Lecture 12

Nonlinear difference equations/dynamical systems (cont'd)

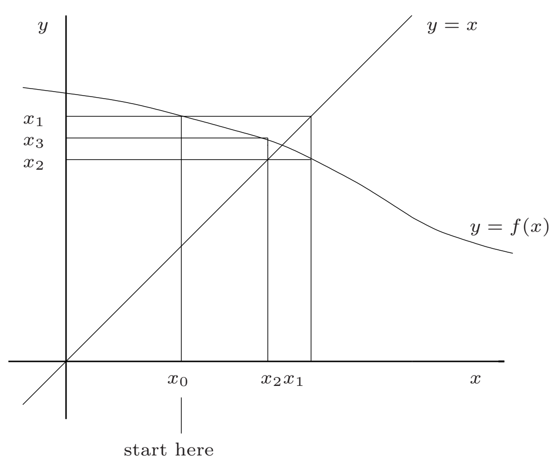
Graphical methods of analyzing iteration (cont'd)

This first part is a repetition of what was presented in Lecture 11.

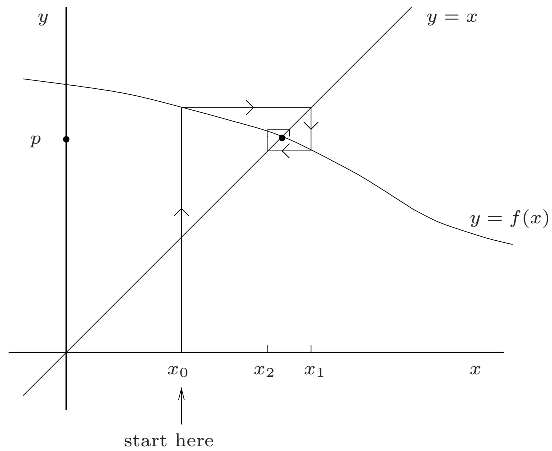
We wish to graphically represent the iteration process defined by

$$x_{n+1} = f(x_n), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$. We begin with a "seed" $x_0 \in \mathbb{R}$ and insert it into the "f machine". The output $x_1 = f(x_0)$ is then reinserted into the "f machine" to produce $x_2 = f(x_1)$, etc. How do we do this graphically? First start with a plot of the graph of $f(x)$ along with the line $y = x$, as shown in the figure on the next page. Now pick a starting point x_0 on the x-axis. Getting $x_1 = f(x_0)$ is easy: You simply find the point $(x_0, f(x_0))$ that has on the graph of $f(x)$ above (or below) the point $x = x_0$. In other words, travel upward (or downward) from $(x_0, 0)$ to $(x_0, f(x_0))$. We now have $x_1 = f(x_0)$. How do we input x_1 into the "f machine" to find $x_2 = f(x_1)$? First, we have to find where x_1 lies on the x-axis. We do this by travelling from the point $(x_0, f(x_0))$ horizontally to the line $y = x$: the point of intersection will be $(f(x_0), f(x_0)) = (x_1, x_1)$. We are now sitting directly above (or below) the point $(x_1, 0)$ on the x-axis, which is patiently waiting to be input into $f(x)$ to produce $x_2 = f(x_1)$. We can travel from (x_1, x_1) to $x_1, 0)$ and then back up (or down) to $(x_1, f(x_1)) = (x_1, x_2)$. From here, we once again travel to the line $y = x$, intersecting it at (x_2, x_2). From here, we travel to the curve $y = f(x)$ to intersect it at (x_2, x_3), etc.. The procedure is illustrated below.



When you have performed this procedure a few times, you will see that including all the lines from intersection points (x_i, x_i) on the line $y = x$ and intersection points (x_i, x_{i+1}) on $y = f(x)$ to the x- and y-axis is unnecessary. In fact, these lines clutter up the figure. We have removed them to produce the figure below, in which the iteration process is presented in a much clearer way.



For rather obvious reasons, figures such as this one are called "cobweb diagrams". It appears as if the iterates x_0, x_1, x_2, x_3 are jumping back and forth, yet "zeroing in" on the point (p, p) at which the graph of f intersects the line $y = x$.

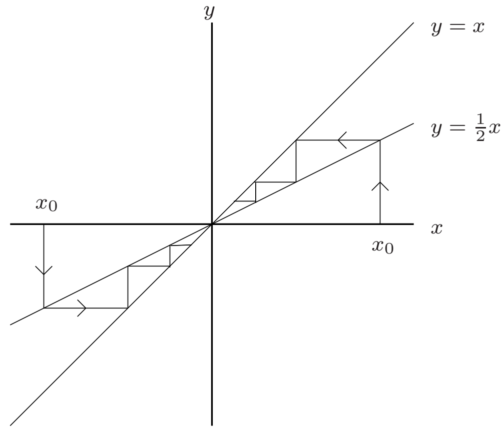
Question: Is there anything special about such a point, i.e., the point (p, p) at which the line $y = x$ intersects the graph of $f(x)$?

Answer: There certainly is! Such a point of intersection must be a fixed point of f since it implies that $f(p) = p$.

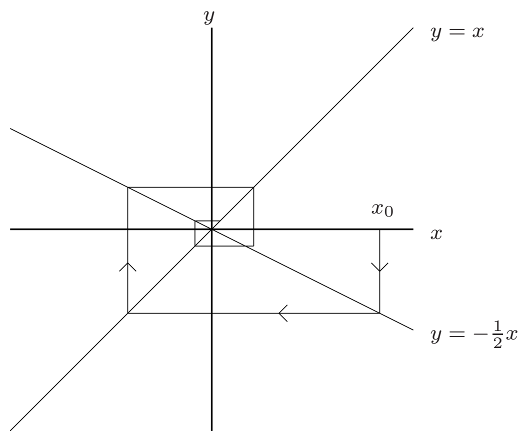
Of course, not all fixed points are attractive such as the one in this diagram: The graphical procedure outlined above will give us some insight into what makes a fixed point attractive or repulsive. Later, we'll establish some theoretical results.

Let us first examine some simple dynamical systems studied earlier, namely the linear ones, using graphical methods.

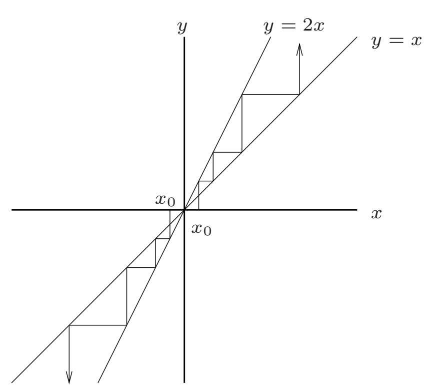
Example 1: $x_{n+1} = cx_n$, $0 < c < 1$. For $x_0 > 1$ or $x_0 < 1$, the graphical method shows the monotonic approach of the $x_n = c^n x_0$ toward the fixed point $x = 0$:



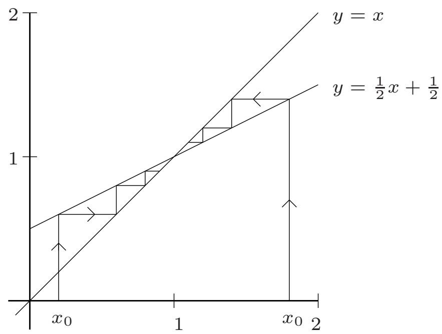
Example 2: $x_{n+1} = cx_n$, $-1 < c < 0$. The iterates x_n oscillate about $x = 0$, $x_n = (-1)^n |c|^n x_0$, with $x_n \to 0$ as $n \to \infty$:



Example 3: $x_{n+1} = cx_n$, $c > 1$. Here, $x_n = c^n x_0$. The iterates travel away from the fixed point 0:



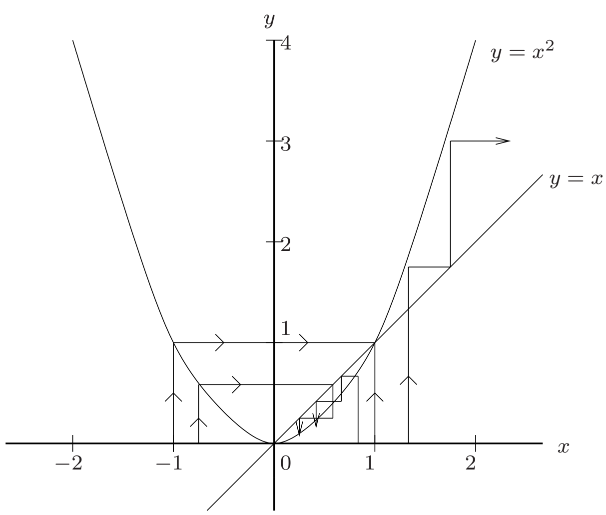
Example 4: $x_{n+1} = cx_n + d$, $0 < c < 1$, $d > 0$



In this example, the two straight lines, $y = x$ and $y = cx + d$, can intersect at only one point, the fixed point of $f(x) = cx + d$, which is $\bar{x} = \frac{d}{1-c}$. Since $|c| < 1$, this fixed point is **attractive**: For any $x_0 \in \mathbb{R}$, $x_n \to \bar{x}$ as $n \to \infty$. Note that this picture looks like that of Example 1, with the fixed point translated from $x = 0$ to $x = \frac{d}{1-c}$. The reader is once again encouraged to examine the iteration of $f(x) = cx + d$ for the cases i) $-1 < c < 0$, ii) $c = -1$ and iii) $c < -1$.

We now examine a few simple nonlinear iteration processes using the graphical method described above. The graphs of nonlinear functions $f(x)$ are not necessarily straight lines – in other words, they can "bend". As such, a function $f(x)$ can have more than one fixed point. Each fixed point $\bar{x}_i$ can behave differently, i.e. one repulsive, the other attractive, so that the dynamics of the iteration process $x_{n+1} = f(x_n)$ can be quite complicated. We illustrate with a few examples.

Example 5: $f(x) = x^2$, i.e. $x_{n+1} = x_n^2$.



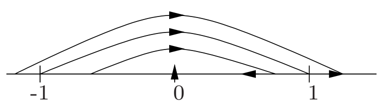
Graphically, we see that the fixed points of $f(x)$ are $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. But let's formally check this:

$$f(x) = x \quad \Longrightarrow \quad x^2 = x \quad \Longrightarrow \quad x(x-1) = 0 \quad \Longrightarrow \quad x = 0, 1\,. \tag{2}$$

Thus, if $x_0 = 0$, then $x_n = 0$ for all $n \geq 0$. If $x_0 = 1$, then $x_n = 1$ for all $n \geq 1$. Let's now look at other groups of points:

- If we begin with a point $x_0 \in (0,1)$, then $x_1 = x_0^2 < x_0$. Likewise $x_0 > x_1 > x_2, \ldots, x_n > x_{n+1}, \ldots$ and $x_n \to 0$ as $n \to \infty$. (In other words, if we keep squaring a number starting in (0,1), we approach zero.)
- If $x_0 \in (-1,0)$, then $x_1 = x_0^2 \in (0,1)$ and the above process is repeated, i.e. $x_n \to 0$ as $n \to \infty$.
- As found earlier, if $x_0 = 1$, then $x_n = 1$. for $n \geq 1$.
- If $x_0 > 1$, then $x_1 = x_0^2 > x_0$ and $x_{n+1} > x_n$, so that $x_n \to +\infty$ as $n \to \infty$.
- If $x_0 = -1$, then $x_1 = (-1)^2 = 1$, a fixed point of f so that all $x_n = 1$ for $n \geq 1$.
- If $x_0 < -1$, then $x_1 = x_0^2 > 1$ and $x_n \to +\infty$ as $n \to \infty$.

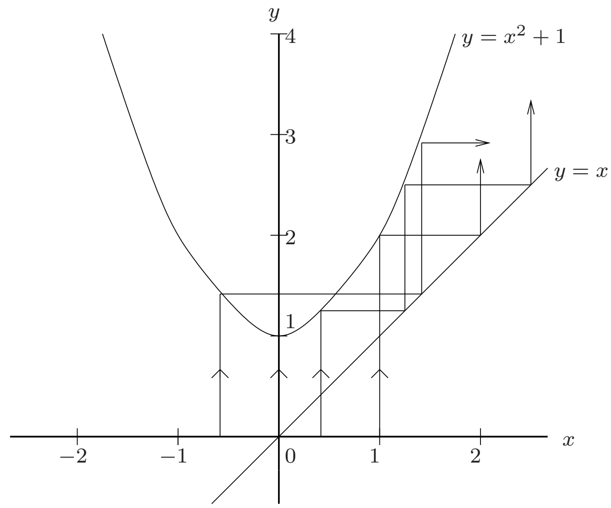
We may try to summarize the dynamics associated with this iteration process, i.e., all of the behaviours listed above, in the "phase portrait' sketched below:



A few other observations:

1. For all points $x_0 \in (-1,1)$, $x_n \to 0$. We say that $\bar{x}_1 = 0$ is an **attractive** fixed point and that the interval $I = (-1,1)$ is its **basin of attraction**.
2. The fixed point $\bar{x}_2 = 1$ is **repulsive** since all points in a neighbourhood of $\bar{x}_2$, $J = (1-\varepsilon, 1+\varepsilon)$ leave J after a finite number of iterations. (Those to the left of 1 go to 0; those to the right of 1 go to $+\infty$.)
3. The point $x = -1$ is mapped to the fixed point $\bar{x}_2 = 1$, and is called a **preperiodic point**, since it remains at $\bar{x}_2 = 1$ in future iterations.
4. For all other initial conditions, $x_0 \in (-\infty, -1) \cup (1, \infty) = \mathbb{R} - [0,1]$, the iterates $x_n \to +\infty$ as $n \to \infty$.

Example 6: $f(x) = x^2 + 1$, i.e. $x_{n+1} = x_n^2 + 1$



Graphically, we see that $f(x)$ has no (real) fixed points. Let's double check:

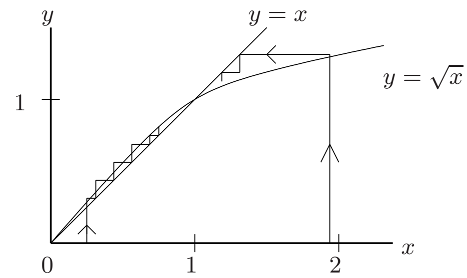
$$f(x) = x \quad \Longrightarrow \quad x^2 + 1 = x \quad \Longrightarrow \quad x^2 - x + 1 = 0 \quad \Longrightarrow \quad x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}\,. \tag{3}$$

Yes, indeed: No real fixed points.

For $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$, $x_n \to +\infty$. For $x_0 > 0$, $x_1 < x_2 < x_3 \ldots, x_n \to +\infty$. For $x_0 < 0$, $x_1 > 0$, so that $x_n \to +\infty$. In summary, for all $x_0 \in \mathbb{R}$, $x_n \to +\infty$ as $n \to \infty$.

Note that if the above parabola is translated downward by a sufficient amount, the graph of f will intersect with the line $y = x$ to produce a fixed point. As the graph is translated downward even further, this fixed point evolves into two fixed points. The dynamics associated with this quadratic family $f_c(x) = x^2 + c$ can become quite complicated and will be the subject of later sections.

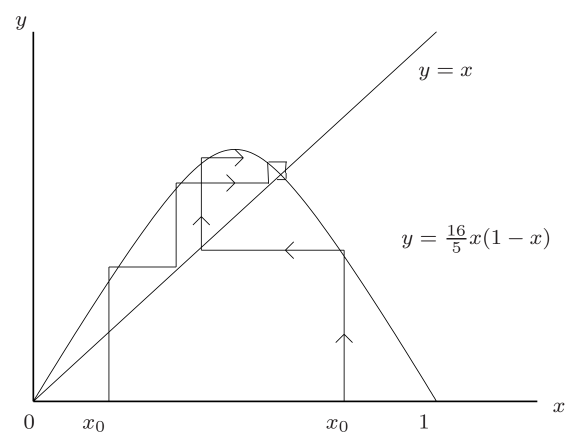
Example 7: $f(x) = \sqrt{x}$, $x \geq 0$, i.e. $x_{n+1} = \sqrt{x_n}$



There are two fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. For $x_0 \in (0,1)$, $x_1 = \sqrt{x_0} > x_0$, $x_2 > 1, \ldots$ so that $x_n \to 1$ as $n \to \infty$. For $x_0 > 1$, $x_1 = \sqrt{x_0} < x_0, \ldots$ $x_{n+1} = \sqrt{x_n} < x_n$, so that $x_n \to 1$ as $n \to \infty$. Thus, $\bar{x}_2 = 1$ is an attractive fixed point and its basin of attraction if $I = (0, \infty)$. (In other words, if you start with a positive number x_0 and keep taking square roots, you approach 1 in the limit.)

We end this section with a couple of more interesting examples, as a motivation for later studies.

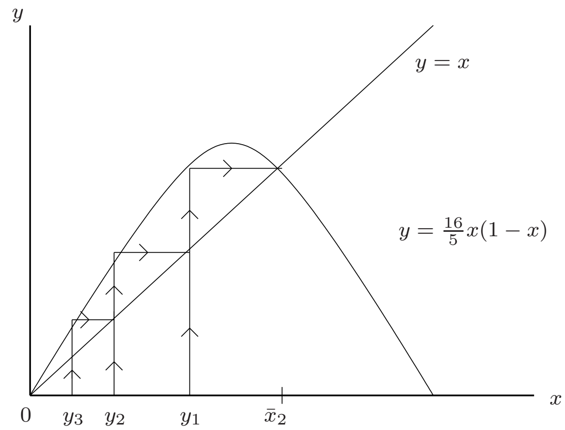
Example 8: $f(x) = \frac{16}{5}x(1-x)$, $x \in [0,1]$



f has two fixed points: $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{11}{16}$ (Exercise). Clearly, if $x_0 = 0$, then $x_n = 0$. If x_0 is near 0, then the first few iterates x_1, x_2 are observed to travel away from 0 toward fixed point $\bar{x}_2$. The points $(x_n, f(x_n))$ travel "up the hump" until they reach roughly the height $\bar{x}_2$, whereupon they are directed toward the right side of the "hump". Graphical methods are insufficient to account for the long term behaviour of the x_n. If we magnify the graph of $f(x)$ near its fixed point $(\bar{x}_2, \bar{x}_2) = \left(\frac{11}{16}, \frac{11}{16}\right)$ and examine the "cobweb nature" of the iteration procedure $x_{n+1} = f(x_n)$, we find that a point x_0 near $\bar{x}_2 = \frac{11}{16}$ is sent by the function f to a point $x_1 = f(x_0)$ on the other side of $\bar{x}_2 = \frac{11}{16}$ and farther away from it, i.e. $|x_1 - \bar{x}_2| > |x_0 - \bar{x}_2|$. In other words, $\bar{x}_2 = \frac{11}{16}$ is a repulsive fixed point: If we choose $x_0 = \bar{x}_2$, then $x_n = \bar{x}_2$, i.e. we remain at $\bar{x}_2$. However, if we choose an x_0 "ε-close" to $\bar{x}_2$, with $\varepsilon > 0$, the iterates x_n travel away from $\bar{x}_2$, no matter how small an ε is chosen.

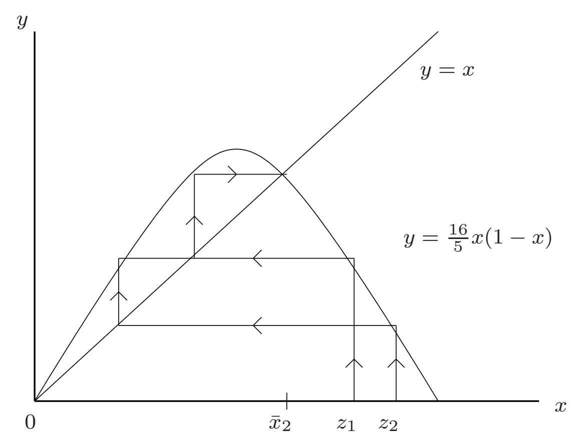
So what happens to the iterates x_n? Numerically, we find that they approach the two-cycle $(p_1, p_2) \cong (0.799, 0.513)$ as $n \to \infty$. We'll study this phenomenon in more detail a few lectures from now.

This is not the end of the story for this example. Even though the fixed point $\bar{x}_2$ is unstable, there is still an infinity of points that get mapped to $\bar{x}_2$ after a finite number of iterations and then stay at $\bar{x}_2$. To determine these points, we simply iterate "backwards", i.e. find graphically the point y_1 such that $f(y_1) = \bar{x}_2$, then find graphically the point y_2 such that $f(y_2) = y_1$ so that $f^2(y_2) = \bar{x}_2$, etc. In this way, we find an infinite set of **preperiodic** points y_n, $n = 1, 2, 3, \ldots$ such that $f^n(y_n) = \bar{x}_2$. Note that $y_n \to 0$ as $n \to \infty$.



In other words, these points can be found at arbitrarily small distances from the (repulsive) fixed point $\bar{x}_1 = 0$.

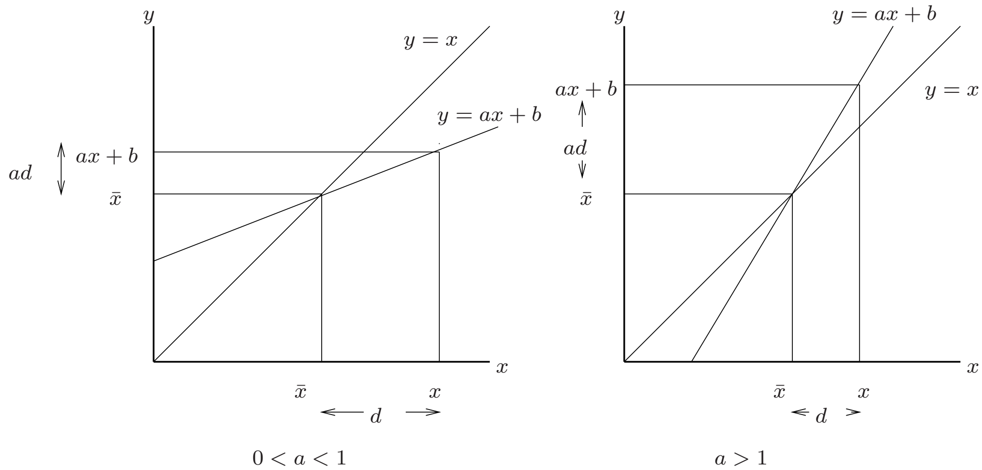
But that's not all! There's an infinity of points "on the other side of the hump near $x = 1$ that also get mapped to $\bar{x}_2$ after a finite number of iterations. We can find them by going "right" instead of left in our "backwards" iteration procedure pictured above. In the figure below, we have found graphically the point z_1 near $n = 1$ such that $f(z_1) = y_1$ so that $f^2(z_1) = \bar{x}_2$ as well as the point z_2 such that $f(z_2) = y_2$ so that $f^3(z_2) = \bar{x}_2$. In this way, we find another infinite set of **preperiodic** points z_n, $n = 1, 2, 3, \ldots$ such that $f^{n+1}(z_n) = \bar{x}_2$. Note also that these points $z_n \to 1$ as $n \to \infty$.



The reader is invited to devise an algorithm to run this iteration procedure "backwards" to numerically generate the sequences $\{y_n\}$ and $\{z_n\}$.

Determining the nature of a fixed point of a nonlinear mapping

We have examined the nature of fixed points for linear (degree one polynomial) mappings $f(x) = ax+b$ on several occasions. For $a \neq 1$, $\bar{x} = \frac{b}{1-a}$ is the unique fixed point of f, i.e. $f(\bar{x}) = \bar{x}$. If $|a| < 1$, $\bar{x}$ is attractive; if $|a| > 1$, $\bar{x}$ is repulsive. When $a = -1$, $\bar{x}$ is neither attractive nor repulsive and is referred to as **neutral** or **indifferent**. The attractivity of repulsivity of $\bar{x}$ as determined by the multiplier "a" can be viewed geometrically, as we show below for the case $a > 0$:



In each of the above cases, pick an $x \neq \bar{x}$ and let d denote the distance between $\bar{x}$ and x, i.e. $d = |x-\bar{x}|$. Then the distance between $f(x) = ax + b$ and $f(\bar{x}) = \bar{x} = a\bar{x} + b$ is

$$\begin{aligned} |f(x) - \bar{x}| &= |ax + b - a\bar{x} - b| \\ &= |a|\,|x - \bar{x}| \\ &= |a|d. \end{aligned} \tag{4}$$

If $|a| < 1$, then $f(x)$ is closer to $\bar{x}$ than x is. If $|a| > 1$, then $f(x)$ is farther from $\bar{x}$ than x is. This is a simple consequence of the slope a of the line $y = ax + b$. Let us now replace x in Eq. (4) with $f(x)$ to give then

$$\begin{aligned} |f^2(x) - \bar{x}| &= |a|\,|f(x) - \bar{x}| \\ &= |a|^2\,|x - \bar{x}| \quad \text{(from previous equation)} \\ &= |a|^2 d\,. \end{aligned} \tag{5}$$

This is the same as applying the function f twice to the original point x near $\bar{x}$. If we keep replacing x with $f(x)$, we obtain, in general,

$$|f^n(x) - \bar{x}| = |a|^n\,|x - \bar{x}| = |a|^n d\,. \tag{6}$$

If $|a| < 1$, then $|a|^n \to 0$ which implies that $|f^n(x) - \bar{x}| \to 0$ as $n \to \infty$, implying that the points $x_n = f^n(x)$ converge to $\bar{x}$ as $n \to \infty$. If $|a| > 1$, then $|f^n(x) - \bar{x}| \to \infty$ as $n \to \infty$. The equalities in

(4)-(6) are a consequence of the fact that the graph of $f(x) = ax + b$ is a **straight line** with constant slope a for all $x \in \mathbb{R}$.

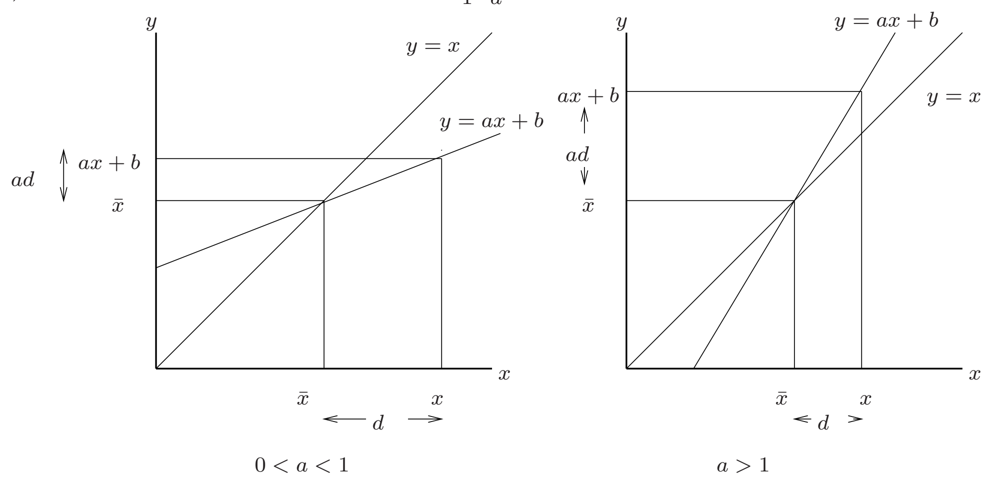
Of course, the graphs of nonlinear functions $f(x)$, e.g. $f(x) = x^2 - 1$, are **not** straight lines. Nevertheless, the examples studied in the previous section suggest that we may be able to consider the graph of f as behaving roughly like a straight line in a neighbourhood of a fixed point. This, of course, has the aroma of considering the **linear approximation** to $f(x)$ at a fixed point $\bar{x}$.

Lecture 13

Nonlinear dynamics (cont'd)

Determining the nature of the fixed point of a nonlinear function

In the previous lecture, we examined once again the attractivity/repulsivity of the fixed point $\bar{x} = \frac{b}{1-a}$ $(a \neq 1)$ of the **linear function** $f(x) = ax + b$. From our previous work in this course, we know that the fixed point $\bar{x}$ is attractive if $|a| < 1$ and repulsive if $|a| > 1$. But the purpose of the analysis in the last lecture was to look at the problem geometrically, in order to obtain a little more insight on the iteration process. We looked at the following geometrical interpretations of the behaviour of the function $f(x) = ax + b$ near its fixed point $\bar{x} = \frac{b}{1-a}$:



Without going into any details – reader is invited to look at the notes for Lecture 12 – we established, in a straightforward geometric way, the following result for the linear case: For an $x \neq \bar{x}$,

$$|f(x) - \bar{x}| = |a||x - \bar{x}|\,. \tag{7}$$

Note:

- The term $|x - \bar{x}|$ on the right side of the above equation is the **distance** between x and the fixed point $\bar{x}$.
- The term $|f(x) - \bar{x}|$ on the left side of the above equation is the **distance** between $f(x)$ and the fixed point $\bar{x}$.

From Eq. (7) we can then conclude the following:

- If $|a| < 1$, then $f(x)$ is **closer** to $\bar{x}$ than x is. Since $f(\bar{x}) = \bar{x}$, this is sufficient to establish that $\bar{x}$ is an **attractive** fixed point.
- If $|a| > 1$, then $f(x)$ is **farther** from $\bar{x}$ than x is which is sufficient to establish that $\bar{x}$ is a **repulsive** fixed point.

This clearly indicates that a, which is the **slope** of the graph of $f(x)$ at $\bar{x}$ – in fact at all points $x \in \mathbb{R}$ – determines whether or not the fixed point $\bar{x}$ is attractive or repulsive.

But recall that we did not stop there. We then replaced x in Eq. (7) with $f(x)$ to establish the relationship between the point $f(x)$ and the point $f(f(x))$ to which it is mapped. And we then replaced x with $f(x)$ once again, and so on and so on to obtain the following result,

$$|f^n(x) - \bar{x}| = |a|^n |x - \bar{x}|, \quad n \geq 1. \tag{8}$$

From this result, we can conclude the following:

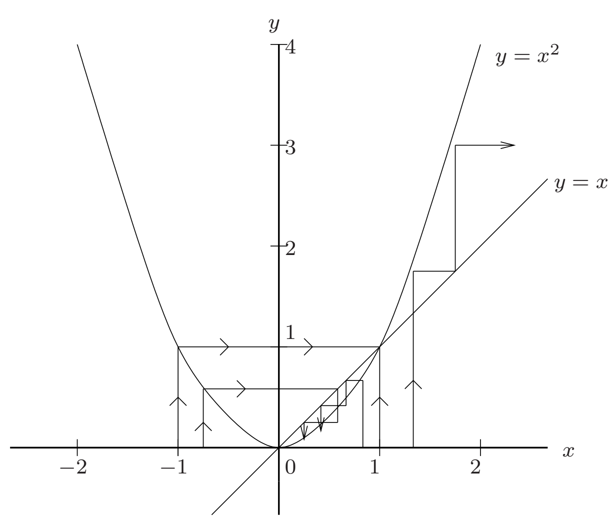
- If $|a| < 1$, then $|a|^n \to 0$ as $n \to \infty$, which implies that $|f^n(x) - \bar{x}| \to 0$ as $n \to \infty$. Therefore $f^n(x) \to \bar{x}$ as $n \to \infty$. In other words, x is an element of the basin of attraction of the attractive fixed point $\bar{x}$.

- If $|a| > 1$, then $|a|^n \to \infty$ as $n \to \infty$, which implies that $|f^n(x) - \bar{x}| \to \infty$ as $n \to \infty$. Therefore the distance between $f^n(x)$ and $\bar{x}$ increases without bound. This is in accordance with the fact that $\bar{x}$ is a **repulsive** fixed point.

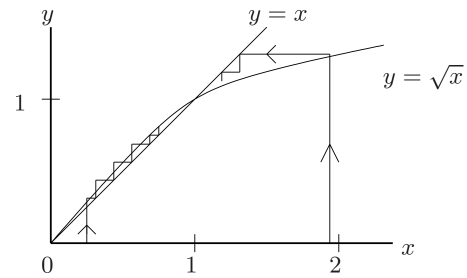
We ended the last Lecture with the following paragraph:

> Of course, the graphs of nonlinear functions $f(x)$, e.g. $f(x) = x^2 - 1$, are **not** straight lines. Nevertheless, the examples studied in the previous section suggest that we may be able to consider the graph of f as behaving roughly like a straight line in a neighbourhood of a fixed point. This, of course, has the aroma of considering the **linear approximation** to $f(x)$ at a fixed point $\bar{x}$.

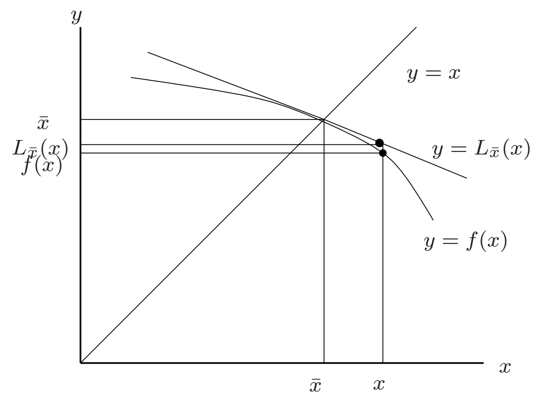
This is what we wish to do in what follows – examine nonlinear functions and the possible use of linear approximations to establish the nature of their fixed points. For example, in the case $f(x) = x^2$ (Example 5 of the previous section) $\bar{x}_1 = 0$ is attractive and $\bar{x}_2 = 1$ is repulsive. Note that $f'(\bar{x}_1) = 0$ and $f'(x_2) = 2$:



In Example 7, $f(x) = \sqrt{x}$, $\bar{x}_1 = 0$ is repulsive, with $f'(0) = +\infty$, and $\bar{x}_2 = 1$ is attractive, with $f'(\bar{x}_2) = \frac{1}{2}$:



We might then conjecture that a linear approximation to $f(x)$ at a fixed point $\bar{x}$ may provide geometric pictures analogous to those sketched above for linear maps.



Recall, from first year Calculus, that the **linearization** of the function f at $x = \bar{x}$ is defined as follows,

$$L_{\bar{x}}(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x})\,. \tag{9}$$

The **linear approximation** to f at $x = \bar{x}$ is then given by

$$f(x) \cong L_{\bar{x}}(x) \quad \text{for } x \text{ near } \bar{x}\,. \tag{10}$$

From these two equations, we have that for x near $\bar{x}$,

$$f(x) \cong f(\bar{x}) + f'(\bar{x})(x - \bar{x})\,. \tag{11}$$

Since $f(\bar{x}) = \bar{x}$, we can rewrite the above relation as follows,

$$f(x) \cong \bar{x} + f'(\bar{x})(x - \bar{x})\,. \tag{12}$$

Now subtract $\bar{x}$ from both sides and take absolute values to obtain,

$$\boxed{|f(x) - \bar{x}| \cong |f'(\bar{x})|\,|x - \bar{x}| \quad \text{for } x \text{ near } \bar{x}\,.} \tag{13}$$

Looking back at Eq. (7), we see that, as expected, $|f'(\bar{x})|$ plays the role of $|a|$ for linear maps. (Of course, $f'(\bar{x})$ **is equal to** a for the linear map $f(x) = ax + b$.) And from the above figure, it looks as if we are replacing the not-necessarily-linear graph of $f(x)$ with its **straight**, i.e., **linear** approximation at $\bar{x}$, at least for x near $\bar{x}$.

There is still a problem with the relation in (13) – the occurrence of "$\cong$" due to the use of the linear approximation. We cannot conclude that $|f(x) - \bar{x}| < |x - \bar{x}|$ if $|f'(\bar{x})| < 1$ since we have no idea of "how good" the approximation "$\cong$" is.

Fortunately, we can bypass this difficulty with the use of the famous **Mean Value Theorem** from first-year Calculus, which we review below.

Theorem 1: (Mean Value Theorem) Let f be differentiable on $[a, b]$. Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{14}$$

Note that the Mean Value Theorem does not, in general, tell us where the point c is located: It only guarantees the existence of at least one such c.

We are now in a position to prove an important result relating the nature of a fixed point $\bar{x}$ of a function f to its derivative $f'(\bar{x})$. In what follows, we assume that f is C^1 on its domain of definition (generally $\mathbb{R}$). "f is C^1" means that "$f'(x)$ is continuous on that domain."

Theorem 2: Let $\bar{x}$ be a fixed point of f such that $|f'(\bar{x})| < 1$. Then there exists an open interval I containing $\bar{x}$ such that for any $x \in I$,

$$\lim_{n\to\infty} f^n(x) = \bar{x}. \tag{15}$$

In other words, $\bar{x}$ is an **attractive fixed point** and the interval I belongs to the **basin of attraction** of $\bar{x}$.

Note: The result involving (15) can be written in another way: For any $x_0 \in I$, the iteration sequence $x_{n+1} = f(x_n)$, $n \geq 0$, behaves as follows:

$$\lim_{n\to\infty} x_n = \bar{x}. \tag{16}$$

Proof: Since $f'(x)$ is assumed to be continuous, the fact that $|f'(\bar{x})| < 1$ implies that there exists an open interval J containing $\bar{x}$ and a constant $0 \leq K < 1$ such that

$$|f'(x)| < K < 1 \quad \text{for all} \quad x \in J \tag{17}$$

(Aside for clarification: $|f'(\bar{x})| < 1$ means that $-1 < f'(\bar{x}) < 1$. Suppose that $f'(\bar{x}) = 1 - \varepsilon$ where $\varepsilon > 0$ is very small. The fact that $f'(x)$ is continuous implies that as we move away from $x = \bar{x}$, the value of $f'(x)$ cannot instantaneously jump from the value $1-\varepsilon<1$ to a value greater than 1. Even if it were to increase to an eventual value greater than one at, say, $x_1 \neq \bar{x}$, it would have to assume all intermediate values between $1-\varepsilon$ and 1 before getting to $x=x_1$. This is a consequence of the Intermediate Value Theorem applied to the continuous function $f'(x)$. The same argument holds for $f(\bar{x})=-1+\varepsilon$, $\varepsilon>0$.)

Let $I \subset J$ be an open interval centered around the point $\bar{x}$, i.e. $I=(\bar{x}-\delta,\bar{x}+\delta)$ for some $\delta>0$. Now choose an $x \in I$, $x \neq \bar{x}$. We now wish to compare the distance $|f(x)-\bar{x}|$ to the distance $|x-\bar{x}|$. From the Mean Value Theorem, Eq. (14), with $a=\bar{x}$ and $b=x$:

$$\frac{f(x)-f(\bar{x})}{x-\bar{x}}=f'(c) \tag{18}$$

for some c between x and $\bar{x}$. Note that this implies that $c \in I$. (We don't know where c is but, as we'll show below, we don't care. It is only important to know that $c \in I$.) We rearrange (18),

$$f(x)-f(\bar{x})=f'(c)(x-\bar{x}), \tag{19}$$

and then substitute $f(\bar{x})=\bar{x}$ to arrive at the equation,

$$f(x)-\bar{x}=f'(c)(x-\bar{x}). \tag{20}$$

Now take absolute values of both sides, i.e.,

$$|f(x)-\bar{x}|=|f'(c)||x-\bar{x}|, \tag{21}$$

From (17), it follows that $|f'(c)|<K$ since c must be in I (from above). We may therefore conclude that

$$|f(x)-\bar{x}| \leq K|x-\bar{x}|, \quad \text{where} \;\; 0 \leq K < 1. \tag{22}$$

In other words, $f(x)$ is closer to $\bar{x}$ than x is. This implies that $f(x) \in I$. Note that $x \in I$ above plays the role of the "seed" x_0 in the iteration process $x_1=f(x_0)$, $x_{n+1}=f(x_n)$. Let us now replace x in (22) with $f(x)$. The result is

$$\begin{aligned} |f(f(x))-\bar{x}| &\leq K|f(x)-\bar{x}| \\ &\leq K^2|x-\bar{x}|, \end{aligned} \tag{23}$$

where the final line is a consequence of (22). We rewrite (23) as

$$|f^2(x)-\bar{x}| \leq K^2|x-\bar{x}|. \tag{24}$$

Note that $f^2(x) \in I$.

The reader should see the pattern: If we apply f to x n times, then we obtain

$$|f^n(x)-\bar{x}| \leq K^n|x-\bar{x}|. \tag{25}$$

Recall, from (17) that $0<K<1$. Since $K^n \to 0$ as $n \to \infty$, we have that $|f^n(x)-\bar{x}| \to 0$ as $n \to \infty$, i.e.

$$\lim_{n\to\infty} f^n(x)=\bar{x}.$$

The proof is complete. ∎

The following theorem is an important result regarding the attractive fixed point $\bar{x}$ and interval I.

Theorem 3: No other fixed points of f lie in the interval I of the previous Theorem.

Proof: By contradiction. Suppose that $\bar{y} \neq \bar{x}$ is another fixed point in I. Apply the Mean Value Theorem to $a = \bar{x}$, $b = \bar{y}$:

$$\frac{f(\bar{y}) - f(\bar{x})}{\bar{y} - \bar{x}} = f'(c)$$

for some $c \in I$ between $\bar{x}$ and $\bar{y}$. Since $f(\bar{y}) = \bar{y}$ and $f(\bar{x}) = \bar{x}$, we are led to the conclusion that $f'(c) = 1$. Since this contradicts (17), the assumption that $\bar{y} \neq \bar{x}$ is false. ■

Remark: Because of its importance in determining whether or not a fixed point $\bar{x}$ of a function $f(x)$ is attractive, the quantity $|f'(\bar{x})|$ is known as the **multiplier** of f at $\bar{x}$.

As might be expected, if the multiplier $|f'(\bar{x})| > 1$ then the fixed point $\bar{x}$ is **repulsive**. The reader is encouraged to prove the following result along similar lines used in the proof of Theorem 2.

Theorem 4: Let $\bar{x}$ be a fixed point of f such that $|f'(\bar{x})| > 1$. Then there exists an open interval I containing $\bar{x}$ such that, if $x \in I$, $x \neq \bar{x}$, then there exists a $k > 0$ such that $f^k(x) \notin I$.

Hint: First show that there exists an interval J such that

$$|f(x) - \bar{x}| \geq L|x - \bar{x}|, \quad \forall x \in J,$$

where $L > 1$.

A quick return to some previous examples to confirm the above results

- Example 5: $f(x) = x^2$ with fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. $f'(\bar{x}_1) = f'(0) = 0$ and $\bar{x}_1 = 0$ is attractive. $f'(\bar{x}_2) = f'(1) = 2$ and $\bar{x}_1 = 0$ is repulsive.

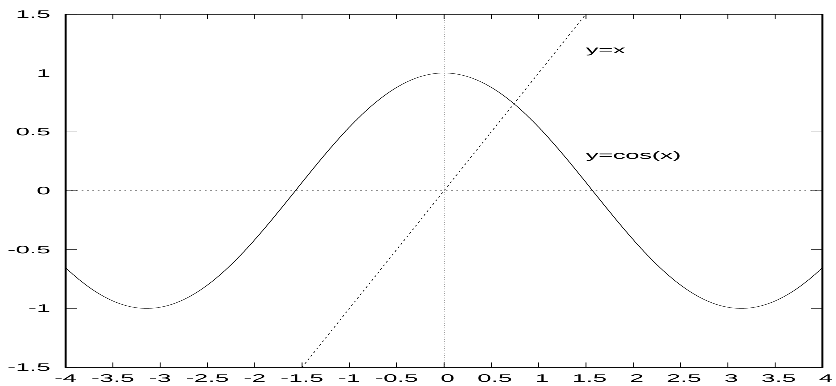
- Example 7: $f(x) = \sqrt{x}$ with fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. $f'(\bar{x}_1) = f'(0)$ does not exist so the test is inapplicable. $f'(\bar{x}_2) = f'(1) = 2$ and $\bar{x}_1 = 0$ is repulsive.

Final Remarks:

From Theorems 2 and 4, the nature of a fixed point $\bar{x}$ of a (nonlinear) function $f(x)$ can, in general, be determined only **locally**, that is, in an immediate neighbourhood of $\bar{x}$. This is often sufficient to ascertain the dynamics of the iteration process $x_{n+1} = f(x_n)$ over larger intervals. Indeed, as mentioned earlier, a function f may have several fixed points with various properties, e.g. repulsive, attractive, neutral/indifferent.

Regarding the term "neutral/indifferent": A fixed point $\bar{x}$ for which $|f'(\bar{x})| = 1$ is said to be **neutral** or **indifferent**. A more detailed analysis must usually be performed in order to determine the behaviour of iterates near such points. A such, theories for neutral/indifferent points are much more complicated. Fortunately, indifferent points play rather minor roles in the applications we wish to consider, representing transition points from one behaviour to another, e.g. from repulsive to attractive.

Example: The function $f(x) = \cos x$ has one – and only one – fixed point $\bar{x} \simeq 0.739085$ as shown in the plot below.



Since $f'(x) = -\sin x$, we have that

$$|f'(\bar{x})| = |\sin \bar{x}|\,. \tag{26}$$

Note that $0 < \bar{x} < \frac{\pi}{2}$. Also recall that the function $\sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$ and that $\sin 0 = 0$ and $\sin(\frac{\pi}{2}) = 1$, which implies that $|\sin \bar{x}| < 1$. Therefore,

$$|f'(\bar{x})| < 1\,, \tag{27}$$

which implies that the fixed point $\bar{x}$ is **attractive**.

In fact, we can do much better in bounding $|f'(\bar{x})|$ away from 1. From the graph, we can see that $0 < \bar{x} < 1$, which implies that

$$|f'(\bar{x})| < |\sin 1| \simeq 0.84147\,, \tag{28}$$

which is a better inequality than (27) since it establishes that $|f'(\bar{x})|$ cannot be quite close to 1 – its value is significantly smaller than 1.

From this result, we expect that for a starting value x_0 sufficiently close to $\bar{x}$, the iteration sequence,

$$x_{n+1} = \cos(x_n)\,, \tag{29}$$

converges to $\bar{x}$ in the limit $n \to \infty$. The reader is invited to check this result with the aid of a calculator or computer. Here is the iteration sequence that results when we choose $x_0 = 1$:

n	x_n
0	1.0000000000
1	0.5403023059
2	0.8575532158
3	0.6542897905
4	0.7934803587
5	0.7013687736
6	0.7639596829
7	0.7221024250
8	0.7504177618
9	0.7314040424
10	0.7442373549
20	0.7391843998
30	0.7390870427
40	0.7390851699
50	0.7390851339
60	0.7390851332
70	0.7390851332

Note that the x_n are approaching the fixed point $\bar{x}$ in an alternating fashion, as is expected, given that the cosine function has a negative slope at and around $\bar{x}$. A graphical sketch of the iteration will produce a true "cobweb" which approaches $\bar{x}$ in the limit $n \to \infty$. Also note that for $n \geq 60$, all iterates x_n agree to at least 10 decimal digits.

Exercise: It is a fact that for **any** starting point x_0 on the real line, the iteration sequence,

$$x_{n+1} = \cos(x_n)\,, \tag{30}$$

converges to the attractive fixed point $\bar{x} \simeq 0.739085$ of the cosine function. Show why this is the case.

Hints:

1. Use the fact that $-1 \leq \cos(x) \leq 1$ to show that regardless of the point $x_0 \in \mathbb{R}$, the point x_1 must lie in the interval $[-1, 1]$.

2. Now show that regardless of where x_1 lies in $[-1, 1]$, x_2 must lie in the interval $[0, 1]$.

3. Finally, show that all x_n, $n \geq 3$, must lie in the interval $[0, 1]$.

4. You can then use the graphical interpretation of iteration to arrive at the final result that $x_n \to \bar{x}$ as $n \to \infty$.

5. From a more rigorous point of view, however, one can use the Mean Value Theorem, in the way that it was used to prove Theorem 2 above, to show that for any point $x \in [0, 1]$, $f^n(x) \to \bar{x}$ as $n \to \infty$.

Lecture 14

Dynamics of the Logistic Quadratic Map

In this section we shall study some dynamics associated with the iteration process $x_{n+1} = f_a(x_n)$ where

$$f_a(x) = ax(1-x), \qquad a > 0. \tag{31}$$

The functions $f_a(x)$ in (31) comprise a one-parameter family of quadratic mappings, referred to as the "logistic maps." Note that we exclude the parameter value $a = 0$ since it corresponds to the trivial and uninteresting map $f_0(x) = 0$, i.e., all points are mapped to zero.

In the discussion that follows, we shall be keeping the parameter a fixed for a given iteration process,

$$x_{n+1} = f_a(x_n)\,. \tag{32}$$

It will be interesting to observe the dynamical processes for various values of a. For example, it will be observed that for each a-value in a certain range, say, $a_0 < a < a_1$, the iterates x_n converge to the fixed point $\bar{x} = 0$ for all starting values $x_0 \in (0,1)$. In another range of a-values, say, $a_1 < a < a_2$, the iterates converge to another fixed point $\bar{x}_2(a)$ whose value depends upon a. It will then be found that for another range of a-values, $a_2 < a < a_3$, almost all iterates with starting points $x_0 \in (0,1)$ approach a two-cycle $\{p_1(a), p_2(a)\}$ the values of which depend upon a. This is reminiscent of the asymptotic behaviour observed in our earlier numerical experiments involving the dynamical system

$$x_{n+1} = cx_n - dx_n^2. \tag{33}$$

In fact, it is sufficient to understand the iteration dynamics of the one-parameter family in (31) in order to understand that of the two-parameter family in (33). To see this, consider the following change of variable,

$$x_n = \alpha y_n\,, \tag{34}$$

in Eq. (33), where the constant $\alpha \in \mathbb{R}$ is to be determined. Substitution into (33) followed by division by α yields

$$\begin{aligned} y_{n+1} &= cy_n - \alpha d y_n^2 \\ &= cy_n\left(1 - \frac{\alpha d}{c} y_n\right). \end{aligned} \tag{35}$$

Thus, setting $\alpha = \frac{c}{d}$, or $x_n = \frac{c}{d} y_n$, yields the "logistic" dynamical system in the y_n:

$$y_{n+1} = cy_n(1-y_n). \tag{36}$$

Apart from the difference in the letter used to denote the parameter, i.e., c vs. a, the iteration procedure in (36) is essentially the iteration of the function $f_a(x)$ in Eq. (31). We would never have seen this in the original d.e. in Eq. (33) unless we performed the change of variable. Note also that the change of variable in Eq. (34) is simply a scaling of the x variable to produce the y variable (or vice versa).

The change of variable performed above shows that we can begin with an x_0, translate it into a y_0, iterate Eq. (36) to produce the y_n and then translate back, if we wish, to give x_n. It is interesting that the dynamics of (36) depends only upon c, and not d, in (33). Furthermore, the dynamics of the y_n is **equivalent** to that of the x_n: If the sequence y_n approaches a fixed point $\bar{y}$, then the sequence x_n approaches a corresponding fixed point $\bar{x} = \frac{c}{d}\bar{y}$. If the sequence y_n approaches an N-cycle, then the sequence x_n approaches the corresponding N-cycle.

For $a > 0$, the functions $f_a(x)$ in (31) are downward pointing parabolas that pass through the points $(0,0)$ and $(1,0)$ and achieve global maximum values at $\left(\frac{1}{2}, \frac{a}{4}\right)$. (Exercise.) Let us determine the fixed point of $f_a(x)$:

$$f_a(x) = x \quad \Longrightarrow \quad ax(1-x) = x\,. \tag{37}$$

Certainly, $x = 0$ satisfies this equation. To find the nonzero fixed point, we divide both sides by $x \neq 0$ to yield the equation,

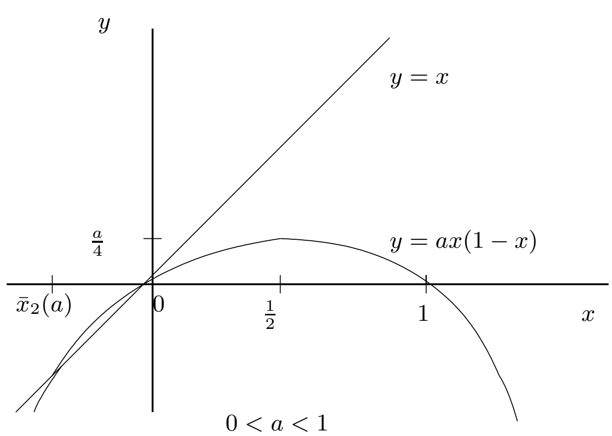
$$a(1-x) = 1 \quad \Longrightarrow \quad 1 - x = \frac{1}{a} \quad \Longrightarrow \quad x = 1 - \frac{1}{a} = \frac{1-a}{a}\,. \tag{38}$$

We shall denote the two fixed points of $f_a(x)$ as follows,

$$\bar{x}_1 = 0, \qquad \bar{x}_2(a) = \frac{a-1}{a}. \tag{39}$$

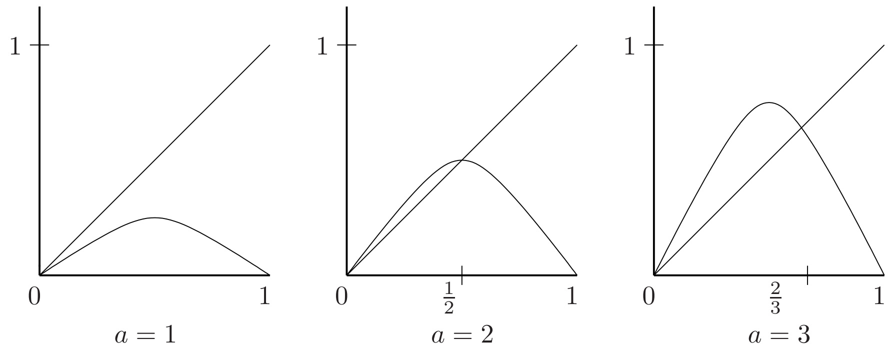
Clearly, the fixed point $\bar{x}_1 = 0$ is independent of a – it remains fixed. Let us now track the motion of the fixed point $\bar{x}_2$:

- For $0 < a < 1$, $\bar{x}_2 < 0$. As $a \to 0^+$, $\bar{x}_2(a) \to -\infty$:

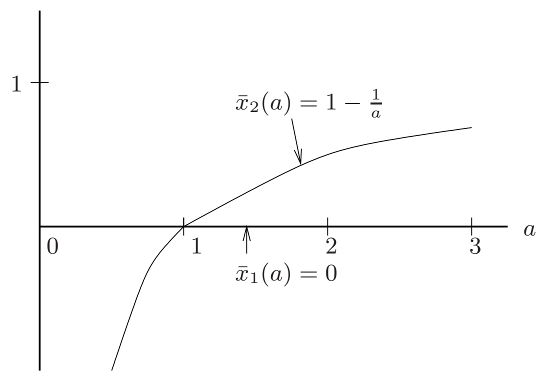


- When $a = 1$, $\bar{x}_1 = \bar{x}_2 = 0$, i.e. the fixed point $\bar{x}_2$ has moved to coincide with $\bar{x}_1$.

- As a increases, $a > 1$, the fixed point $\bar{x}_2(a)$ moves away from 0 toward the right. At $a = 2$, $\bar{x}_2(2) = \frac{1}{2}$. At $a = 3$, $\bar{x}_2(3) = \frac{2}{3}$. The position of $\bar{x}_2$ for some particular values of $a \geq 1$ are shown below.



The positions of the two fixed points as a function of a are plotted below:



We now compute the multipliers $|f_a'(\bar{x}_i(a))|$ of these fixed points in order to determine if and when each of them is attractive or repulsive. From (31),

$$f_a'(x) = a - 2ax.$$

Therefore, from (39):

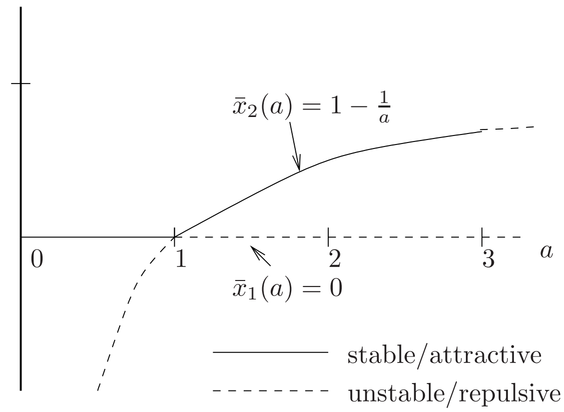
i) $\bar{x}_1 = 0$: $\qquad f_a'(\bar{x}_1) = a.$

Therefore $\bar{x}_1 = 0$ is attractive for $0 \leq a < 1$, indifferent for $a = 1$, and repulsive for $a > 1$.

ii) $\bar{x}_2 = \frac{a-1}{a}$ $\qquad f_a'(\bar{x}_2) = 2 - a.$

The fixed point $\bar{x}_2$ is attractive when $|2 - a| < 1$, neutral/indifferent when $|2 - a| = 1$ and repulsive when $|2 - a| > 1$. We leave it as an exercise for the reader to show that $\bar{x}_2$ is repulsive for $0 < a < 1$, indifferent for $a = 1$, attractive for $1 < a < 3$, indifferent for $a = 3$, and repulsive for $a > 3$. (This can be done by either looking at the inequalities involving absolute values, or by simply examining the term $2 - a$, and hence $|2 - a|$ as a increases from 0.)

Before looking at the iteration dynamics of sequences, it is useful to summarize all of the information that we have obtained so far regarding the two fixed points $\bar{x}_i$ of the logistic map $f_a(x)$. We shall include the information on the attractive/repulsive properties of the fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{a-1}{a}$ of the logistic map $f_a(x)$ in addition to their locations, as shown earlier. This is done in the figure below. Dotted lines indicate repulsive/unstable behaviour and solid lines denote attractive/stable behaviour.



This figure may be considered as a kind of "atlas" of the behaviour of the two fixed points $\bar{x}_i$. Later, we shall add to this figure some other important dynamical behaviour.

We shall now be primarily interested in the iteration dynamics of iteration of f_a over the interval $[0, 1]$, where $f_a(x)$ is non-negative. One could argue that this is the only region of concern as far as applications to population dynamics are concerned, since populations are non-negative. Also note that if $x \notin [0, 1]$, then $f_a(x) \notin [0, 1]$. In other words, if we start with a point outside [0,1], then we can never get into the interval [0,1]. As summarize below, the dynamics of iteration outside the interval [0,1] is rather straightforward and uninteresting:

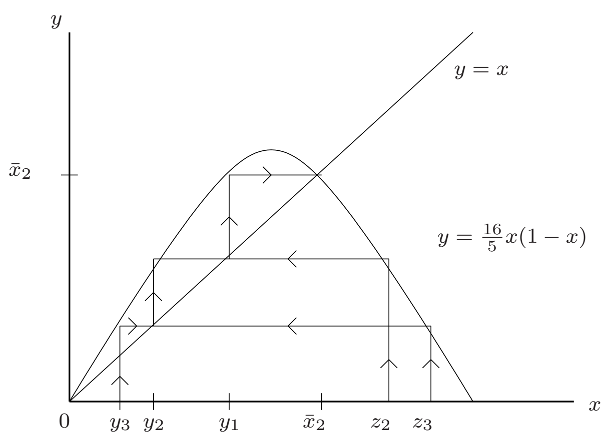
- For $0 < a < 1$ (see plot on previous page), points $x \in (\bar{x}_2(a), 0)$ will travel to $\bar{x}_1 = 0$ under iteration. Likewise, points $x \in (1, 1 - \bar{x}_2(a))$ are mapped to the interval $(\bar{x}_2(a), 0)$, after which they travel to $\bar{x}_1 = 0$ under iteration. Of course, $\bar{x}_2(a)$ is a fixed point and the point $x = 1 - \bar{x}_2(a)$ is mapped to $\bar{x}_2(a)$. All other points $x \notin [\bar{x}_2(a), 1 - \bar{x}_2(a)]$ travel to $-\infty$ under iteration of $f_a(x)$ (graphical exercise).

- When $a \geq 1$, all points $x \notin [0, 1]$ travel toward $-\infty$ under iteration of f_a, i.e. $f_a^n(x) \to -\infty$ as $n \to \infty$ for $x \notin [0, 1]$.

We can now focus on the dynamics of the iteration process $x_{n+1} = f_a(x_n)$ for $x_0 \in [0, 1]$. This is possible since f_a maps $[0, 1]$ into itself for $0 \leq a \leq 4$. When $a = 4$, f_a maps $[0, 1]$ *onto* itself. (We'll return to this later.) Let us now examine the dynamics as a increases from 0 to 3:

- Recall that for $0 \leq a < 1$, $f_a(x)$ has only one fixed point in [0,1], namely, $\bar{x}_1 = 0$. Moreover, for all $0 \leq a \leq 1$, this fixed point is **attractive**. From a look at the graph of $f_a(x)$ for these a-values, we see that for *any* $x_0 \in [0,1]$, $x_n \to 0$ as $n \to \infty$, where $x_{n+1} = f_a(x_n)$. (From a population viewpoint, the species is destined for extinction, as is expected, since $0 < a < 1$.) When $a = 1$, the fixed point $\bar{x}_1 = 0$ is neutral. Nevertheless, all iterates $x_n \to 0$ for all $x_0 \in [0,1]$.

- As a increases from 1, the fixed point $\bar{x}_1 = 0$ is **repulsive** and $\bar{x}_2 = \dfrac{a-1}{a}$ assumes the role of the **attractive fixed point**. The reader is invited to determine the motion of various sets of points on the interval $[0,1]$ as they approach $\bar{x}_2$ under iteration. For example, in the case $a = 2$: (i) if $x_0 \in \left(0, \frac{1}{2}\right)$, then $x_1 < x_2 < x_3, \dots$, i.e. $x_n \to \frac{1}{2}$ monotonically; (ii) if $x_0 \in \left(\frac{1}{2}, 1\right)$, then $x_1 \in \left(0, \frac{1}{2}\right)$ and $x_n \to \frac{1}{2}$ monotonically. For $2 < a < 3$, the motion of iterates is not as straightforward since the fixed point value $\bar{x}_2(a)$ now lies below the maximum value $f_a\left(\frac{1}{2}\right) = \frac{a}{4}$, as shown in the figure below. By running the iteration of f "backwards" as was done in Example 8 of Section 3.3, we discover a set of points $y_1, y_2, \dots$ such that $f^k(y_k) = \bar{x}_2$. These points y_k are called **preperiodic** points since they eventually land on a periodic orbit – in this case the fixed point $\bar{x}_2$ – after a finite number of iterations. Note that $y_k \to 0$ as $k \to \infty$, indicating that we can find points y_k arbitrarily close to the point $x = 0$. And for each point y_k, $k \geq 2$, there is a point z_k on the other side of the fixed point $\bar{x}_2$, that maps to $\bar{x}_2$ after k iterations, as shown in the diagram below. Also note that $z_k \to 1$ as $k \to \infty$. This rather complicated behaviour is made possible by the fact that the function $f(x)$ is "2-to-1": for any value $y \in \left[0, \frac{a}{4}\right)$, there are two distinct points x_1 and x_2 such that $f(x_1) = f(x_2) = y$. As in Example 8, Section 3.3, the reader is invited to construct a numerical algorithm to determine the points y_k and z_k that are sent to $\bar{x}_2$ after exactly k iterations.



$a > 2$: Preperiodic points y_k and z_k that map to $\bar{x}_2$ after k iterations.

All other points $x \notin \{0, 1, y_k, z_k, k \geq 1\}$ approach $\bar{x}_2$ as f_a is iterated, but never reach $\bar{x}_2$. When they are mapped sufficiently close to x_2, they oscillate about $\bar{x}_2$ as they approach it. Graphically, they trace out a "cobweb" that spirals inward to the point $(\bar{x}_2, \bar{x}_2)$: The iterates x_n jump from one side of the fixed point to the other approaching it in the process.

- When $a = 3$ – see the diagram on Page 89 – the fixed point $\bar{x}_2 = \frac{2}{3}$ is no longer attractive but "neutral". For an $x \in (0,1)$, the forward orbit, $x_n = f^n(x)$ does converge to $\bar{x}_2$. However, for any $a > 3$, the fixed point $\bar{x}_2$ is repulsive. The natural question is, "Where do the iterates go?" A glimpse of this behaviour was provided by Example 8 of Section 3.3. We shall analyze the dynamics in a later section.

In closing this section, let us present once again the "atlas" which summarizes the attractive/repulsive properties of the fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{a-1}{a}$ of the logistic map along with their locations, considered as functions of the parameter a. Dotted lines indicate repulsive/unstable behaviour and solid lines denote attractive/stable behaviour.