Lecture 9

Linear systems of difference equations

Solutions of homogeneous linear systems with constant coefficients

Introduction

The following discussion is meant to provide a motivation for the method that will eventually be presented to solve systems of linear d.e.s with constant coefficients. It will be covered "lightly". The student is **not** expected to remember this material.

We continue our discussion of systems of first order difference equations with constant coefficients in $\mathbb{R}^2$ which have the general form,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad n \geq 0. \tag{1}$$

In vector-matrix notation, the above system assumes the form,

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n \,, \tag{2}$$

with solution

$$\mathbf{x}_n = \mathbf{A}^n \mathbf{x}_0, \,. \tag{3}$$

By writing the equations for x_{n+2} and y_{n+2} in Eq. (1) and performing appropriate eliminations of variables, it can be shown (a nice exercise) that the individual sequences $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ satisfy a common second-order homogeneous difference equation with constant coefficients, namely,

$$\begin{aligned} x_{n+2} + px_{n+1} + qx_n &= 0 \\ y_{n+2} + py_{n+1} + qy_n &= 0 \,, \end{aligned} \tag{4}$$

where

$$p = -\mathrm{tr}\mathbf{A} = -(a_{11} + a_{12}) \,, \quad q = \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \,. \tag{5}$$

(This does not imply that the solutions for x_n and y_n are identical since the initial conditions for the two equations will be different in general.) Recall that the eigenvalues of $\mathbf{A}$, λ_1 and λ_2, must satisfy the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which, in the two-dimensional case, becomes the quadratic equation,

$$\lambda^2 - (\mathrm{tr}\mathbf{A})\lambda + (\det \mathbf{A}) = 0 \quad \text{or} \quad \lambda^2 + p\lambda + q = 0 \,. \tag{6}$$

In other words, the roots of the characteristic equation for the second-order d.e.'s in (4) are precisely the eigenvalues of $\mathbf{A}$, i.e. $m_1 = \lambda_1$ and $m_2 = \lambda_2$.

From the previous discussions on solutions to second-order d.e.'s such as (4), one would be tempted to write that the general solutions of the d.e.'s in Eq. (4) are (assuming, for simplicity, that $\lambda_1 \neq \lambda_2$)

$$\begin{aligned} x_n &= C_{11}\lambda_1^n + C_{12}\lambda_2^n, \\ y_n &= C_{21}\lambda_1^n + C_{22}\lambda_2^n, \end{aligned} \tag{7}$$

where the C_{ij}, $1 \leq i, j \leq 2$, are arbitrary real constants. There is a problem with these solutions, however. From Eq. (3) we know that a given set of **two** initial values, $\mathbf{x}_0 = (x_0, y_0)^T$, determines a unique solution sequence $\mathbf{x}_n$ to Eq. (2). However, the specification of unique solutions to the d.e.'s Eq. (7) would appear to require **four** initial values since there are four arbitrary constants C_{ij}. The resolution of this apparent paradox lies in the fact that the four constants C_{ij} are not entirely independent of each other, as we show below.

Let us examine the equation $\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0$, i.e. $n = 0$ in (2), written out in full form, using the forms of the solutions in Eq. (7):

$$\begin{pmatrix} C_{11}\lambda_1 + C_{12}\lambda_2 \\ C_{21}\lambda_1 + C_{22}\lambda_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} C_{11} + C_{12} \\ C_{21} + C_{22} \end{pmatrix}. \tag{8}$$

Now rewrite (8) as follows,

$$\lambda_1 \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} + \lambda_2 \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix}. \tag{9}$$

Eq. (9) is satisfied if $\mathbf{v}_1 = (C_{11}, C_{21})^T$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ_1 (or simply, a "λ_1-eigenvector of $\mathbf{A}$") and $\mathbf{v}_2 = (C_{12}, C_{22})^T$ is a λ_2-eigenvector of $\mathbf{A}$:

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \lambda_1 \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \lambda_2 \mathbf{v}_2 \end{aligned} \tag{10}$$

(There is a slight technical complication regarding Eq. (9) in that we have assumed the existence of two eigenvectors, which is true if $\lambda_1 \neq \lambda_2$. In the case $\lambda_1 = \lambda_2 = \lambda_1$ it is possible that only one eigenvector exists. In this case, we can still satisfy Eq. (9) with the choice $C_{11} = C_{12}$ and $C_{21} = C_{22}$, however a general solution to the d.e. has not been formed. We shall not address this complication for the moment.)

Now recall one additional property of eigenvectors: If $\mathbf{v}$ is a λ-eigenvector of matrix $\mathbf{A}$, then any constant multiple of $\mathbf{v}$, i.e., $\mathbf{u} = C\mathbf{v}$, for any $C \in \mathbb{R}$, is a λ-eigenvector of $\mathbf{A}$. The proof of this result is simple:

$$\mathbf{A}\mathbf{u} = A(C\mathbf{v}) = C\mathbf{A}\mathbf{v} = C\lambda\mathbf{v} = \lambda(C\mathbf{v}) = \lambda\mathbf{u}. \tag{11}$$

Thus, there is no **unique** λ-eigenvector of a matrix $\mathbf{A}$. The fact that $\mathbf{v}_1 = (C_{11}, C_{21})^T$ is a λ_1-eigenvector of $\mathbf{A}$ implies that C_{11} and C_{21} are not independent of each other. As such, there will be **only one degree of freedom** – or one arbitrary constant – associated with the vector $(C_{11}, C_{21})^T$.

The same is true for the λ_2-eigenvector $\mathbf{v}_2 = (C_{12}, C_{22})^T$. Therefore, the general solutions for x_n and y_n in Eq. (7) can be written as

$$\boxed{\mathbf{x}_n = C_1\lambda_1^n\mathbf{v}_1 + C_2\lambda_2^n\mathbf{v}_2, \quad n \geq 0,} \tag{12}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are, respectively, λ_1- and λ_2-eigenvectors of $\mathbf{A}$, and C_1 and C_2 are arbitrary constants. (Note that in practical applications, as we show below, we first choose $\mathbf{v}_1$ and $\mathbf{v}_2$, and then solve for particular values of C_1 and C_2. How we choose $\mathbf{v}_1$ and $\mathbf{v}_2$ is not important. Some authors like to construct normalized eigenvectors.)

Note the similarity between Eq. (12) and the general solution of a second order d.e. with constant coefficients studied earlier,

$$Y_k = C_1m_1^k + C_2m_2^k, \quad k \geq 0. \tag{13}$$

In fact, **Eq.** (12) **can be viewed as a kind of vector analogue of these solutions, since it gives the solutions of the two second-order d.e.'s in Eqs.** (4)**.** Note that these d.e.'s are **not independent**: They are coupled by the fact that they arose from a common linear first-order system, Eq. (1).

As stated earlier, Eq. (12) is applicable to the two cases in which the eigenvalues of $\mathbf{A}$ are distinct, i.e. real and complex conjugate eigenvalues. We shall also discuss the special case of equal real roots.

Case 1: Distinct real eigenvalues $\lambda_1 \neq \lambda_2$

In this case, two distinct and linearly independent eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, exist. As a result,

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2\,. \tag{14}$$

Recall that, in general, the **span** of a given set of vectors $\{\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_n\}$ is defined as the set of all possible linear combinations of these vectors, i.e.

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n\}. \tag{15}$$

In this case, Eq. (12) gives the general solution to the linear system.

But let's look at the problem in another way, by using the fact that the eigenvectors of the matrix $\mathbf{A}$, $\mathbf{v}_1$ and $\mathbf{v}_2$, span $\mathbb{R}^2$. This means that given a starting point $\mathbf{x}_0 = (x_0, y_0)^T$, there exists a unique pair of numbers, say, a_1 and a_2 such that

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2\,. \tag{16}$$

Now substitute this starting point into the linear system in Eq. (2):

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 \\ &= \mathbf{A}[a_1\mathbf{v}_1 + a_2\mathbf{v}_2] \\ &= a_1\mathbf{A}\mathbf{v}_1 + a_2\mathbf{A}\mathbf{v}_2 \\ &= a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2\,. \end{aligned} \tag{17}$$

Now apply the matrix $\mathbf{A}$ to compute $\mathbf{x}_2$:

$$\begin{aligned}\mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 \\ &= \mathbf{A}[a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2] \\ &= a_1\lambda_1\mathbf{A}\mathbf{v}_1 + a_2\lambda_2\mathbf{A}\mathbf{v}_2 \\ &= a_1\lambda_1^2\mathbf{v}_1 + a_2\lambda_2^2\mathbf{v}_2\,. \end{aligned} \tag{18}$$

The reader should see the pattern. After n applications of $\mathbf{A}$, we have

$$\mathbf{x}_n = a_1\lambda_1^n\mathbf{v}_1 + a_2\lambda_2^n\mathbf{v}_2\,. \tag{19}$$

The pair of coefficients (a_1, a_2) corresponded to the initial point $\mathbf{x_0} = (x_0, y_0)^T$. By varying the initial point, we can generate all solutions to the d.e., which implies that a_1 and a_2 may be considered as arbitrary constants. This leads to the result that the general solution of the d.e. is

$$\mathbf{x}_n = C_1\lambda_1^n\mathbf{v}_1 + C_2\lambda_2^n\mathbf{v}_2, \quad n \geq 0, \tag{20}$$

in agreement with Eq. (12).

Example:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

The first step is to find the eigenvalues and associated eigenvectors of $\mathbf{A}$. To find the eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

yielding the quadratic equation

$$\lambda^2 - 5\lambda + 6 = 0$$

with roots $\lambda_1 = 3$, $\lambda_2 = 2$.

Now find the λ_1-eigenvector, which we shall denote as $\mathbf{v} = (v_1, v_2)^T$. From the equation

$$\mathbf{A}\mathbf{v} = 3\mathbf{v}\,,$$

we have, in component form,

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

yielding the equations

$$\begin{aligned} -2v_1 + 2v_2 &= 0 \\ -v_1 + v_2 &= 0, \end{aligned}$$

both of which imply that $v_1 = v_2$. If we choose $v_1 = 1$, then $v_2 = 1$, yielding $\mathbf{v}_1 = (1,1)^T$.

It's not a bad idea to check the result before proceeding. (We won't do this every time in these notes, but it's still a good idea.) Subsituting our result into the LHS of the d.e.,

$$\mathbf{A}\mathbf{v} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\mathbf{v}\,,$$

which confirms the result.

Similarly, for the λ_2-eigenvector,

$$\mathbf{A}\mathbf{v} = 2\mathbf{v}$$

implies that

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This yields

$$-v_1 + 2v_2 = 0,$$

implying that $v_1 = 2v_2$. If we choose $v_2 = 1$, then $\mathbf{v}_2 = (2,1)^T$. Thus, the general solution of this linear system is

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = C_1 3^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 2^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Suppose that the initial condition is $(x_0, y_0) = (3,1)$. We must find C_1 and C_2 which satisfies the above vector equation for $n = 0$. In component form,

$$\begin{aligned} 3 &= C_1 + 2C_2 \\ 1 &= C_1 + C_2\,. \end{aligned} \tag{21}$$

There are a number of ways to solve this linear system, including the formal "Cramer Rule" approach. That being said, if we subtract the second equation from the first, we obtain

$$2 = C_2\,. \tag{22}$$

Substitution of this result into either equation yields $C_1 = -1$. As a result, the solution of the linear system satisfying the initial condition $(x_0, y_0) = (3,1)$ is

$$\begin{aligned} x_n &= -3^n + 2^{n+2} \\ y_n &= -3^n + 2^{n+1}. \end{aligned}$$

Now suppose that the initial condition is $(x_0, y_0) = (1,1)$. We then find that $C_1 = 1$ and $C_2 = 0$. In other words

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 3^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we interpret the pair $(x_n, v_n)^T$ as a vector, then the solution is simply the vector $(1,1)$ being "stretched" or expanded by factors 3^n. However, let us recall that $\mathbf{v}_1(1,1)^T$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 3$. The motion of the (x_n, y_n) indicates that we are moving in the direction of the eigenvector $\mathbf{v}_1$ away from (0,0). In fact, if our initial condition (x_0, y_0) represented **any** multiple of $\mathbf{v}_1$, the points (x_n, y_n) would all move in the direction of $\mathbf{v}_1$ since:

$$\mathbf{A}^n(C\mathbf{v}_1) = C\mathbf{A}^n\mathbf{v}_1 = C\lambda_1^n\mathbf{v}_1 = \lambda_1^n(C\mathbf{v}_1). \tag{23}$$

Let us define the subspace $E_1 \subset \mathbb{R}^2$ as follows:

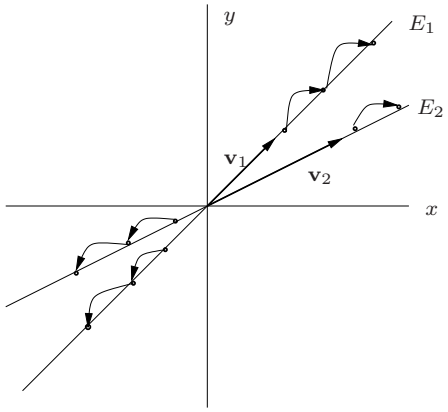
$$E_1 = \left\{(x,y) \in \mathbb{R}^2 | \, (x,y) = c\mathbf{v}_1 \quad \text{for some} \quad c \in \mathbb{R}\right\}. \tag{24}$$

We may think of E_1 as the "λ_1-eigenspace" of $\mathbf{A}$. If $P = (x,y) \in E_1$, then the vector $\bar{OP}$ is a λ_1-eigenvector of $\mathbf{A}$. If we choose the initial point $\mathbf{x}_0 = (x_0, y_0)^T \in E_1$, then all points $\mathbf{x}_n = (x_n, y_n)^T$ given by $\mathbf{x}_n = \mathbf{A}^n x_0$ lie in E_1. We say that $E_1 \subset \mathbb{R}^2$ is an **invariant subspace** of $\mathbb{R}^2$ under the action of $\mathbf{A}$.

Likewise, if we define the "λ_2-eigenspace" of $\mathbf{A}$ as

$$E_2 = \left\{(x,y) \in \mathbb{R}^2 \mid (x,y) = c\mathbf{v}_2 \quad \text{for some} \quad c \in \mathbb{R}\right\} \tag{25}$$

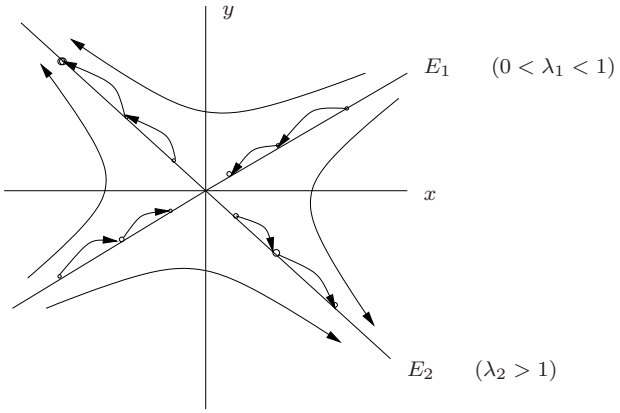
Then E_2 is an invariant subspace of $\mathbb{R}^2$. If we choose an $(x_0, y_0) \in E_2$, then all points (x_n, y_n) in the solution of our linear system will lie in E_2. We illustrate these spaces below.



Note that $E_1 \cap E_2 = \{(0,0)\}$. Since the general solution involves linear combinations of the expanding vectors $\lambda_1^n\mathbf{v}_1$ and $\lambda_2^n\mathbf{v}_2$, it is clear that all the other points $(x_n, y_n) \in \mathbb{R}^2$, i.e. those not in either of E_1 or E_2, will be sent away from the origin and off to infinity. In short, the phase portrait for this system is a sheared version of the portrait on Page 57 for the case $\mathbf{A}$ diagonal with eigenvalues $\lambda_1 > 1$, $\lambda_2 > 1$. The invariant spaces have been rotated from their positions on the x- and y-axes to their positions on the lines $y = x$ and $y = \frac{1}{2}x$.

Let us now consider a general case where both eigenvalues are positive, but $\lambda_1 < 1$ and $\lambda_2 > 1$. Then, points lying in the λ_1-eigenspace E_1 will be mapped toward the origin $(0,0)$, whereas points in

E_2 will be mapped away from the origin. This feature, along with its effects on points that do not lie in either of these two spaces, is sketched below.



This looks like a "sheared" version of the phase portrait on Page 60. The x-axis of the diagram on Page 60 has been moved to the line E_1 above. The y-axis has been moved to E_2.

The invariance of the eigenspaces E_1 and E_2 and its effect on the phase portraits suggests that these spaces provide "natural coordinate axes" for the phase portraits. In fact, when $\mathbf{A}$ is diagonal, i.e.

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

then E_1 is, in fact, the x-axis, since $\mathbf{v}_1 = (1,0)^T$ is a λ_1-eigenvector of $\mathbf{A}$. Likewise, E_2 is the y-axis. This picture is very pleasant to us since, in the usual Cartesian coordinate system, we choose $\mathbf{i} = (1,0)^T$ and $\mathbf{j} = (0,1)^T$ as basis vectors. When $\mathbf{A}$ is non-diagonal (which implies that $\mathbf{i}$ and $\mathbf{j}$ are not eigenvectors), then the natural basis vectors to use are eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. These ideas will be explored further in a later section.

Lecture 10

Linear systems of difference equations (cont'd)

Let us now discuss the results of the previous lecture in another way. Before we consider difference equations, however, we'll mention an important general point. Suppose that $\mathbf{A}$ is an $n \times n$ real matrix, and suppose that λ is a real eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{v}$, i.e.,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}\,. \tag{26}$$

The eigenvector $\mathbf{v}$ is often called the λ-eigenvector of $\mathbf{A}$. This implies that

$$\mathbf{A}^2\mathbf{v} = \mathbf{A}(\mathbf{A}v) = \mathbf{A}(\lambda\mathbf{v}) = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}\,. \tag{27}$$

Repeated application yields

$$\mathbf{A}^n\mathbf{v} = \lambda^n\mathbf{v}\,. \tag{28}$$

Now define the following one-dimensional subspace of $\mathbb{R}^n$, often called the λ-eigenspace of $\mathbf{A}$,

$$E = \text{span}\{\mathbf{v}\} = \{t\mathbf{v}\,,\ t \in \mathbb{R}\}\,. \tag{29}$$

Now let $\mathbf{x} \in E$, which implies that $\mathbf{x} = t\mathbf{v}$ for some $t \in \mathbb{R}$. Then

$$\mathbf{A}\mathbf{x} = \mathbf{A}(t\mathbf{v}) = t\mathbf{A}\mathbf{v} = t\lambda\mathbf{v} = \lambda(t\mathbf{v}) = \lambda\mathbf{x} \in E\,. \tag{30}$$

This proves that **the matrix A maps the subspace E to itself.** As such, E is called an **invariant set** of $\mathbf{A}$. In fact, E is an **invariant subspace** since its elements satisfy the requirements of a linear vector space. It is a one-dimensional linear space. Furthermore, from Eq. (28), if you start at a point in E and apply $\mathbf{A}$ to it iteratively, you remain in E.

Let us now return to our first order linear system of difference equations in $\mathbb{R}^2$, written in the following vector-matrix form,

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n\,. \tag{31}$$

Given an initial condition $\mathbf{x}_0 = (x_0, y_0)^T \in \mathbb{R}^2$, the solution of the above d.e. is

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0\,, \quad n \geq 0\,. \tag{32}$$

Now assume that $\mathbf{A}$ has a real eigenvalue $\lambda \in \mathbb{R}$. Once again let

$$E = \text{span}\{\mathbf{v}\} = \{t\mathbf{v}\,,\ t \in \mathbb{R}\}\,. \tag{33}$$

Here is the important point: If the initial point $\mathbf{x}_0 \in E$, then from Eq. (28), all points $\mathbf{x}_n \in E$. But there is more: From Eq. (32),

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0 = \lambda^n\mathbf{x}_0\,, \quad n \geq 0\,. \tag{34}$$

Now take magnitudes of these vectors,

$$\|\mathbf{x}_n\| = |\lambda|^n \|\mathbf{x}_0\|\,, \tag{35}$$

where $\|\mathbf{v}\|$ denotes the **magnitude** or **Euclidean length** of the vector $\mathbf{v} \in \mathbb{R}^2$. Note that the magnitude of λ determines the behaviour of the iterates $\mathbf{x}_n$:

1. If $|\lambda| < 1$, then $\|\mathbf{x}_n\| \to 0$ as $n \to \infty$. In this case, E is called a **stable** subspace since the points $\mathbf{x}_n$ approach the fixed point $\mathbf{0}$.

2. If $|\lambda| > 1$, then $\|\mathbf{x}_n\| \to \infty$ as $n \to \infty$. In this case, E is called an **unstable** subspace since the iterates $\mathbf{x}_n$ move away from the fixed point $\mathbf{0}$.

3. If $|\lambda| = 1$, then $\|\mathbf{x}_n\| = \|\mathbf{x}_0\|$ for $n \geq 0$.

We now return to Case 1 of the previous lecture, where the **eigenvalues** λ_1 and λ_2 of $\mathbf{A}$ are assumed to be **real** and **distinct** with associated $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, i.e.,

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \lambda_1 \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \lambda_2 \mathbf{v}_2\,. \end{aligned} \tag{36}$$

In this case, $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis of $\mathbb{R}^2$. Recall that given a starting point $\mathbf{x}_0 = (x_0, y_0)^T \in \mathbb{R}^2$, there exists a unique pair of numbers a_1 and a_2 such that

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2\,. \tag{37}$$

From this equation, we then derived the following expression for the iterates $\mathbf{x}_n$:

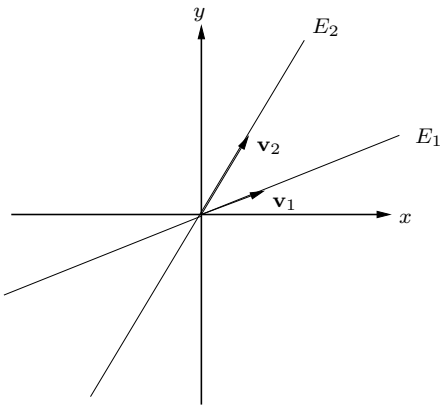
$$\mathbf{x}_n = a_1\lambda_1^n\mathbf{v}_1 + a_2\lambda_2^n\mathbf{v}_2\,. \tag{38}$$

Note that each $\mathbf{x}_n$ is expressed in terms of the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Now define the following subspaces of $\mathbb{R}^2$:

$$\begin{aligned} E_1 &= \text{span}\{\mathbf{v}_1\} = \{t\mathbf{v}_1\,,\ t \in \mathbb{R}\} \\ E_2 &= \text{span}\{\mathbf{v}_2\} = \{t\mathbf{v}_2\,,\ t \in \mathbb{R}\}\,. \end{aligned} \tag{39}$$

A generic situation is sketched below.



From our earlier discussion, each of the subspaces E_1 and E_2 is invariant under the action of $\mathbf{A}$. If we start at a point $\mathbf{x}_0 \in E_1$, we remain in E_1 under iteration of $\mathbf{A}$. But this case corresponds to setting $a_2 = 0$ in Eq. (37): The point $a_1\mathbf{v}_1$ must lie in E_1. But if $a_2 = 0$, then from Eq. (38), $\mathbf{x}_n$ also lies in E_1 for all $n \geq 0$.

Similarily, if we start at a point $\mathbf{x}_0 \in E_2$, we remain in E_2 under iteration of $\mathbf{A}$. In this case $a_1 = 0$.

There are two very simple cases that can be immediately derived from this analysis.

Case No. 1: $|\lambda_1| < 1$ and $|\lambda_2| < 1$

In this case:

1. If $\mathbf{x}_0 \in E_1$, then $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0 = \lambda_1^n\mathbf{x}_0 \to \mathbf{0}$ as $n \to \infty$.
2. If $\mathbf{x}_0 \in E_2$, then $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0 = \lambda_2^n\mathbf{x}_0 \to \mathbf{0}$ as $n \to \infty$.

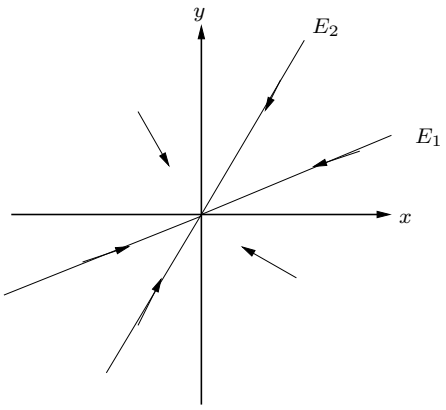
Both E_1 and E_2 are **stable** subspaces. As such, the iterates move toward $\mathbf{0}$ along the lines defining the E_i. If the point $\mathbf{x}_0$ does not lie in any of the spaces E_i, i.e., if both a_1 and a_2 are nonzero in Eq. (38), it should not be difficult to see that $\mathbf{x}_n \to \mathbf{0}$ as $n \to \infty$. Therefore, the fixed point $\bar{\mathbf{x}} = \mathbf{0}$ is **attractive**. A generic situation is sketched below.

Case No. 2: $|\lambda_1| > 1$ and $|\lambda_2| > 1$

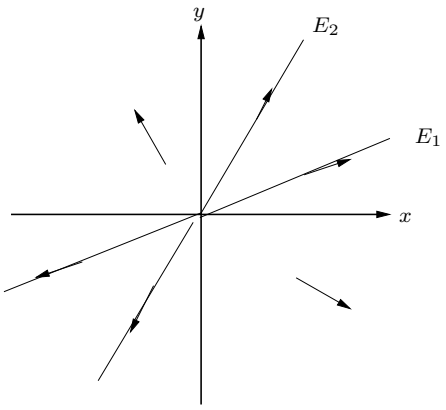
In this case:

1. If $\mathbf{x}_0 \in E_1$, then $\|\mathbf{x}_n\| = \|\mathbf{A}^n\mathbf{x}_0\| = |\lambda_1|^n\|\mathbf{x}_0\| \to \infty$ as $n \to \infty$.
2. If $\mathbf{x}_0 \in E_2$, then $\|\mathbf{x}_n\| = \|\mathbf{A}^n\mathbf{x}_0\| = |\lambda_2|^n\|\mathbf{x}_0\| \to \infty$ as $n \to \infty$.

Both E_1 and E_2 are **unstable** subspaces. In this case, the iterates move away from the point $\mathbf{x} = \mathbf{0}$. If $\mathbf{x}_0$ does not lie in either of the spaces E_1 or E_2, then both a_1 and a_2 are nonzero in Eq. (38). With a little bit of work – we omit it here – it can be shown that $\|\mathbf{x}_n\| \to \infty$ as $n \to \infty$. Therefore, the fixed point $\bar{\mathbf{x}} = \mathbf{0}$ is **repulsive**. A generic situation is sketched below.



Case No. 1: $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$.



Case No. 2: $\lambda_1 > 1$ and $\lambda_2 > 1$.

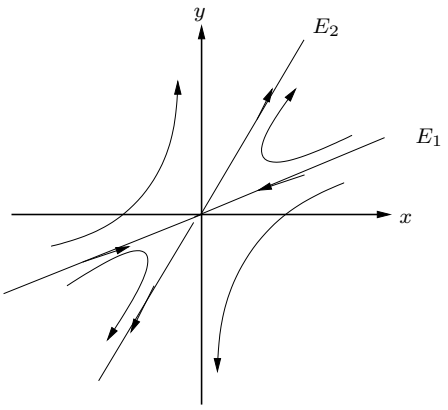
Case No. 3: $|\lambda_1| < 1$ **and** $|\lambda_2| > 1$

In this case:

1. If $\mathbf{x}_0 \in E_1$, then $\|\mathbf{x}_n\| = \|\mathbf{A}^n\mathbf{x}_0\| = |\lambda_1|^n\|\mathbf{x}_0\| \to 0$ as $n \to \infty$. E_1 is a **stable** subspace.

2. If $\mathbf{x}_0 \in E_2$, then $\|\mathbf{x}_n\| = \|\mathbf{A}^n\mathbf{x}_0\| = |\lambda_2|^n\|\mathbf{x}_0\| \to \infty$ as $n \to \infty$. E_2 is an **unstable** subspace.

Therefore, the fixed point $\bar{\mathbf{x}}_0$ is once again a **saddle point**. A generic situation is sketched below. Note that in all of the three figures above, we have designated the motion, or "flow", of the solutions or iterates, $\mathbf{x}_n$, in terms of smooth lines, as opposed to jumps from one dot to the next, as was done in previous lectures. This is perfectly fine, even in the case of negative eigenvalues, where the iterates alternate from one side of the origin to the other while either approaching the origin or moving away from it.



Case No. 3: $0 < \lambda_1 < 1$ and $\lambda_2 > 1$.

Case 2: Complex Eigenvalues

We now consider the case when the 2×2 (real) matrix $\mathbf{A}$ defining the d.e.,

$$\mathbf{x}_{n+1}\mathbf{A} = \mathbf{x}_n \,, \tag{40}$$

has a complex eigenvalue λ with associated complex eigenvector $\mathbf{v}$, i.e.,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \,, \quad \lambda \in \mathbb{C} \,, \quad \mathbf{v} \in \mathbb{C}^2 \,. \tag{41}$$

Now take complex conjugates of both sides of (41) to obtain

$$\mathbf{A}\mathbf{v}^* = \lambda^*\mathbf{v}^* \,, \tag{42}$$

which implies that λ^*, the complex conjugate of λ, is also an eigenvalue of $\mathbf{A}$, with associated eigenvector $\mathbf{v}^*$, the complex conjugate of $\mathbf{v}$. (Note that this result is possible because the matrix $\mathbf{A}$ is real.)

The fact that the two eigenvalues are complex conjugates of each other implies that they are "connected" to each other. Unlike the real-eigenvalue case studied earlier, there are no one-dimensional invariant subspaces here. The two complex conjugate eigenvalues will define a two-dimensional invariant subspace which, in this case, is the entire space $\mathbb{R}^2$ over which the difference equation is defined.

Example:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

yields the characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0$$

with roots $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$.

Find an eigenvector for $\lambda_1 = 1 + i$:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1+i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

yielding $-v_2 = iv_1$. If we choose $v_1 = 1$, then $\mathbf{v}_1 = (1, -i)^T$. Similarly, we find, as expected that $\mathbf{v}_2 = (1, i)^T$ is a $(1 - i)$-eigenvector.

Since $\lambda \neq \lambda^*$ (it is assumed that λ is complex, i.e., its imaginary component is nonzero), we could, in principle, employ the same solution as we did for distinct real roots in Case 1 from the last lecture. The general solution to the d.e. in (40) would then be given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = C_1(1+i)^n \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2(1-i)^n \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

If we impose real initial conditions (x_0, y_0), then C_1 and C_2 will be determined so that (x_n, y_n) will be real. However, having the solution in such a complex form is not very helpful, especially if one desires a picture of how the points (x_n, y_n) are behaving in two-dimensional, real "phase space". It is convenient to extract real solutions from the above, much in the same way as was done for scalar d.e.'s with complex roots.

We now consider the general case where $\mathbf{A}$ is real with complex conjugate eigenvalues λ and λ^*. (It doesn't matter which of the complex eigenvalues we consider to be λ.) In Cartesian form, we'll let λ be represented as follows,

$$\lambda = a + ib\,, \qquad a, b \in \mathbb{R}\,. \tag{43}$$

so that the other eigenvalue is

$$\lambda^* = a - ib\,. \tag{44}$$

It will also be useful to express the complex eigenvalue in polar form, i.e.,

$$\lambda = re^{i\theta} = r(\cos\theta + i\sin\theta)\,, \tag{45}$$

where

$$r = \sqrt{a^2 + b^2}\,, \qquad \tan\theta = \frac{b}{a}\,. \tag{46}$$

From the polar form comes the following result which will be useful below,

$$\lambda^n = re^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))\,. \tag{47}$$

We'll also express the eigenvector $\mathbf{v}$ of λ in terms of real and imaginary components,

$$\mathbf{v} = \mathbf{v}_R + i\mathbf{v}_I\,, \tag{48}$$

where $\mathbf{v}_R$ and $\mathbf{v}_I$ are real-valued vectors in $\mathbb{R}^2$. This implies that

$$\mathbf{v}^* = \mathbf{v}_R - i\mathbf{v}_I \tag{49}$$

is the λ^*-eigenvector of $\mathbf{A}$.

We now employ a standard result from linear algebra: **The vectors $\mathbf{v}_R$ and $\mathbf{v}_I$ are linearly independent and therefore form a basis for $\mathbb{R}^2$.**

Aside: Briefly, the proof of this result is as follows. The complex vectors $\mathbf{v}$ and $\mathbf{v}^*$ are linearly independent in the space of complex ordered pairs $\mathbb{C}^2$. (This rests on the assumption that $\mathbf{v}_I \neq \mathbf{0}$ - otherwise, the eigenvalues are real and equal, as are the eigenvectors!) This implies that the vector equation,

$$C_1\mathbf{v} + C_2\mathbf{v}^* = \mathbf{0}\,, \tag{50}$$

is true if and only if $C_1 = C_2 = 0$. From this, after a little algebra (separate equation into real and imaginary components) that

$$c_1\mathbf{v}_R + c_2\mathbf{v}_I = 0 \tag{51}$$

is true if and only if $c_1 = c_2 = 0$.

As we did in the case of distinct real roots, let us express the initial point $\mathbf{x}_0$ of our solution to (40) in terms of this basis, i.e.

$$\mathbf{x}_0 = C_1\mathbf{v}_R + C_2\mathbf{v}_I, \tag{52}$$

The solution to the d.e. in (40) is will then given by

$$\begin{aligned}
\mathbf{x}_n &= \mathbf{A}^n\mathbf{x}_0 && (53)\\
&= \mathbf{A}^n(C_1\mathbf{v}_R + C_2\mathbf{v}_I) && (54)\\
&= C_1\mathbf{A}^n\mathbf{v}_R + C_2\mathbf{A}^n\mathbf{v}_I. && (55)
\end{aligned}$$

It therefore remains to determine the action of $\mathbf{A}^n$ on the basis vectors $\mathbf{v}_R$ and $\mathbf{v}_I$. Since $\mathbf{v}$ is a λ-eigenvector of $\mathbf{A}$, we have

$$\mathbf{A}^n\mathbf{v} = \lambda^n\mathbf{v} \tag{56}$$

or

$$\mathbf{A}^n(\mathbf{v}_R + i\mathbf{v}_I) = \lambda^n(\mathbf{v}_R + i\mathbf{v}_i). \tag{57}$$

Since $\mathbf{A}$ is real, we may equate real and imaginary parts of both sides of this equation to give

$$\mathbf{A}^n\mathbf{v}_R = \mathrm{Re}(\lambda^n\mathbf{v}), \qquad \mathbf{A}^n\mathbf{v}_I = \mathrm{Im}(\lambda^n\mathbf{v}). \tag{58}$$

From Eq. (47), we have that

$$\lambda^n\mathbf{v} = r^n[\cos(n\theta) + i\sin(n\theta)][\mathbf{v}_R + i\mathbf{v}_I]$$

so that

$$\begin{aligned}\mathrm{Re}(\lambda^n \mathbf{v}) &= r^n[\cos(n\theta)\mathbf{v}_R - \sin(n\theta)\mathbf{v}_I] \\ \mathrm{Im}(\lambda^n \mathbf{v}) &= r^n[(\sin n\theta)\mathbf{v}_R + (\cos n\theta)\mathbf{v}_I].\end{aligned} \tag{59}$$

Substitution of these results into Eq. (55) yields the following,

$$\begin{aligned}\mathbf{x}_n &= C_1\mathbf{A}^n\mathbf{v}_R + C_2\mathbf{A}^n\mathbf{v}_I \\ &= C_1\mathrm{Re}(\lambda^n\mathbf{v}) + C_2\mathrm{Im}(\lambda^n\mathbf{v}) \\ &= C_1 r^n[\cos(n\theta)\mathbf{v}_R - \sin(n\theta)\mathbf{v}_I] + C_2 r^n[\sin(n\theta)\mathbf{v}_R + \cos(n\theta)\mathbf{v}_I]\,,\end{aligned} \tag{60}$$

which can be rearranged to produce the final result,

$$\mathbf{x}_n = r^n[C_1\cos(n\theta) + C_2\sin(n\theta)]\mathbf{v}_R + r^n[-C_1\sin(n\theta) + C_2\cos(n\theta)]\mathbf{v}_I. \tag{61}$$

This represents the general solution with C_1 and C_2 as the arbitrary constants. Given any initial condition $\mathbf{x}_0$, we can find a unique pair of values C_1, C_2 so that Eq. (52) holds.

This solution may also be expressed in phase-shifted form:

$$\mathbf{x}_n = Ar^n\left[\cos(n\theta + B)\mathbf{v}_R - \sin(n\theta + B)\mathbf{v}_I\right] \tag{62}$$

where

$$A = \sqrt{C_1^2 + C_2^2}, \qquad \tan\theta = -\frac{C_2}{C_1}. \tag{63}$$

In this case, A and B play the role of arbitrary constants.

These formulas look quite formidable! Instead of trying to work with memorized formulas, it is perhaps better to begin with the eigenvalue λ in polar form, Eq. (45), and the eigenvector $\mathbf{v}$ written as a sum of real and imaginary vectors, Eq. (48), letting the mathematics take over from there.

To illustrate, let's return to the example introduced earlier, and choose $\lambda = (1+i)$ so that $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$, i.e.,

$$\lambda = 2^{1/2}e^{i\frac{\pi}{4}} = 2^{1/2}\left[\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right]\,.$$

Since $\mathbf{v} = (1,-i)^T$, we have

$$\mathbf{v}_R = (1,0)^T\,, \quad \mathbf{v}_I = (0,-1)^T\,.$$

Then,

$$\mathbf{A}^n\mathbf{v} = \lambda^n\mathbf{v} = 2^{n/2}\left[\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right]\left[\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\-1\end{pmatrix}\right],$$

so that

$$\begin{aligned}\mathbf{A}^n\mathbf{v}_R &= \mathrm{Re}[\mathbf{A}^n\mathbf{v}] \\ &= 2^{n/2}\begin{pmatrix}\cos(\frac{n\pi}{4}) \\ \sin(\frac{n\pi}{4})\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\mathbf{A}^n\mathbf{v}_I &= \mathrm{Im}[\mathbf{A}^n\mathbf{v}] \\ &= 2^{n/2}\begin{pmatrix}\sin(\frac{n\pi}{4}) \\ -\cos(\frac{n\pi}{4})\end{pmatrix}.\end{aligned}$$

If

$$\begin{pmatrix}x_0 \\ y_0\end{pmatrix} = C_1\begin{pmatrix}1 \\ 0\end{pmatrix} + C_2\begin{pmatrix}0 \\ -1\end{pmatrix}.$$

then from Eq. (55), we have

$$\mathbf{x}_n = C_1\mathbf{A}^n\mathbf{v}_R + C_2\mathbf{A}^n\mathbf{v_I}\,,$$

which, in component form, becomes

$$\begin{pmatrix}x_n \\ y_n\end{pmatrix} = 2^{n/2}\begin{pmatrix}C_1\cos(\frac{n\pi}{4}) + C_2\sin(\frac{n\pi}{4}) \\ \\ C_1\sin(\frac{n\pi}{4}) - C_2\cos(\frac{n\pi}{4})\end{pmatrix}.$$

(This is in agreement with (61).) Suppose that the initial condition is $(x_0, y_0) = (1, 0)$. Setting $n = 0$ we have

$$\begin{aligned}1 &= C_1 \\ 0 &= -C_2.\end{aligned}$$

Therefore, the solution is

$$\begin{pmatrix}x_n \\ y_n\end{pmatrix} = 2^{n/2}\begin{pmatrix}\cos\left(\frac{n\pi}{4}\right) \\ \sin\left(\frac{n\pi}{4}\right)\end{pmatrix}.$$

In phase-shifted form, the general solution to this problem is given as (Exercise):

$$\begin{pmatrix}x_n \\ y_n\end{pmatrix} = A2^{n/2}\begin{pmatrix}\cos\left(\frac{n\pi}{4} + B\right) \\ \sin\left(\frac{n\pi}{4} + B\right)\end{pmatrix}.$$

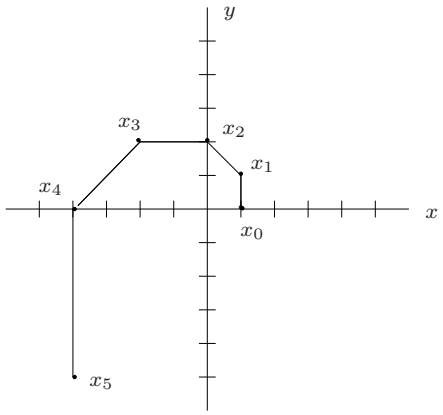
Using the initial condition $(x_0, y_0) = (1, 0)$ and setting $n = 0$ gives the equations

$$\begin{aligned}1 &= A\cos(B) \\ 0 &= A\sin(B).\end{aligned}$$

Thus, $\tan(B) = 0$, implying that $B = 0$. Therefore $A = 1$ and we arrive at the same solution as above.

The reader should check these results by (1) computing $\mathbf{x}_n$ for $n = 1, 2, 3$ using the above solution and (2) computing the $\mathbf{x}_n$ directly from the linear system using the initial values $(x_0, y_0) = (1, 0)$. (It is always a good idea to check solutions in this way.)

Note that in the general solution above, the points $(\cos(\frac{n\pi}{4}), \sin(\frac{n\pi}{4}))$ lie on the unit circle. As n increases by one, we move along the circle in a counterclockwise direction over an angle of $\theta = \frac{\pi}{4}$. However, the actual points (x_n, y_n) lie at distance $2^{n/2}$ from the origin. The combination of rotation and outward motions produces an outward spiralling motion of the points (x_n, y_n) as shown below.



General comments regarding the asymptotic behaviour of solutions for the case of complex eigenvalues

Let's now return to the general solution for the case of complex eigenvalues, Eq. (61) above, and reproduced here for convenience,

$$\mathbf{x}_n = r^n[C_1\cos(n\theta) + C_2\sin(n\theta)]\mathbf{v}_R + r^n[-C_1\sin(n\theta) + C_2\cos(n\theta)]\mathbf{v}_I\,, \tag{64}$$

Although rather formidable, one can deduce the general qualitative, as well as asymptotic, behaviour of such solutions from this equation.

First of all, the factors in square brackets are oscillatory. This is perhaps even better illustrated in the phase-shifted form of the solution in Eq. (62), reproduced here,

$$\mathbf{x}_n = Ar^n\left[\cos(n\theta + B)\mathbf{v}_R - \sin(n\theta + B)\mathbf{v}_I\right] \tag{65}$$

As such, the solution will be also oscillating in terms of its $\mathbf{v}_R$ and $\mathbf{v}_I$ components. The "jump angle" is θ, the argument of the complex eigenvalue λ.

The asymptotic behaviour of solutions, however, is determined by the multiplicative factor r^n. Recall that $r = |\lambda|$, the magnitude of the complex eigenvalue λ (and, of course, its conjugate λ^*):

1. If $|\lambda| = r < 1$, then $r^n \to 0$, implying that all solutions $\mathbf{x}_n \to 0$ as $n \to \infty$, generally in a spiralling manner.

2. If $r = |\lambda| > 1$, then $r^n \to \infty$, implying that solutions will exhibit the "oscillatory divergence" mentioned in an earlier lecture. This means that there will be an infinite sequence of indices $n_1 < n_2 < n_3 \cdots$, with $n_k \to \infty$ as $k \to \infty$ such that $\|\mathbf{x}_{n_k}\| \to \infty$ as $k \to \infty$. That being said, we can say, somewhat "loosely", that $\|\mathbf{x}_n\| \to \infty$.

3. If $r = |\lambda| = 1$, then the elements $\mathbf{x}_n$ are bounded: They don't go off to infinity, nor do they go to zero.

Note also that in the case of complex (conjugate) eigenvalues, there are no "preferred" one-dimensional eigenspaces E_1 and E_2 as was the case for real roots. The existence of complex eigenvalues implies a rotation in two-dimensional phase space. Of course, no rotation exists if $\theta = 0$, but this would imply that the roots are no longer complex.

Case 3: Equal real eigenvalues $\lambda_1 = \lambda_2 = \lambda$

This case was not covered in class, so you are not responsible for it in any examination.

In this case, it is not guaranteed that the 2×2 matrix $\mathbf{A}$ has two linearly independent λ-eigenvectors. If $\mathbf{A}$ is a diagonal 2×2 matrix, i.e. $\mathbf{A} = \lambda\mathbf{I}$, then $v_1 = (1, 0)^T$ and $\mathbf{v}_2 = (0, 1)^T$

are two such λ-eigenvectors and the general solution to the system $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is

$$\mathbf{x}_n = C_1\lambda^n\mathbf{v}_1 + C_2\lambda^n\mathbf{v}_2. \tag{66}$$

However, if $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{67}$$

then only one λ-eigenvector, $\mathbf{v} = (1, 0)^T$ (or any multiple thereof) exists. This is the typical situation for matrices with equal eigenvalues. For $\mathbf{A}$ in (67), one would be tempted to assume a solution of the form

$$\mathbf{x}_n = C_1\lambda^n\mathbf{v} + C_2 n\lambda^n\mathbf{v}, \tag{68}$$

following the procedure used for second order d.e.'s whose characteristic equations have equal roots. Unfortunately, this form of $\mathbf{x}_n$ does **not** satisfy the d.e. $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$, as the reader may wish to verify. The term $C_1\lambda^n\mathbf{v}$ works, but there is a problem with the second term.

We briefly outline a method to construct the general solution for 2×2 matrices $\mathbf{A}$ with equal (real) eigenvalues but only one eigenvector. The method can be extended, in a systematic but more complicated way, to matrices $\mathbf{A}$ of higher dimension.

First of all, we let E_v denote the λ-eigenspace of $\mathbb{R}^2$ spanned by $\mathbf{v}$:

$$E_v \{(x, y) \mid (x, y) = c\mathbf{v} \quad \text{for some} \quad c \in \mathbb{R}\}. \tag{69}$$

Now define the matrix

$$\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}. \tag{70}$$

For the matrix $\mathbf{A}$ given in (67),

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{71}$$

There are two important properties regarding $\mathbf{B}$ in (70):

1) If $\mathbf{r} \in E_v$, then $\mathbf{B}\mathbf{r} = \mathbf{0}$. This follows from the fact that $\mathbf{A}\mathbf{r} = \lambda\mathbf{r}$.

2) $\mathbf{B}^2 = \mathbf{0}$, i.e. $\mathbf{B}$ is a nilpotent matrix. (One can easily confirm that this is true for the example in (71).) This is a result from linear algebra.

Now let $\mathbf{u}$ be a vector **not** in the eigenspace E_v, i.e. $u \in \mathbb{R}^2 - E_v$. Note that $\mathbf{u} \neq \mathbf{0}$ since $\mathbf{0} \in E_v$. Then:

Theorem 1: $\mathbf{B}$ maps $\mathbf{u}$ into E_v, i.e. $\mathbf{B}\mathbf{u} \in E_v$.

Proof: Let $\mathbf{w} = \mathbf{B}\mathbf{u}$. then $\mathbf{B}\mathbf{w} = \mathbf{B}(\mathbf{B}\mathbf{u}) = \mathbf{B}^2\mathbf{u} = \mathbf{0}\mathbf{u} = \mathbf{0}$. Then $\mathbf{w} \in E_v$ from 1) above. ■

Let $\mathbf{v}$ be the λ-eigenvector of $\mathbf{A}$ we chose earlier. Now let $\mathbf{u}$ be the vector in $\mathbb{R}^2 - E_v$ such that

$$\mathbf{B}\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}. \tag{72}$$

Since $\mathbf{u} \notin E_v$, it follows that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Then, given any initial value $\mathbf{x}_0 = (x_0, y_0)$, we may find particular values of C_1 and C_2 such that

$$\mathbf{x}_0 = C_1\mathbf{v} + C_2\mathbf{u} \tag{73}$$

Since the solution of the linear d.e. $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is $\mathbf{x}_n = \mathbf{A}^n x_0$, we have

$$\mathbf{x}_n = C_1\mathbf{A}^n\mathbf{v} + C_2\mathbf{A}^n\mathbf{u}. \tag{74}$$

The first term is easy since $\mathbf{v} \in E_v$ implies that $\mathbf{A}^n\mathbf{v} = \lambda^n\mathbf{v}$. It remains to compute $\mathbf{A}^n\mathbf{u}$ for the $\mathbf{u}$ vector in $\mathbb{R}^2 - E_v$ satisfies Eq. (72). To do this, we rewrite $\mathbf{A}$ as follows:

$$\begin{aligned} \mathbf{A} &= \lambda\mathbf{I} + \mathbf{A} - \lambda\mathbf{I} \\ &= \lambda\mathbf{I} + \mathbf{B}. \end{aligned} \tag{75}$$

Then

$$\begin{aligned} \mathbf{A}^n &= [\lambda\mathbf{I} + \mathbf{B}]^n \\ &= \lambda^n\mathbf{I} + n\lambda^{n-1}\mathbf{B}, \end{aligned} \tag{76}$$

where we have expanded the RHS using the binomial theorem and the fact that $\mathbf{B}$ and $\mathbf{I}$ commute, as well as the fact that $\mathbf{B}^2 = \mathbf{0}$, implying that $\mathbf{B}^k = \mathbf{0}$ for $k \geq 2$.

From (76),

$$\begin{aligned} \mathbf{A}^n\mathbf{u} &= \lambda^n\mathbf{u} + n\lambda^{n-1}\mathbf{B}\mathbf{u} \\ &= \lambda^n\mathbf{u} + n\lambda^{n-1}\mathbf{v}. \end{aligned} \tag{77}$$

Thus, the general solution in (74) becomes

$$\mathbf{x}_n = C_1\lambda^n\mathbf{v} + C_2[\lambda^n\mathbf{u} + n\lambda^{n-1}\mathbf{v}], \quad n \geq 0. \tag{78}$$

We summarize the procedure:

i) Assuming that $\mathbf{A}$ has equal eigenvalues and only one eigenvector can be found, let $\mathbf{v}$ be such an eigenvector, i.e. $A\mathbf{v} = \lambda\mathbf{v}$.

2) Find a $\mathbf{u}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$.

3) The general solution to the linear system $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is given by Eq. (78).

Let us return to the example $\mathbf{A}$ in (67), with $\lambda = 2$. The matrix $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}$ is given in (71). We have chosen $\mathbf{v} = (1, 0)^T$. It now remains to find a $\mathbf{u}$ such that $\mathbf{Bu} = \mathbf{v}$, or

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or

$$0u_1 + u_2 = 1$$
$$0u_1 + 0u_2 = 0.$$

Thus, $u_2 = 1$ and u_1 is arbitrary. We choose $u_1 = 0$ for simplicity so that $\mathbf{u} = (0, 1)^T$. The general solution of $\mathbf{x}_{n+1} = \mathbf{A}x_n$ is from (78),

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = C_1 2^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \left[2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + n2^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right],$$

or

$$x_n = C_1 1^n + C_2 n 2^{n-1}$$
$$y_n = C_2 2^n.$$

Example:

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = 3$. If we try to find an eigenvector $\mathbf{v}$ for $\lambda = 3$:

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

implying that $v_2 = -v_1$. Let us choose $v_1 = 1$, $v_2 = -1$ so that $\mathbf{v} = (1, -1)^T$. We now look for a vector $\mathbf{u}$ such that $(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \mathbf{v}$, i.e.

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$u_1 + u_2 = 1$$
$$-u_1 - u_2 = -1.$$

If we choose $u_2 = 0$, then $u_1 = 1$, so that $\mathbf{u} = (1, 0)$. Then the general solution to the linear system $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is;

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = C_1 3^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 3^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 n 3^{n-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

If the initial condition $\mathbf{x}_0 = (x_0, y_0) = (2, 1)$, then we solve for C_1 and C_2; setting $n = 0$ above

$$\begin{aligned} C_1 + C_2 &= 2 \\ -C_1 &= 1 \end{aligned}$$

or $C_1 = -1$, $C_2 = 3$. After some simplification, the solution is

$$\begin{aligned} x_n &= (n+2)3^n \\ y_n &= -(n-1)3^n. \end{aligned}$$

Summary and Behaviour of Solutions

We summarize the results of the previous section for the two-dimensional linear system $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ according to the behaviour of the eigenvalues of A:

Case 1: Distinct real eigenvalues $\lambda_1 \neq \lambda_2$

General solution is

$$\mathbf{x}_n = C_1\lambda_1^n\mathbf{v}_1 + C_2\lambda_2^n\mathbf{v}_2$$

where $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

Case 2: Complex conjugate eigenvalues $\lambda_2 = \lambda_1^*$

Let $\lambda = re^{i\theta}$ and $\mathbf{v} = \mathbf{v}_R + i\mathbf{v}_I$ be a (complex) λ-eigenvector of $\mathbf{A}$. Then, in phase-shifted form,

$$\mathbf{x}_n = Ar^n[\cos(n\theta + B)\mathbf{v}_R - \sin(n\theta + B)\mathbf{v}_I].$$

Case 3: Equal real eigenvalues $\lambda_1 = \lambda_2 = \lambda$

i) If two linearly independent λ-eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$ exist, then

$$\mathbf{x}_n = C_1\lambda^n\mathbf{v}_1 + C_2\lambda^n\mathbf{v}_2.$$

Otherwise

ii) If $\mathbf{v}$ is a λ-eigenvector of $\mathbf{A}$, then

$$x_n = C_1\lambda^n\mathbf{v} + C_2[\lambda^n\mathbf{u} + n\lambda^{n-1}\mathbf{v}]$$

where the vector $\mathbf{u}$ satisfies the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$.

Behaviour of Solutions as $n \to \infty$

As in the case of second-order linear d.e.'s, the long-term behaviour of solutions is determined by the magnitudes of the eigenvalues λ_i. We can conclude that:

i) If $|\lambda_i| < 1$ for $i = 1, 2$, then all solutions $\mathbf{x}_n \to \mathbf{0}$ as $n \to \infty$.

ii) If $|\lambda_i| > 1$ for $i = 1, 2$, then, for all solutions (except $\mathbf{x}_n = \mathbf{x}_0 = \mathbf{0}$), $||x_n|| \to +\infty$ as $n \to \infty$.

Converting A Second-Order Homogeneous Difference Equation Into a Homogeneous Linear System

At the beginning of Lecture 9, we showed that a two-dimensional first-order linear system of difference equations in the variables $\mathbf{x}_n = (x_n, y_n)$ can be transformed into a system of two separate second-order difference equations, one in the x_n and one in the y_n. These two d.e.s, however, had the same coefficients.

We now show that a second-order d.e. can be transformed into an equivalent linear system of two first-order d.e.s. Those familiar with differential equations (DE) will see an analogy here – a second-order DE can be transformed into an equivalent first-order linear system. Let

$$w_{n+2} + pw_{n+1} + qw_n = 0 \tag{79}$$

be the second-order linear homogeneous d.e. in the variable w_n. Now define the variables

$$\begin{aligned} x_n &= w_n \\ y_n &= w_{n+1}, \qquad n \geq 0. \end{aligned} \tag{80}$$

Then

$$x_{n+1} = w_{n+1} = y_n \tag{81}$$

and

$$\begin{aligned} y_{n+1} = w_{n+2} &= -pw_{n+1} - qw_n \\ &= -py_n - qx_n. \end{aligned} \tag{82}$$

Thus we have the following linear system in x_n and y_n:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{83}$$

Note that the eigenvalues of the matrix $\mathbf{A}$ defining this linear system again coincide with the roots of the characteristic equation of the second-order system in (79).

Lecture 11

Linear systems of difference equations: The finale

A model of red blood cell production

We close this chapter with a look at a simple difference equation model of blood cell production which is discussed briefly in the book by L. Edelstein-Keshet (Problem 2, Page 27). Here, we simply quote the relevant text from the book:

> "In the circulatory system, the red blood cells (RBCs) are constantly being destroyed and replaced. Since these cells carry oxygen throughout the body, their number must be maintained at some fixed level. Assume that the spleen filters out and destroys a certain fraction of the cells daily and that the bone marrow produces a number proportional to the number lost on the previous day. What would be the cell count on the nth day?"

If we let R_n denote the number of red blood cells (RBCs) in circulation on day n then, with a little thought, one could come up with a second-order difference equation in R_n since, from the above text, the quantity R_{n+1} will depend not only on R_n but on R_{n-1}. That being said, we shall proceed with the analysis presented in the book by Edelstein-Keshet and show that, eventually, one can arrive at the second-order d.e..

The following parameters will be adopted in this model:

- R_n, the number of RBCs in circulation on day n,
- M_n, the number of RBCs produced by bone marrow on day n,
- f, the fraction of RBCs removed by the spleen,
- γ, the "production constant," which is the number of RBCs produced per number of RBCs destroyed.

It should not be too difficult to see that the difference equations for R_n and M_n are

$$\begin{aligned} R_{n+1} &= (1-f)R_n + M_n \\ M_{n+1} &= \gamma f R_n \,. \end{aligned} \tag{84}$$

Briefly, the reasoning for these equations is as follows:

- The number of RBCs which are destroyed by the spleen on day n is fF_n. This means that $R_n - fF_n = (1-f)R_n$ RBCs survive into the next generation. To this number we must add the M_n RBCs produced by the bone marrow on day n.

- Once again, the number of RBCs destroyed by the spleen on day n is fR_n. In this model, the number of RBCs produced by the bone marrow is $\gamma \times (fR_n)$. One might wonder whether this number of cells is produced on day n or day $n+1$, i.e., is this number M_n or M_{n+1}?. In this model, we are simply taking this number and adding it to the next generation of RBCs, which means that we consider the number to define M_{n+1}.

The matrix $\mathbf{A}$ which defines the system of d.e.'s in (84) is

$$\mathbf{A} = \begin{pmatrix} 1-f & 1 \\ \gamma f & 0 \end{pmatrix} . \tag{85}$$

The eigenvalues of this matrix are (Exercise)

$$\lambda_{1,2} = \frac{1}{2}(1-f) \pm \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f} \, . \tag{86}$$

For $\gamma > 0$ and $0 < f < 1$, it should not be difficult to see (recall our discussion of the discrete plant production model) that

$$\lambda_1 = \frac{1}{2}(1-f) + \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f} > 1 - f > 0 \tag{87}$$

and

$$\lambda_2 = \frac{1}{2}(1-f) - \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f} < 0 \, . \tag{88}$$

Before discussing the condition that the number of red blood cells remains roughly constant in time, note that the d.e. in Eq. (84) may be easily converted into a second order d.e. in R_n. From the second equation in the system, it follows that $M_n = \gamma f R_{n-1}$, which may be substituted into the first equation in the system,

$$\boxed{R_{n+1} = (1-f)R_n + \gamma f R_{n-1} \, .} \tag{89}$$

This is the second-order difference equation that could have, "with a little thought," been derived from the text taken from the book by Edelstein-Keshet. Briefly, the number of RBCs on day $n+1$ is given by sum of (i) the number $(1-f)R_n$ of RBCs on day n that are not removed by the spleen and (ii) the number $\gamma f R_{n-1}$ produced by bone marrow on day $n-1$.

By increasing all indices by 1 – or, equivalently, replacing n with $n+1$ – in Eq. (89), we arrive at the following homogeneous, constant coefficient d.e. in R_n,

$$R_{n+2} - (1-f)R_{n+1} - \gamma f R_n = 0 \, . \tag{90}$$

The characteristic equation associated with this d.e. is

$$m^2 - (1-f)m + \gamma f = 0 \, . \tag{91}$$

The roots of this characteristic equation are precisely the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$ above. We would expect this result since the second order d.e. in R_n was derived from the first order system.

Recalling that $\lambda_1 > 0$ and $\lambda_2 < 0$, we can conclude that the general solution of the d.e. in (90) has the form,

$$R_n = C_1\lambda_1^n + C_2\lambda_2^n\,. \tag{92}$$

An analysis of this model is performed in the Problem Set of Prof. Edelstein-Keshet's book: Problem 16, Page 33. We have already solved Problems 16(a) and 16(b).

Problem 16(c) of the textbook is concerned with "homeostasis," i.e., the requirement that the total number of red blood cells, R_n, should remain roughly constant in time. One way of achieving this result is to require that the positive root λ_1 be equal to 1. From Eq. (87), we have

$$1 = \frac{1}{2}(1-f) + \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f}\,, \tag{93}$$

which is equivalent to

$$1 - \frac{1}{2}(1-f) = \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f}\,. \tag{94}$$

Now square both sides,

$$1 - (1-f) + \frac{1}{4}(1-f)^2 = \frac{1}{4}(1-f)^2 + \gamma f\,, \tag{95}$$

which simplifies to

$$f = \gamma f \quad \Longrightarrow \quad \gamma = 1\,, \tag{96}$$

since we have assumed that $f > 0$. Recall that γ is the "production constant," implying that the spleen should produce one RBC for every RBC destroyed in order to keep the total number of RBCs constant. This seems to make sense.

In the case that $\gamma = 1$, the second root is

$$\begin{aligned}
\lambda_2 &= \frac{1}{2}(1-f) - \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f} \\
&= \frac{1}{2}(1-f) - \frac{1}{2}\sqrt{(1-f)^2 + 4f} \\
&= \frac{1}{2}(1-f) - \frac{1}{2}\sqrt{(1+2f+f^2} \\
&= \frac{1}{2}(1-f) - \frac{1}{2}(1+f) \\
&= -f\,.
\end{aligned} \tag{97}$$

The general solution, then, has the form

$$\begin{aligned}
R_n &= A\lambda_1^n + B(-f)^n \\
&= A + B(-f)^n\,,
\end{aligned} \tag{98}$$

where A and B are constants. This implies that in this condition of homeostasis, the red blood cell count will oscillate between the values $A - fB$ and $A + fB$. If $|fB| = fB$ is very small, then the RBC count will oscillate a very small amount about the value A.

An introduction to nonlinear difference equations/dynamical systems

Introductory remarks

This section of the course will be concerned with first-order difference equations having the form,

$$x_{n+1} = f(x_n)\,, \tag{99}$$

where f is a nonlinear function of its argument(s). Equation (99) is also referred to as a *dynamical system*. The iterates x_n can be real numbers, i.e. $x_n \in \mathbb{R}$, or vectors, $x_n \in \mathbb{R}^n$. In the latter case, $f : \mathbb{R}^n \to \mathbb{R}^n$ and Eq. (99) represents a system of first-order difference equations. There is no problem in restricting our attention to first-order difference equations since any n-th order d.e. can be rewritten as a system of first order d.e.'s. Our purpose is to examine some general types of behaviour for nonlinear dynamical systems that distinguish them from linear systems. The treatment of the subject is necessarily introductory. The book, *An Introduction to Chaotic Dynamical Systems*, by R. Devaney, a reference for the course which will be posted on LEARN, is highly recommended to those who would like to investigate the subject in more detail. (The book is addressed to undergraduates with a solid advanced calculus and linear algebra background.)

The basic goal of dynamical systems theory is to understand the eventual or asymptotic behaviour of an iterative process such as in Eq. (99). The reader will note that we were already doing this in earlier chapters when we asked whether or not solutions to linear difference equations decayed to zero or diverged to infinity (either monotonically or by oscillation) as $n \to \infty$. Of particular interest are constant solutions to (99), i.e. $x_n = \bar{x}$, $n = 0, 1, 2, \ldots$, which correspond to *fixed points* of the function f, i.e. $f(\bar{x}) = \bar{x}$. (In the case of linear homogeneous d.e.'s, the only fixed point is $\bar{x} = 0$.) One is interested to know if such solutions are **stable** or **unstable**, i.e. if we pick an initial condition x_0 near $\bar{x}$, do the iterates approach $\bar{x}$ (**stable** or **attractive** fixed point) or do they travel away from $\bar{x}$ (**unstable** or **repulsive** fixed point).

In the case of nonlinear dynamical systems, as we shall see below, there the possibility of another class of interesting solutions to (99), namely periodic cycles. For example, there may exist a set of N distinct points $p_1, p_2, \ldots, p_N$ such that $f(p_1) = p_2$, $f(p_2) = p_3, \ldots$, $f(p_N) = p_1$. The set $\{p_1, p_2, \ldots, p_N\}$ is known as a periodic N-cycle. Once again, one is interested to know whether such a cycle is stable/attractive or unstable/repulsive, i.e. are points nearby attracted to these cycles (stable) or are they repelled away from them (unstable).

First we introduce some useful notation. Given an $x_0 \in \mathbb{R}$, the set of points $\{x_0, x_1, x_2, \ldots\}$ or $\{x_n\}_{n=0}^{\infty}$ determined by the iteration procedure in (99) is known as the **forward orbit** of x_0. The subject of **backward orbits** will be treated later. It is also convenient to consider the forward orbit

of x_0 in the following way:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) = f^{\circ 2}(x_0) \\ &\vdots \\ x_n &= f^{\circ n}(x_0), \end{aligned}$$

where $f^{\circ n}(x) = (f \circ f \circ \cdots \circ f)(x)$ (n times) denotes the n-fold composition of the function f with itself. (Note that $f^{\circ n}$ does *not* mean $f(x)$ raised to the nth power, i.e. $[f(x)]^n$, a function that we shall never use in this course.) By definition, $f^{\circ(n+1)}(x) = f(f^{\circ n}(x))$. That being said, we shall relax the notation for convenience and, simply let $f^n(x)$ denote $f^{\circ n}(x)$.

Examples:

1) $f(x) = cx$. Then

$$\begin{aligned} f^2(x) &= f(f(x)) = c(cx) = c^2x \\ f^3(x) &= f(f^2(x)) = c(c^2x) = c^3x \\ &\vdots \\ f^n(x) &= c^n x. \end{aligned}$$

2) $f(x) = x^2$. Then

$$\begin{aligned} f^2(x) &= f(f(x)) = (x^2)^2 = x^4, \\ f^3(x) &= f(f^2(x)) = (x^4)^2 = x^8, \\ &\vdots \\ f^n(x) &= x^{2^n}. \end{aligned}$$

A motivating example – the Logistic difference equation

We return to the simple population models of Chapter 1 and introduce terms that model the effects of limited resources and competition among members of the species, thereby limiting growth. Let us recall a rather simple continuous model from Calculus – the **logistic equation**, given by the differential equation (DE)

$$\frac{dx}{dt} = ax - bx^2, \tag{100}$$

where $a, b > 0$. The nonlinear term, $-bx^2$, is a simple model for competition: for small x, $x \ll 1$, it is negligible but as x increases, it becomes more significant. If x is "too large" then it will dominate the growth term ax, causing the population to decrease. We can understand the qualitative behaviour of solutions to Eq. (100) without actually solving the DE. (This is the essence of qualitative analysis.) First, let us rewrite this DE as

$$\frac{dx}{dt} = ax\left(1 - \frac{b}{a}x\right) \tag{101}$$

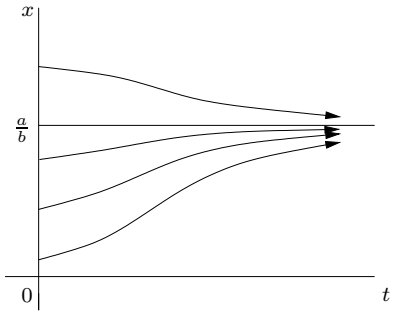
and note that there are two constant or "equilibrium" solutions: (1) the trivial solution $\bar{x}_1(t) = 0$ and (2) the solution $\bar{x}_2(t) = \frac{a}{b}$. The trivial solution, of course, is not a desirable solution as far as the species is concerned. As for the second solution $\bar{x}_2(t) = \frac{a}{b}$, note that as $b \to 0$, i.e. with lesser effects of competition/limited resources, $\frac{a}{b} \to \infty$. Likewise, as b, the effect of competition, increases the quantity $\frac{a}{b}$ decreases. This equilibrium value is known as the **carrying capacity** of the biological system.

Without even solving the DE in (101), we can deduce the qualitative behaviour of its other solutions. This is known as **qualitative analysis** of the DE:

1) If $x(t) > \frac{a}{b}$, then $\frac{dx}{dt} < 0$, implying that $x(t)$ is decreasing, approaching $\bar{x}_2(t)$ asymptotically.

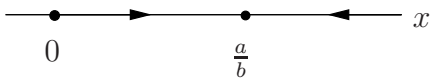
2) If $0 < x(t) < \frac{a}{b}$, then $\frac{dx}{dt} > 0$, implying that $x(t)$ is increasing, approaching $\bar{x}_2(t)$ asymptotically.

The qualitative behaviour of solution curves is sketched below.



Solution curves to logistic differential equation

(Negative solutions were not sketched since they are not physically relevant here.) The "phase portrait" summary of this solution behaviour may be sketched as follows:



Phase portrait for solution curves of logistic DE

In other words, $x = 0$ is an *unstable* fixed-point or equilibrium solution: If we begin with an initial condition $x(0) = x_0$ near zero, then the corresponding solution $x(t)$ will increase away from 0. On the other hand, the equilibrium solution $x_1 = \frac{a}{b}$, which corresponds to the carrying capacity of the system is **stable**. In fact, with the exception of the solution $\bar{x}_1(t) = 0$, all solutions $x(t)$ approach it as $t \to \infty$ – it is an **attractive** equilibruium value.

With the above continuous model as motivation, let us now consider the discrete population model

$$x_{n+1} = cx_n - dx_n^2, \tag{102}$$

where $c, d > 0$. In fact, recall that when $d = 0$, population growth was possible only when $c > 1$. Once again, the term $-dx_n^2$ is an attempt to introduce effects of competition due to limited resources. When $d = 0$ and $c > 1$, then $x_n = c^n x_0 \to \infty$ as $n \to \infty$. Note that Eq. (102) has the form $x_{n+1} = f(x_n)$ where

$$f(x) = cx - dx^2. \tag{103}$$

For the general case $c, d > 0$, the reader can verify that:

1. $f(0)$=0
2. $f(x)$ is increasing on the interval $[0, \frac{c}{2d}]$,
3. $f(x)$ is decreasing on $[\frac{c}{2d}, \infty)$.
4. $f\left(\frac{c}{d}\right) = 0$.
5. $f(x) < 0$ for $x > \frac{c}{d}$.

A generic plot of the graph of $f(x)$ is shown below. We shall be interested only in the case that the iterates x_n are non-negative, in which case we focus our attention on the interval $\left[0, \frac{c}{d}\right]$.

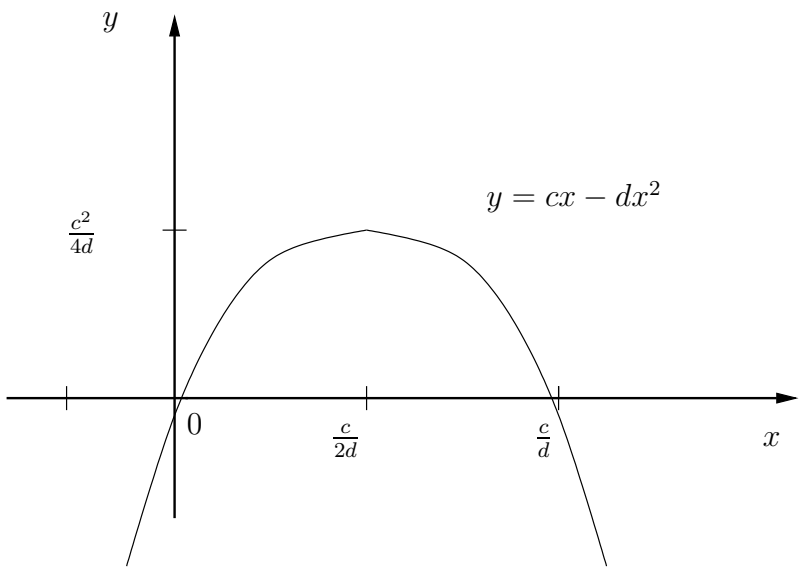
When one wishes to analyze the dynamics of iteration of a nonlinear mapping $f(x)$, the first thing to do is to look for **fixed points** of $f(x)$, i.e., points $\bar{x}$ such that

$$f(\bar{x}) = \bar{x}\,. \tag{104}$$

You have already seen the reason for this: If one chooses $x_0 = \bar{x}$, a fixed point of f, then $x_n = \bar{x}$ for all $n \geq 0$, i.e., on remains at $\bar{x}$.

Clearly $x = 0$ is a fixed point of $f(x)$ in Eq. (103) since $f(0) = 0$. We'll label this fixed point as $\bar{x}_1 = 0$. There is another fixed point which is nonzero:

$$f(x) = x \quad \Longrightarrow \quad cx - dx^2 = x\,. \tag{105}$$



Sketch of graph of $f(x) = cx - dx^2$.

Dividing by $x \neq 0$ yields,

$$c - dx = 1 \quad \Longrightarrow \quad x = \frac{c-1}{d}\,. \tag{106}$$

We'll label this fixed point as $\bar{x}_2 = \frac{c-1}{d}$. Being nonzero, it is the more interesting of the two equilibrium values and is known as the **'carrying capacity** of the difference equation. If we start at it by setting $x_0 = \bar{x}_2$, then we stay there, i.e., $x_n = \bar{x}_2$.

One additional point: Note that as d increases, the fixed point/carrying capacity $\bar{x}_2 = \frac{c-1}{d}$ decreases, in accordance with the idea that d is a measure of competition, hence the limitedness of the resources. Conversely, as d decreases, the carrying capacity $\bar{x}_2 = \frac{c-1}{d}$ increases. These behaviours were also observed for the differential equation. (Note that the carrying capacities of the differential equation and the difference equation are different, even though the two equations look similar in form.)

The question that we shall answer later is: "Are these fixed points **attractive** or **repulsive**, i.e., **stable** or **unstable**?"

Let us now examine the behaviour of iteration sequences $\{x_n\}$ for various values of the parameters c and d to see how nonlinear dynamical systems can differ from linear ones. Below are presented the results of some numerical experiments. **The reader is invited not only to perform these computations to verify them but also to perform other numerical experiments.**

Numerical Experiment No. 1:

$$x_{n+1} = cx_n - dx_n^2\,, \tag{107}$$

where $c, d > 0$.

We set $c = 2$ and vary d. For each value of d reported in the table below, the iteration procedure was begun with $x_0 = 0.25$. The iterates x_1 to x_{1000} were examined. Note that for $d = 0$, $x_n \to +\infty$, as expected since $x_n = c^n x_0 = 2^n x_0$ in this case. For $0 < d \leq 5.0$, iterates x_n are observed to converge to the fixed point $\bar{x}_2 = \frac{c-1}{d}$.

c	d	Behaviour of x_n as $n \to \infty$	$\bar{x}_2 = \frac{c-1}{d}$
2.0	0.0	$x_n \to \infty$	∞
2.0	0.1	$x_n \to 10.0$	10.0
2.0	0.2	$x_n \to 5.0$	5.0
2.0	0.5	$x_n \to 2.0$	2.0
2.0	1.0	$x_n \to 1.0$	1.0
2.0	2.0	$x_n \to 0.5$	0.5
2.0	5.0	$x_n \to 0.2$	0.2
2.0	7.0	$x_n \to 0.142857\ldots$	$0.142857\ldots$
2.0	8.0	$\boxed{x_n \to 0.0}$	0.125

It is observed that for $x_0 = 0.25$, the iterates x_n tend to the fixed point $\bar{x}_2 = \frac{c-1}{d}$ for $c = 2.0$ and d values up to, but not including, $d = 8.0$. At $d = 8.0$, the iterates are observed to travel to the fixed point $\bar{x}_1 = 0$. If, for $d = 8.0$, we change the initial condition to $x_0 = 0.1$, which is closer to zero than 0.25, then the x_n tend once again to the fixed point $\bar{x}_2 = \frac{c-1}{d} = 0.125$. In all cases, the populations x_n tend toward limiting values as $n \to \infty$.

Numerical Experiment No. 2:

$$x_{n+1} = cx_n - dx_n^2\,, \tag{108}$$

where $c, d > 0$.

We fix the parameter $d = 2.5$ and now vary c from 2.0 upwards, once again with the initial condition $x_0 = 0.25$. For $2.0 \leq c \leq 2.8$, we observe that the iterates x_n approach the fixed point $\bar{x}_2 = \frac{c-1}{d}$. However, at $c \cong 3.0$, an interesting phenomenon occurs. The iterates x_n do not approach a **single** fixed point but instead approach an oscillation between **two distinct values** – a **two-cycle** $\{p_1, p_2\}$. The fixed point value observed at $c = 2.8$ appears to have split into two values, for example, $\{0.773, 0.836\}$ at $c = 3.0$. As c is increased to 3.2, the two values comprising the two-cycle $\{p_1, p_2\}$ are moving away from each other. This continues up to $c = 3.4$. At $c = 3.5$, the two-cycle disappears and the iterates x_n approach a **four-cycle** of values $\{p_1, p_2, p_3, p_4\}$. At $c \cong 3.55$, the four-cycle is replaced by an **eight-cycle**. At $c \cong 3.56$, the eight-cycle becomes a **sixteen-cycle**. At $c \cong 3.59$, the iterates x_n are observed to travel over the interval $[0.334, 1.289]$ in a seemingly random, unperiodic manner. The results of this experiment are summarized in the next table.

c	d	$x_n \to$
2.0	2.5	$0.4 = \bar{x}_2$ fixed point
2.5	2.5	$0.6 = \bar{x}_2$ "
2.75	2.5	$0.7 = \bar{x}_2$ "
2.8	2.5	$0.72 = \bar{x}_2$ "
3.0	2.5	{0.773, 0.836} 2-cycle
3.1	2.5	{0.692, 0.941} "
3.2	2.5	{0.657, 1.023} "
3.4	2.5	{0.615, 1.145} "
3.5	2.5	{0.536, 1.158, 0.701, 1.225} 4-cycle
3.55	2.5	8-cycle
3.56	2.5	16-cycle
3.59	2.5	"chaotic motion" over [0.334, 1.289]

The reader is invited to explore this behaviour in more detail. For example, with a little more care (and precision in the floating point computations) one may be able to locate more accurately the value of c at which asymptotic motion towards a fixed point is replaced by motion towards a two-cycle.

The evolution of behaviour observed in the above table is an example of the classic "pitchfork bifurcation and cascade to chaos," a subject of intense research over the past twenty or so years. We shall attempt to analyze a little piece of this scheme in a later section.

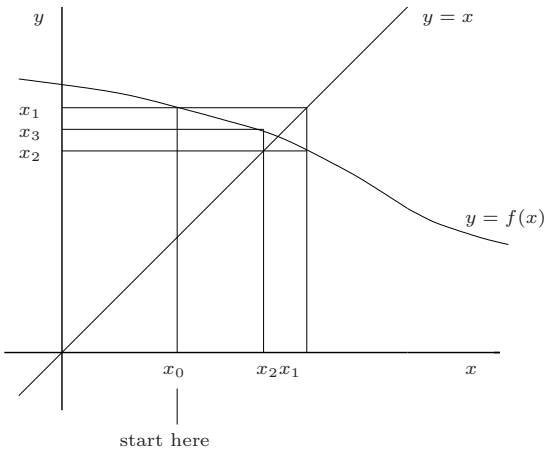
Numerical experiments such as those presented above indicate that there is some rich dynamical behaviour associated with the iteration of nonlinear functions. **Numerical experiments, however, are fascinating but they do not provide a full understanding of what is responsible for such interesting behaviour. A better understanding will be provided with the help of (a) some Calculus as well as (b) some graphical methods of analyzing iteration schemes.** We begin with a discussion of (b) in the next section.

Graphical methods of analyzing iteration

We wish to graphically represent the iteration process defined by

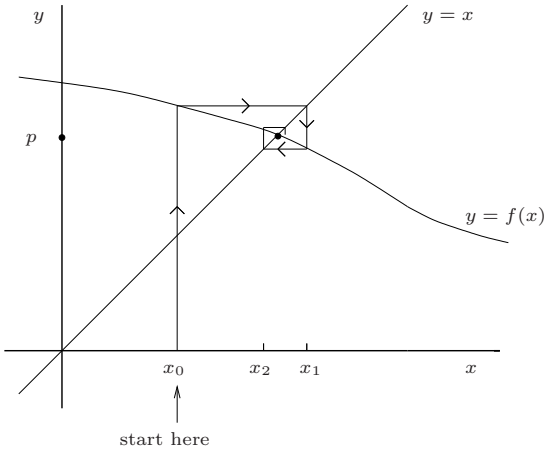
$$x_{n+1} = f(x_n), \tag{109}$$

where $f : \mathbb{R} \to \mathbb{R}$. We begin with a "seed" $x_0 \in \mathbb{R}$ and insert it into the "f machine". The output $x_1 = f(x_0)$ is then reinserted into the "f machine" to produce $x_2 = f(x_1)$, etc. How do we do this graphically? First start with a plot of the graph of $f(x)$ along with the line $y = x$, as shown in the figure on the next page. Now pick a starting point x_0 on the x-axis. Getting $x_1 = f(x_0)$ is easy: You simply find the point $(x_0, f(x_0))$ that has on the graph of $f(x)$ above (or below) the point $x = x_0$. In other words, travel upward (or downward) from $(x_0, 0)$ to $(x_0, f(x_0))$. We now have $x_1 = f(x_0)$. How do we input x_1 into the "f machine" to find $x_2 = f(x_1)$? First, we have to find where x_1 lies on the x-axis. We do this by travelling from the point $(x_0, f(x_0))$ horizontally to the line $y = x$: the point of intersection will be $(f(x_0), f(x_0)) = (x_1, x_1)$. We are now sitting directly above (or below) the point $(x_1, 0)$ on the x-axis, which is patiently waiting to be input into $f(x)$ to produce $x_2 = f(x_1)$. We can travel from (x_1, x_1) to $x_1, 0)$ and then back up (or down) to $(x_1, f(x_1)) = (x_1, x_2)$. From here, we once again travel to the line $y = x$, intersecting it at (x_2, x_2). From here, we travel to the curve $y = f(x)$ to intersect it at (x_2, x_3), etc.. The procedure is illustrated below.



When you have performed this procedure a few times, you will see that including all the lines from intersection points (x_i, x_i) on the line $y = x$ and intersection points (x_i, x_{i+1}) on $y = f(x)$ to

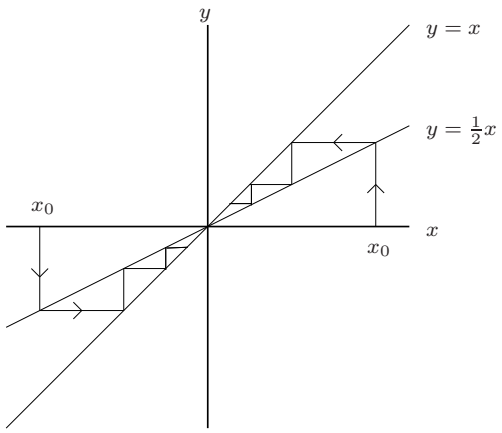
the x- and y-axis is unnecessary. In fact, these lines clutter up the figure. We have removed them to produce the figure below, in which the iteration process is presented in a much clearer way.



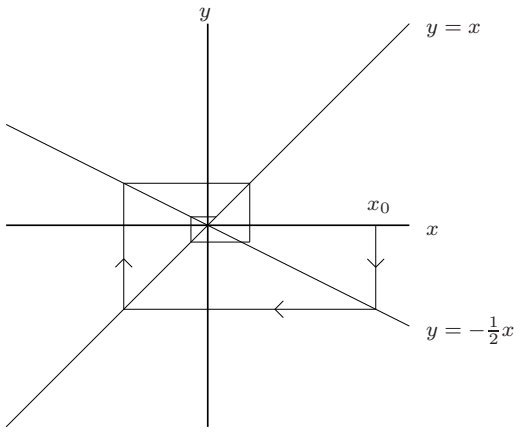
For rather obvious reasons, figures such as this one are called "cobweb diagrams". It appears as if the iterates x_0, x_1, x_2, x_3 are jumping back and forth, yet "zeroing in" on the point (p, p) at which the graph of f intersects the line $y = x$. Such a point of intersection must be a fixed point of f since it implies that $f(p) = p$. Of course, not all fixed points are attractive as the one in this diagram: The graphical procedure outlined above will give us some insight into what makes a fixed point attractive or repulsive.

Let us first examine some simple dynamical systems studied earlier, namely the linear ones, using graphical methods.

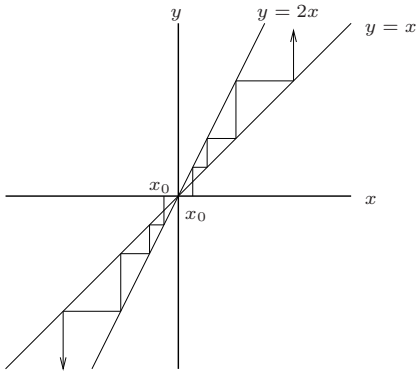
Example 1: $x_{n+1} = cx_n$, $0 < c < 1$. For $x_0 > 1$ or $x_0 < 1$, the graphical method shows the monotonic approach of the $x_n = c^n x_0$ toward the fixed point $x = 0$:



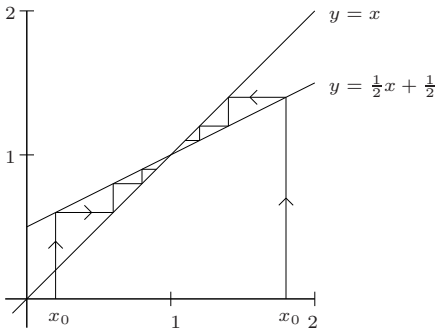
Example 2: $x_{n+1} = cx_n$, $-1 < c < 0$. The iterates x_n oscillate about $x = 0$, $x_n = (-1)^n|c|^n x_0$, with $x_n \to 0$ as $n \to \infty$:



Example 3: $x_{n+1} = cx_n$, $c > 1$. Here, $x_n = c^n x_0$. The iterates travel away from the fixed point 0:



Example 4: $x_{n+1} = cx_n + d$, $0 < c < 1$, $d > 0$



In this example, the two straight lines, $y = x$ and $y = cx + d$, can intersect at only one point, the fixed point of $f(x) = cx + d$, which is $\bar{x} = \frac{d}{1-c}$. Since $|c| < 1$, this fixed point is **attractive**: For any $x_0 \in \mathbb{R}$, $x_n \to \bar{x}$ as $n \to \infty$. Note that this picture looks like that of Example 1, with the fixed point translated from $x = 0$ to $x = \frac{d}{1-c}$. The reader is once again encouraged to examine the iteration of $f(x) = cx + d$ for the cases i) $-1 < c < 0$, ii) $c = -1$ and iii) $c < -1$.