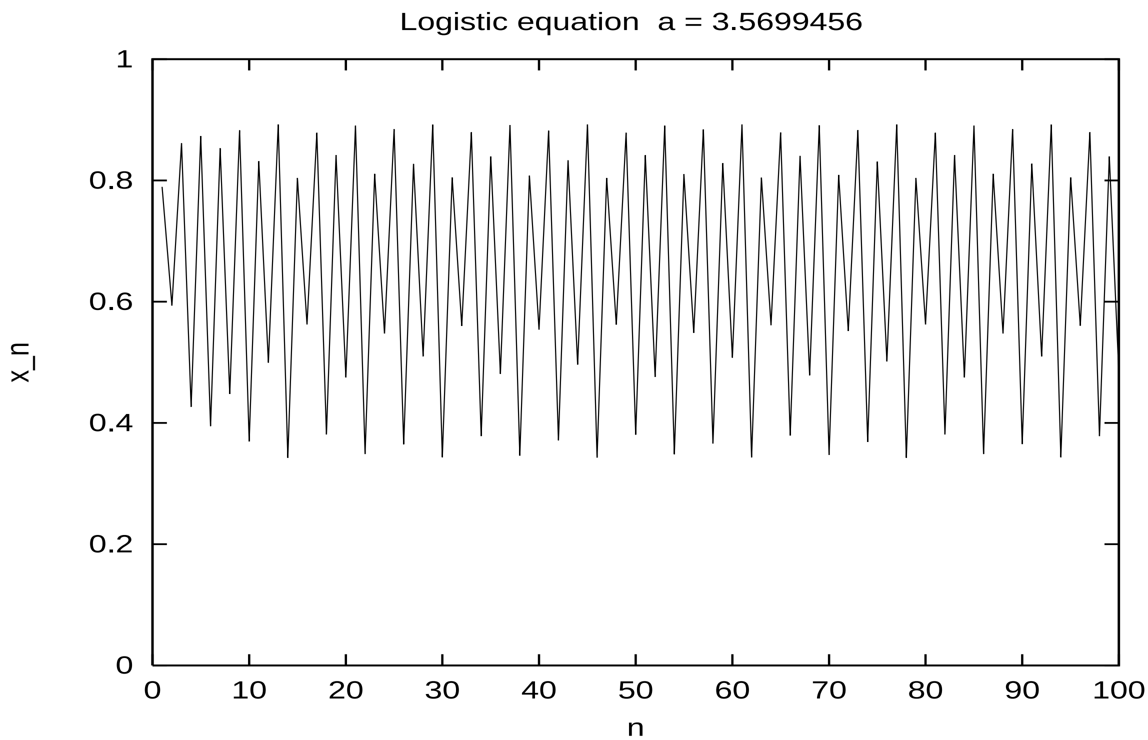
Behaviour at $a_\infty = 3.5699456$

We now return to the question of the behaviour of the iteration scheme $x_{n+1} = f_a(x_n)$ at the limit point $a_\infty = \lim_{k\to\infty} a_k$, where the a_k are bifurcation points at which 2^{k-1}-cycles give rise to 2^k-cycles. The limiting operation $\lim_{k\to\infty} a_k = a_\infty$ is accompanied by a procedure involving a "limit" of sets of points – here, the sets S_k, $k = 1, 2, 3, \ldots$, are composed of points that make up the periodic 2^k-cycle. Note that each set S_k is composed of a finite number of points. The "limiting" operation $\lim_{k\to\infty} S_k$ will involve a "closure" of subsequences of points that have limits. As a result, not only will the limiting set "S_∞" have an infinite number of points but S_∞ will have an **uncountable infinity** of points. In other words, S_∞ will have as many points as there are real numbers in the interval $[0, 1]$! We shall show how such a strange "fractal" set can be produced below.

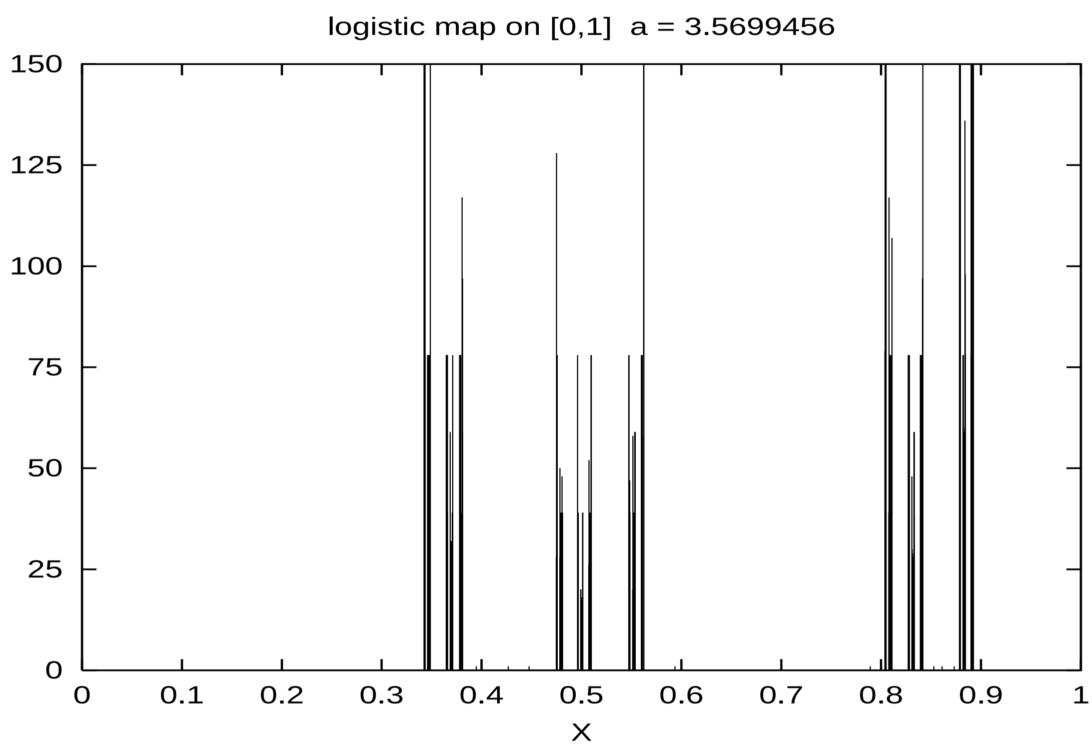
On the next page is plotted a typical iteration sequence $\{x_n\}$. Note that the "signal" x_n vs. n does not look periodic, yet it does not look far from being periodic. In fact, it can be shown that this motion is "chaotic" on the fractal attractor set S_∞. In an attempt to show the fractal nature of the set S_∞ to which the iterates x_n are attracted, the bottom figure on the next page is a "'histogram" plot of 10000 consecutive iterates x_n. The plot is useful to record both the location of iterates as well as the frequency at which they visit regions. 5000 bins representing the subintervals $I_k = [b_{k-1}, b_k)$, $k = 1, 2, \ldots 500$, where $b_k = k\Delta x$, $\Delta x = 1/5000$, were used. Initially, a bin-counting vector was set to zero, i.e. $\text{ibin}(i) = 0$, $1 \le i \le 5000$. Picking an x_0 value (here $x_0 = 1/3$), the iteration process $x_{n+1} = f_a(x_n)$ was begun. For each $n \ge 1$, the location of x_n, i.e. the bin I_m in which x_n fell ($m = \text{int}(x_n/\Delta x) + 1$) was determined and then the counter incremented by 1, i.e. ibin $(m) := \text{ibin}(m) + 1$.

The resolution of the histogram plot shown in the figure – 5000 bins – is still too course to reveal the fractal nature of the set S_∞ on which the x_n travel. Nevertheless, one may notice a kind of self-similar nesting of "gaps" in the set S_∞: The portion of S_∞ that lies in the region $0.47 < x < 0.56$ looks like a scaled down version of the entire set S_∞ lying in the region $0.33 < x < 0.89$.

In order to better understand the idea of a fractal set on the real line, we now introduce a classical example: the so-called "ternary Cantor set" on $[0, 1]$. The Cantor set is constructed by means of a "middle-thirds dissection" procedure. We begin with the unit interval $J_0 = [0, 1]$:

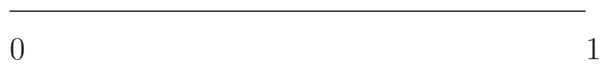


Plot of the first 100 iterates $x_{n+1} = f_a(x_n)$, $a = 3.5699456$



Histogram plot of 10000 iterates to show their distribution over [0,1]

J_0:



Now remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from this set to produce the set $J_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$:

J_1:

Repeat the procedure, removing the "middle-thirds" open sets $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from J_1 to produce the set $J_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$:

J_2:

Now continue this procedure to produce the sets $J_3, J_4, \ldots$. In the limit $k \to \infty$, the sets J_k converge (in a suitable fashion) to a limiting set, C, the ternary Cantor set on $[0, 1]$:

This set C is a fascinating set. It contains points that one may not have imagined to lie on this set, such as $x = \frac{1}{4}$. It also contains an uncountable infinity of points: The points in C can be put into a one-to-one correspondence with points on the real interval $[0, 1]$. This is quite surprising, especially in light of the fact that we have removed virtually all of the interval $[0, 1]$ in order to produce C. In fact, let us attempt to compute the "length" of the Cantor set C by examining the lengths of the sets J_n that were used to construct C:

Length of	$J_0 = [0, 1]$	is	$L_0 = 1$
Length of	$J_1 : 2 \times \frac{1}{3}$		$L_1 = \frac{2}{3}$
Length of	$J_2 : 4 \times \frac{1}{9}$		$L_2 = \frac{4}{9}$
	$\vdots$		
Length of	$J_n : 2^n \cdot \frac{1}{3^n}$		$L_n = \left(\frac{2}{3}\right)^n$

(Another way to obtain the above result is to note that the length of each set J_n is two-thirds the length of its predecessor J_{n-1} since one-third of the latter set was removed.) Note that

$$\lim_{n \to \infty} L_n = 0 \,. \tag{17}$$

In other words, the Cantor set has zero length! (We obtain the same result by computing the total length of intervals that have been removed from $[0, 1]$ in order to construct C. This length is

$$\begin{aligned}\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots &= \frac{1}{3}\left[1 + \frac{2}{3} + \frac{4}{9} + \cdots\right] \\ &= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} \\ &= 1.\end{aligned}$$

Thus, length of $C = 1 - 1 = 0$.)

The Cantor set also exhibits an interesting property of "self-similarity". The subset $C_1 \subset C$ that lies on the interval $\left[0, \frac{1}{3}\right]$, is an exact scaled-down copy of C. If we were to magnify C_1 by a factor of 3 – by taking each point x_i of C_1 and moving it to $3x_1$ – then we would obtain C. Similarity, $C_2 \subset C$, the subset of C that lies on the interval $\left[\frac{2}{3}, 1\right]$, is an exact scaled-down copy of C. We may write

$$C = C_1 \cup C_2,$$

where C_1 and C_2 are shrunken, or contracted, copies of C. In other words, the Cantor set is a union of contracted copies of itself. This is a feature exhibited by many "fractal" sets. Note that we have not, as of yet, formally defined the term "fractal" but shall return to this subject in a later section of this course.