Lecture 6

Linear difference equations with constant coefficients

We continue our discussion of the following second order linear difference equation with constant coefficients,

$$y_{k+2} + py_{k+1} + qy_k = 0\,, \quad k \geq 0\,, \tag{1}$$

where p and q are real constants. Recall that by assuming a solution of the form $y_k = m^k$, we obtained the characteristic equation,

$$m^2 + pm + q = 0\,. \tag{2}$$

We now consider the third and final case for the nature of roots of this characteristic equation.

Case 3: Complex conjugate roots $m_1 = m_2^*$

This is the case when $p^2 - 4q < 0$ so that

$$m_{1,2} = -\tfrac{1}{2}p \pm \tfrac{1}{2}i\sqrt{4q-p^2} = a \pm bi\,, \quad a = -\tfrac{1}{2}p\,, \quad b = \tfrac{1}{2}\sqrt{4q-p^2}\,. \tag{3}$$

Since $m_1 \neq m_2$ (otherwise $b = 0$ and we have $m_1 = m_2 = a$, equal real roots, i.e., Case 2 from the previous lecture) the solutions $y_k^{(1)} = m_1^k$ and $y_k^{(2)} = m_2^k$ automatically form a fundamental set since their determinant $W_0 \neq 0$ – see previous lecture on these determinants. Therefore, we can write the general solution as

$$\begin{aligned} Y_k &= C_1 m_1^k + C_2 m_2^k \\ &= C_1(a+bi)^k + C_2(a-bi)^k. \end{aligned} \tag{4}$$

This complex-valued solution will yield any particular solution with prescribed initial conditions Y_0 and Y_1. Since we are interested primarily in real-valued solutions to our difference equations it is convenient to extract two linearly independent real-valued solutions from (4). This is easily done by decomposing one solution, say $y_k = m_1^k$ into real and imaginary parts, i.e.

$$y_k = m_1^k = u_k + iv_k\,, \quad k \geq 0\,, \tag{5}$$

where $u_k, v_k \in \mathbb{R}$. From the following facts:

1. The difference equation in Eq. (1) is **linear** and
2. its coefficients, p and q are **real**,

it follows that **the sequences** $u = \{u_k\}$ **and** $v = \{v_k\}$ **will be real-valued solutions of the d.e. in (1).** Let us substitute the solution y_k as given in Eq. (5) into the d.e. in Eq. (1) to verify this statement:

$$\begin{aligned} y_{k+2} + py_{k+1} + y_k &= (u_{k+2} + iv_{k+2}) + p(u_{k+1} + iv_{k+1}) + q(u_k + iv_k) \\ &= (u_{k+2} + pu_{k+1} + qu_k) + i(v_{k+2} + pv_{k+1} + qv_k) \\ &= 0\,. \end{aligned} \tag{6}$$

Since this result must hold for all $k \geq 0$, it follows that both the real and imaginary parts of the RHS of the above equation are zero, i.e.,

$$\begin{aligned} u_{k+2} + pu_{k+1} + qu_k &= 0 \\ v_{k+2} + pv_{k+1} + qv_k &= 0\,, \quad k \geq 0\,, \end{aligned} \tag{7}$$

which confirms that the real-valued sequences, $u = \{u_k\}$ and $v = \{v_k\}$, are solutions of the d.e. in (1).

From Eq. (5), we must now extract the real and imaginary parts, u_k and v_k, respectively, from the complex solution m_1^k for $k \geq 0$. At first, this might look like a formidable task. But it is not so bad - we simply have to resort to the **polar form** of a complex variable. The polar form of the root $m_1 = a + bi$, is found as follows:

$$m_1 = a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta), \tag{8}$$

where

$$r = \sqrt{a^2 + b^2}, \quad \tan\theta = \frac{b}{a}. \tag{9}$$

Then

$$m_1^k = r^k(\cos k\theta + i\sin k\theta) = r^k\cos k\theta + i\sin k\theta\,. \tag{10}$$

This gives our two linearly independent solutions in Eq. (5):

$$\boxed{u_k = r^k\cos k\theta, \qquad v_k = r^k\sin k\theta,} \tag{11}$$

so that the general solution is

$$\boxed{Y_k = C_1 r^k\cos k\theta + C_2 r^k\sin k\theta.} \tag{12}$$

In some books, you will see the general solution written as

$$Y_k = Ar^k\cos(k\theta + B), \tag{13}$$

where A and B play the role of arbitrary constants. The two forms in Eqs. (12) and (13) are equivalent by the relation (Exercise):

$$A = \sqrt{C_1^2 + C_2^2}, \qquad \tan B = -\frac{C_2}{C_1}. \tag{14}$$

The reader familiar with DEs may once again see a connection between the above method of extracting real-valued solutions and that employed for second-order constant-coefficient homogeneous DEs in the case of complex conjugate roots of the characteristic equation.

Example: The d.e.

$$y_{k+2} - 2y_{k+1} + 2y_k = 0 \tag{15}$$

has the characteristic equation

$$m^2 - 2m + 2 = 0\,, \tag{16}$$

with roots,

$$m = \frac{1}{2}[2 \pm \sqrt{4-8}] = 1 \pm i\,. \tag{17}$$

We'll let $m_1 = 1 + i$ and $m_2 = 1 - i$. Here $r = |m_1| = |m_2| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$ so that

$$m_1 = 1 + i = \sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right). \tag{18}$$

From (12), the general solution to the d.e. is

$$Y_k = C_1 2^{k/2}\cos\left(\tfrac{k\pi}{4}\right) + C_2 2^{k/2}\sin\left(\tfrac{k\pi}{4}\right). \tag{19}$$

(a) Suppose that the initial conditions are $Y_0 = 1$, $Y_1 = 1$. Then

$$\begin{aligned} k = 0: &\qquad C_1 + C_2 \cdot 0 = 1 \\ k = 1: &\qquad C_1\sqrt{2}\cdot\frac{1}{\sqrt{2}} + C_2\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1 \Longrightarrow C_1 + C_2 = 1\,, \end{aligned}$$

so that $C_1 = 1$, $C_2 = 0$. The particular solution with initial values $y_0 = 1$, $y_1 = 1$ is therefore

$$y_k = 2^{k/2}\cos\left(\tfrac{k\pi}{4}\right).$$

In the phase-shifted form, Eqs. (13) and (14), $A = 1$, $B = 0$.

A plot of the first 14 elements of this sequence is shown in the figure below. Since $-1 \le \cos x \le 1$ for all $x \in \mathbb{R}$, the sequence elements y_k must lie between the upper and lower envelopes $2^{k/2}$ and $-2^{k/2}$, respectively, both of which grow exponentially in amplitude with k. The cosine term produces an oscillation of the y_k with k. A plot of the y_k for $0 \le k \le 14$ is given below. The element $y_{15} = 128$ lies beyond the range covered by the plot.

(b) If the initial conditions are $Y_0 = 1$, $Y_1 = 2$, then

$$\begin{aligned} k = 0: &\qquad C_1 + C_2 \cdot 0 = 1 \\ k = 1: &\qquad C_1\sqrt{2}\cdot\frac{1}{\sqrt{2}} + C_2\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 2 \Longrightarrow C_1 + C_2 = 2\,, \end{aligned}$$

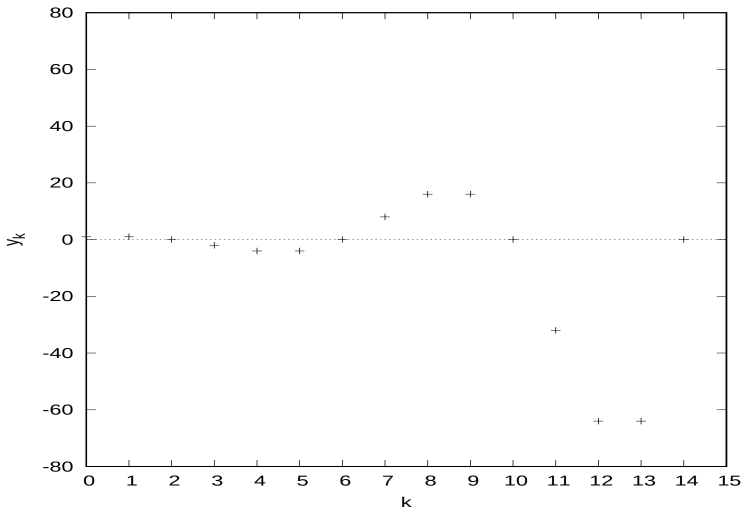
so that $C_1 = C_2 = 1$. The particular solution with initial values $y_0 = 1$ and $y_1 = 1$ is therefore

$$y_k = 2^{k/2}\left[\cos\left(\tfrac{k\pi}{4}\right) + \sin\left(\tfrac{k\pi}{4}\right)\right].$$

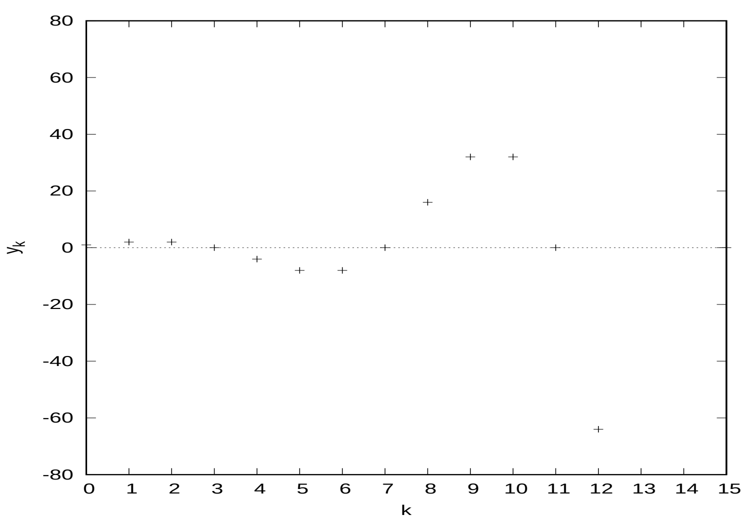
In phase-shifted form, $A = \sqrt{2}$ and $B = -\frac{\pi}{4}$, so that

$$y_k = 2^{\frac{k+1}{2}}\cos\left((k-1)\tfrac{\pi}{4}\right).$$

This solution is plotted in the figure below. A comparison with the previous plot shows that its amplitude is greater (by a factor of $\sqrt{2}$ and that it is shifted by one unit to the right because of the factor $(k-1)$ in the argument of the cosine function.



Solution $y_k = 2^{k/2}\cos\left(\frac{k\pi}{4}\right)$ to Example (a) with initial conditions $y_0 = 1$, $y_1 = 1$.



Solution $y_k = 2^{(k+1)/2}\cos\left((k-1)\frac{\pi}{4}\right)$ to Example (b) with initial conditions $y_0 = 1$, $y_1 = 2$.

Problems

1. Find the general solution of each of the following homogeneous difference equations:

   (a) $y_{k+2} - y_k = 0$

   (b) $2y_{k+2} - 5y_{k+1} + 2y_k = 0$

   (c) $y_{k+2} + 2y_{k+1} + y_k = 0$

   (d) $9y_{k+2} - 6y_{k+1} + y_k = 0$

   (e) $3y_{k+2} - 6y_{k+1} + 4y_k = 0$

   (f) $y_{k+2} + 6y_{k+1} + 25y_k = 0$

2. For each of the equations in Problem 5, find a particular solution satisfying the initial conditions $y_0 = 0$ and $y_1 = 1$.

Answers:

1. (a) $Y_k = C_1 + C_2(-1)^k$. (c) $Y_k = (C_1 + C_2 k)(-1)^k$.
   (e) Roots: $1 \pm \frac{\sqrt{3}}{3} i = \frac{2}{\sqrt{3}} \left(\cos \frac{\pi}{6} \pm i \sin \frac{\pi}{6}\right), y_k = A \left(\frac{2}{\sqrt{3}}\right)^k \cos\left(\frac{k\pi}{6} - B\right)$.

2. (a) $y_k = \frac{1}{2}[1 - (-1)^k]$. (c) $y_k = k(-1)^{k+1}$. (e) $y_k = \sqrt{3}\left(\frac{2}{\sqrt{3}}\right)^k \sin \frac{k\pi}{6}$.

Asymptotic Behaviour of Solutions

In many applications, one may be more interested in the general qualitative behaviour of solutions to difference equations (and differential equations) than in determining particular solutions that match prescribed sets of initial conditions. For example, if the d.e. models the spread of an infectious disease, one would be very interested to know the ranges of parameters that ensure that the number of infected organisms y_k, tends to zero as $k \to \infty$. On the other hand, if the d.e. models the evolution of a stock or bond pricing, one would be interested to know the conditions that ensure that the price y_k increases.

In the previous lecture, we general solution to homogeneous first-order d.e.'s of the form

$$y_{k+1} + ay_k = 0\,, \tag{20}$$

has the form

$$y_k = C(-a)^k = y_0(-a)^k\,, \quad k \leq 0\,. \tag{21}$$

The constant "$-a$" is the root of the first order characteristic equation

$$m + a = 0 \tag{22}$$

associated with the d.e.. Clearly, the long-term behaviour of the solution y_k depends upon the magnitude of a:

(a) If $|a| < 1$, then for all solutions, $|y_k| \to 0$ as $k \to \infty$, implying that $y_k \to 0$.

(b) If $|a| > 1$, then $|y_k| \to +\infty$ as $k \to \infty$. If $a < -1$, then the y_k approach $+\infty$ or $-\infty$ monotonically (depending upon $C = y_0$). If $a > 1$, then the y_k oscillate and diverge.

(c) If $|a| = 1$, then $|y_k| = |C| = |y_0|$. If $a = -1$, then all $y_k = y_0$. If $a = 1$, then the y_k oscillate, $y_k = (-1)^k y_0$.

These types of behaviour were already observed in Chapter 1.

For second-order homogeneous d.e.'s with constant coefficients of the form,

$$y_{k+2} + py_{k+1} + qy_k = 0\,, \tag{23}$$

recall that the general solutions are determined by the roots of the associated quadratic characteristic equation,

$$m^2 + pm + q = 0\,. \tag{24}$$

(a) Distinct real roots, $m_1 \neq m_2$: $Y_k = C_1 m_1^k + C_2 m_2^k$,

(b) Equal real roots, $m_1 = m_2 = m$: $Y_k = C_1 m^k + C_2 k m^k$,

(c) Complex conjugate roots, $m_1 = re^{i\theta}$, $m_2 = m_1^*$: $Y_k = Ar^k \cos(k\theta + B)$.

The simplest situation occurs when both roots have magnitudes less than one. With reference to the above cases, we have:

(a) $|m_1| < 1$ and $|m_2| < 1$ (distinct real roots): $m_1^k \to 0$, $m_2^k \to 0$ as $k \to \infty$,

(b) $|m_1| = |m_2| = |m| < 1$ (equal real roots): $m^k \to 0$, $km^k \to 0$ as $k \to \infty$,

(c) $|m_1| = |m_2| = r < 1$ (complex conjugate roots): $r^k \to 0$ as $k \to \infty$.

(The fact that $km^k \to 0$ for $|m| < 1$ is a result from first-year Calculus.) We therefore have the following result:

Theorem 1: Suppose that both roots of the characteristic equation in (24) have magnitudes less than 1, i.e. $|m_1| < 1$ and $|m_2| < 1$. Then all solutions of Eq. (23), regardless of initial conditions, decay to zero as $k \to \infty$, i.e. $y_k \to 0$ as $k \to \infty$.

Note: The above theorem includes the trivial solution $y_k = 0$.

Example: The d.e.

$$y_{k+2} - y_{k+1} + \frac{1}{4} y_k = 0,$$

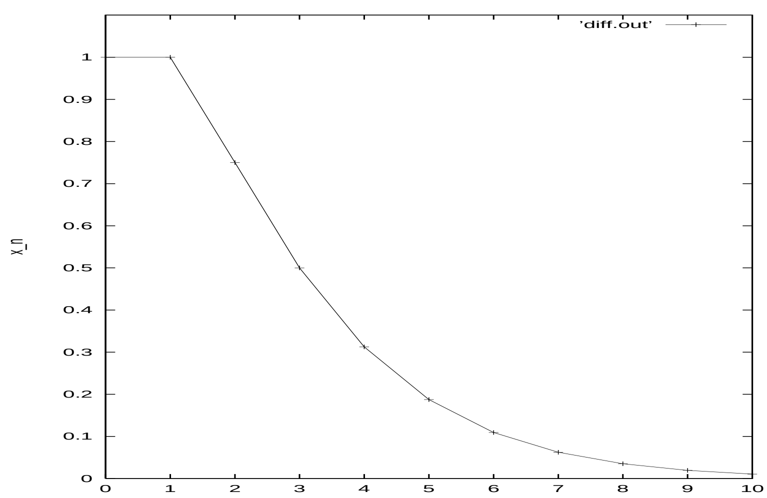
examined in the previous lecture and found to have general solution,

$$Y_k = C_1 \left(\frac{1}{2}\right)^k + C_2 k \left(\frac{1}{2}\right)^k.$$

We plot the first few values of the solution

$$y_k = \left(\frac{1}{2}\right)^k + k \left(\frac{1}{2}\right)^k$$

below:



The decay of solutions y_k to zero need not be monotonic as they are in the above plot. For example, consider the d.e.

$$y_{k+2} + y_{k+1} + \frac{1}{4} y_k = 0$$

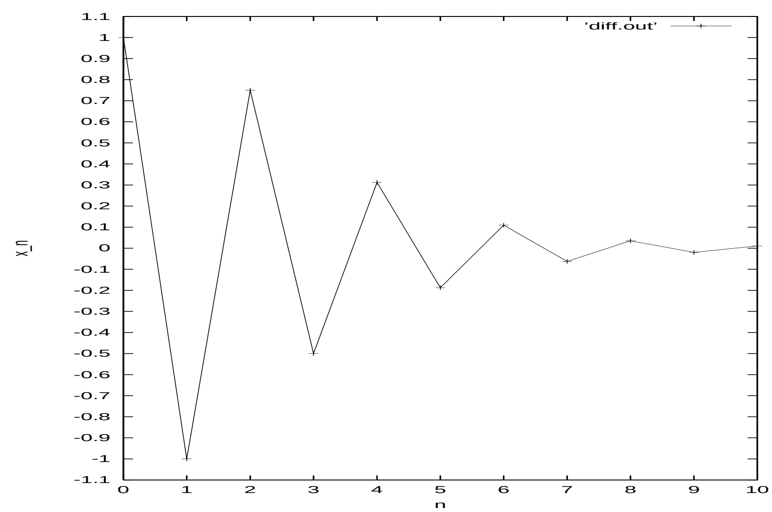
with characteristic equation

$$m^2 + m + \frac{1}{4} = \left(m + \frac{1}{2}\right)^2 = 0.$$

The general solution is

$$Y_k = C_1 \left(-\frac{1}{2}\right)^k + C_2 k \left(-\frac{1}{2}\right)^k.$$

A plot of the solution corresponding to $C_1 = C_2 = 1$ is shown below:



In this case, the y_k exhibit a "damped oscillation" to zero. The reader is invited to investigate other examples numerically.

It may be tempting to make generalizations about the opposite case, i.e. when all roots of the characteristic equation have magnitudes greater than one. In this case $|m_i|^k \to \infty$ as $k \to \infty$. One would suspect that the *magnitudes* $|y_k|$ of solutions will tend to infinity as $k \to \infty$. The detailed behaviour of such "solution blowups" will vary, depending upon the signs of the roots (and, in Case (a), whether the two roots have the same sign or different signs) and the initial conditions. We simply mention here that there are two principal methods of "solution blowup", as was observed for first-order d.e.'s at the end of Chapter 1:

i) Monotonic divergence, i.e. $y_k \to \infty$ or $y_k \to -\infty$ as $k \to \infty$.

Example: The initial value problem,

$$y_{k+2} - 3y_{k+1} + 2y_k = 0\,, \quad y_0 = 1\,,\ y_1 = 2\,,$$

studied in the previous lecture (Lecture 5), with solution

$$y_k = 2^k \,.$$

Here, $y_k \to \infty$ as $k \to \infty$.

ii) Oscillatory divergence, where the y_k oscillate with increasing amplitude as $k \to \infty$.

Example: The initial value problem,

$$y_{k+2} - 2y_{k+1} + 2y_k = 0 \,, \quad y_0 = 1 \,,\ y_1 = 1 \,,$$

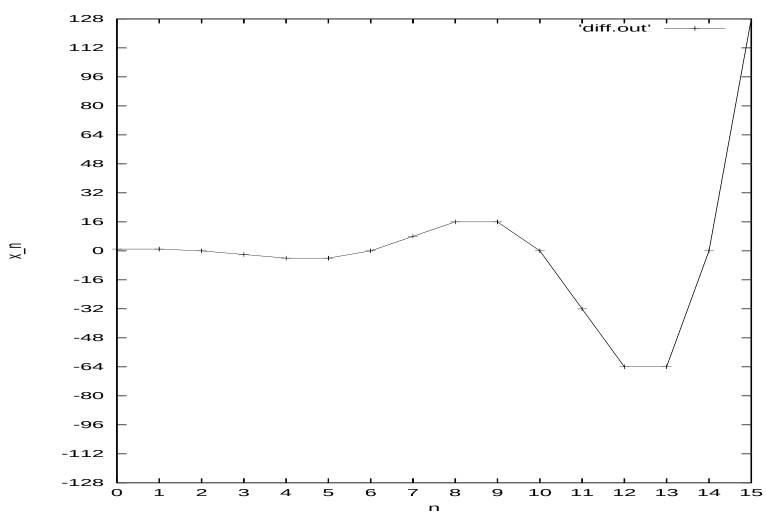
studied earlier in this lecture, with solution

$$y_k = 2^{k/2} \cos\left(\frac{k\pi}{4}\right) \,.$$

The term $\cos\left(\frac{k\pi}{4}\right)$ is periodic: The first eleven values of this sequence are:

$$1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, \dots \,.$$

These values are then multiplied by the terms $2^{k/2}$ to produce an oscillating sequence, sketched below:



The behaviour of the sequence elements is rather complicated. Note that we cannot simply state that $|y_k| \to \infty$ since the sequence y_k assumes that value 0 infinitely often, at $k = 2, 6, 10, \dots$ or $k = 2+4n$, $n = 0, 1, 2 \dots$. If we allowed the index k to assume continuous real values, $k = x \in \mathbb{R}$, then the curve $y(x) = 2^{x/2} \cos\left(\frac{\pi}{4}x\right)$ would oscillate between the envelope curves $g_{\pm}(x) \pm 2^{x/2}$. The sequence y_k given above is the result of sampling the $y(x)$ curve at integer values of x.

Finally, we discuss briefly the case of distinct real roots, with $|m_1| > 1$ and $|m_2| < 1$. Recall that the general solution is

$$Y_k = C_1 m_1^k + C_2 m_2^k. \tag{25}$$

- **Case 1:** $C_1 = 0$. Then $|Y_k| \to 0$ as $k \to \infty$.

- **Case 2:** $C_1 \neq 0$. Then

$$Y_k = C_1 m_1^k \left[1 + \frac{C_2}{C_1} \left(\frac{m_2}{m_1} \right)^k \right]. \tag{26}$$

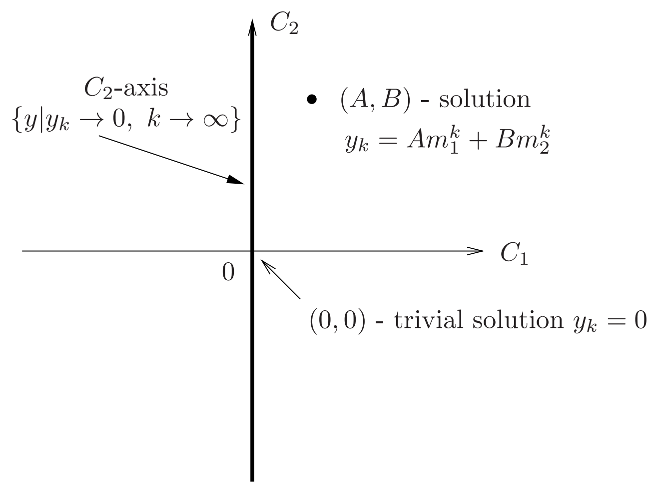
  Since $\left|\frac{m_2}{m_1}\right| < 1$, it follows that $\left(\frac{m_2}{m_1}\right)^k \to 0$ as $k \to \infty$, so that

$$Y_k \cong C_1 m_1^k \;\text{ as }\; k \to \infty\,. \tag{27}$$

  In other words, for $C_1 \neq 0$, the behaviour of the Y_k is dominated by the root m_1. If $m_1 > 1$, then the Y_k will diverge monotonically to $\pm\infty$. If $m_1 < -1$, the Y_k will eventually oscillate so that $|Y_k| \to \infty$.

The general solution Y_k in Eq. (25) is a **two-parameter family** of functions. One must specify an ordered pair of values, say, $(C_1, C_2) = (A, B) \in \mathbb{R}^2$ to isolate a particular solution from this family. Such an ordered pair may be isolated by the imposition of a particular initial condition of the form $Y_0 = A_0$ and $Y_1 = A_1$ in Eq (25).

We can represent this two-parameter family of general solutions Y_k in (25) as points (C_1, C_2) in a "parameter-space plane" as shown below:



This "parameter space" can be viewed as a kind of "atlas" that can be used to summarize important properties of solutions. For example, as seen in Case 1 above, the C_2-axis, i.e. the set of points $(0, C_2) \in \mathbb{R}^2$, for all $C_2 \in \mathbb{R}$, represents the collection of all solutions for which $y_k \to 0$ as $k \to \infty$ (including the trivial solution).

The usual measure of the size of a subset S of the plane is its area. Since the area of a line, in this case the C_2 axis, is zero, we may state that "almost all" solutions Y_k diverge, with the exception of a set of "measure zero" in the plane ("zero area"). To summarize, if a solution y_k has a non-zero ($C_1 \neq 0$) component in the dominant root $|m_1| > 1$, then it is guaranteed to diverge as $k \to \infty$. We shall return to these ideas in a later chapter.

Example: We conclude with a return to the following d.e.,

$$x_{n+2} = x_{n+1} + x_n \,, \tag{28}$$

examined in Lecture 2. We showed that with the initial conditions $x_0 = 0$, $x_1 = 1$, the sequence $\{x_k\}$ corresponds to the famous set of Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, \ldots$. This d.e. can be rewritten in the following form,

$$x_{n+2} - x_{n+1} - x_n = 0 \,, \tag{29}$$

which is a linear second-order homogeneous d.e. with constant coefficients. The characteristic equation for this d.e. is

$$m^2 - m - 1 = 0 \tag{30}$$

with distinct real roots,

$$m_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}, \quad m_2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}. \tag{31}$$

Note that $|m_1| \cong 1.61$ and $|m_2| \cong 0.61$ so that m_1 is the dominant root. The general solution to the d.e. is

$$Y_k = C_1 m_1^k + C_2 m_2^k. \tag{32}$$

If we impose the initial conditions $x_0 = 0$, $x_1 = 1$, then, after some algebra, $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$, so that, in terms of the general solution, the Fibonacci numbers p_n are given by

$$x_n = \tfrac{1}{\sqrt{5}} \left[\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} \right)^n - \left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5} \right)^n \right]. \tag{33}$$

(At first glance, this expression might seem strange because of the presence of the irrational coefficient $\sqrt{5}$. However, you can verify by binomial expansion of each term that perfect cancellation of irrational terms takes place and that x_n is an integer for all n.) Because of the dominance of the root m_1 we may, from Eq. (27) of our previous discussion, conclude that

$$x_n \cong C_1 m_1^k = \tfrac{1}{\sqrt{5}} \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} \right)^n \quad \text{as} \quad n \to \infty. \tag{34}$$

From this, it follows that

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{C_1 m_1^{k+1}}{C_1 m_1^k} = m_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}. \tag{35}$$

This result may also be obtained as follows: Take the original d.e. (28) in the x_n and divide by x_{n+1} to obtain

$$\frac{x_{n+2}}{x_{n+1}} = 1 + \frac{x_n}{x_{n+1}}. \tag{36}$$

Assuming that a limit of the ratios exists, calling it L, we have, in the limit $n \to \infty$,

$$L = 1 + \frac{1}{L} \tag{37}$$

or

$$L^2 - L - 1 = 0\,, \tag{38}$$

which is identical to the characteristic equation (30) of our original d.e. in (28), with roots $L = m_1$, $L = m_2$. Since $m_2 < 0$, only the root $L = m_1$ is admissible, in agreement with the result in (35).

Lecture 7

Linear difference equations with constant coefficients (cont'd)

Analysis of a discrete model for the propagation of annual plants

In this section we formulate a very simple model to describe the propagation of annual plants. The results of the preceding section are then used to understand the qualitative behaviours of solutions to this model and how they are affected by the biological parameters introduced. Our treatment of this model follows the discussion in the book by Edelstein-Keshet (posted on LEARN).

Annual plants produce seeds at the end of a summer. The plants die, leaving dormant seeds that must survive a winter (or more) to give rise to a new generation. In the following spring, a fraction of the seeds germinate. Some seeds might remain dormant for a year or more before germinating. Others may be lost to predation, disease or weather. The survival of the species depends upon the successful renewal of a sufficiently large population.

In the model that follows, we assume that the annual plants produce seeds at the end of their growth season, say August, after which the plants die. A fraction of these seeds survive the winter and some of these seeds germinate in the beginning of the next season, say May, producing the next generation of plants. The fraction that germinates depends upon the age of the seeds. In this simple model, we assume that seeds can survive over only two winters.

We shall let P_n denote the number of plants that are growing in a given generation n. The following parameters will be needed:

γ - the number of seeds produced per plant in August

σ - the fraction of seeds that survive a given winter

α - the fraction of seeds that survive one winter and germinate in May

β - the fraction of seeds that survive two winters and germinate in May.

Let us now set up a difference equation for the variables P_n. Clearly, P_n is determined by both P_{n-1} (seeds that survive one winter) and P_{n-2} (seeds that survive two winters). The contributions are as follows:

(a) From P_{n-1}: In generation $n-1$, γP_{n-1} seeds are produced. Of these, $\sigma\gamma P_{n-1}$ seeds survive the winter. Of these seeds, $\alpha\sigma\gamma P_{n-1}$ germinate to produce plants in generation n.

(b) From P_{n-2}: In generation $n-2$, γP_{n-2} seeds are produced. Of these, $\sigma\gamma P_{n-2}$ seeds survive the winter. Of these seeds, $\alpha\sigma\gamma P_{n-2}$ germinate to produce plants in generation $n-1$ and $(1-\alpha)\sigma\gamma P_{n-2}$ remain dormant. Of these seeds, $\sigma(1-\alpha)\sigma\gamma P_{n-2}$ survive the second winter and $\beta\sigma(1-\alpha)\sigma\gamma P_{n-2}$ seeds germinate to produce plants in generation n.

Combining these results, we have the relation

$$P_n = \alpha\gamma\sigma P_{n-1} + \beta\gamma\sigma^2(1-\alpha)P_{n-2} \tag{39}$$

which we shall rewrite as

$$\boxed{P_{n+2} - \alpha\gamma\sigma P_{n+1} - \beta\gamma\sigma^2(1-\alpha)P_n = 0.} \tag{40}$$

This is a second-order linear d.e. with constant coefficients having the form,

$$P_{n+2} + pP_{n+1} + qP_n = 0\,, \tag{41}$$

where

$$p = -\alpha\gamma\sigma, \quad q = -\beta\sigma^2\gamma(1-\alpha). \tag{42}$$

The roots of the characteristic equation $m^2 + pm + q = 0$ are

$$m_{1,2} = \tfrac{1}{2}\alpha\gamma\sigma \pm \tfrac{1}{2}\sqrt{(\alpha\gamma\sigma)^2 + 4\beta\sigma^2\gamma(1-\alpha)}\,. \tag{43}$$

Recalling that the asymptotic behaviour of solutions P_n to a second-order difference equation with constant coefficients is determined by the roots m_1 and m_2 of its associated characteristic equation, we must investigate Eq. (43) more carefully.

Before proceeding, however, we shall make some assumptions on the fraction α, namely, that it can be neither 0 nor 1 in value, i.e., $0 < \alpha < 1$. Why?

- If $\alpha = 0$, then no seeds that survive one winter will germinate the following year. The only contribution to the plant population P_{n+2} comes from seeds produced two generations ago. This is a rather unrealistic situation. In principle, the resulting difference equation equation in (39),

  $$P_{n+2} - \beta\gamma\sigma^2 P_n = 0\,, \tag{44}$$

  is easily solved but it yields the rather unrealistic result that we have two separate sequences of generations, $P_{n,even}$ and $P_{n,odd}$.

- The case $\alpha = 1$ corresponds to another unrealistic assumption, namely, that all seeds that survive one winter will germinate during the following month of May, meaning that no seeds would be available for the next generation. Our difference equation in (39) would then reduce to

  $$P_n = \gamma\sigma P_{n-1}\,, \tag{45}$$

  which is a linear first-order difference equation that is easily solved, i.e.,

  $$P_n = (\gamma\sigma)^n P_0\,. \tag{46}$$

  This case, however, defeats the purpose of our wish to study the effects of more than one generation contributing to a population.

Therefore, in summary, we shall assume that

$$0 < \alpha < 1 . \tag{47}$$

and now proceed with a first "line of attack". Note that the second term in the square root of (43) is positive, i.e.,

$$4\beta\sigma^2(1-\alpha) > 0 , \tag{48}$$

which, in turn, implies that

$$\sqrt{(\alpha\gamma\sigma)^2 + 4\beta\sigma^2\gamma(1-\alpha)} > \alpha\gamma\sigma . \tag{49}$$

(This, admittedly, is a rather "gross" approximation but we'll start with it. Later, we'll modify it, i.e., "fine tune" it.) From Eq. (43), it follows that

$$m_1 = \tfrac{1}{2}\alpha\gamma\sigma + \tfrac{1}{2}\sqrt{(\alpha\gamma\sigma)^2 + 4\beta\sigma^2\gamma(1-\alpha)} \; > \; \alpha\gamma\sigma \; > \; 0 \tag{50}$$

and

$$m_2 = \tfrac{1}{2}\alpha\gamma\sigma - \tfrac{1}{2}\sqrt{(\alpha\gamma\sigma)^2 + 4\beta\sigma^2\gamma(1-\alpha)} \; < \; 0 . \tag{51}$$

Note that for $A > 0$ and $B > 0$, we have that

$$|A - B| < |A + B| . \tag{52}$$

(Proof: Square both sides of the inequality.) From this inequality, we may conclude that

$$|m_2| < |m_1| . \tag{53}$$

The asymptotic behaviour of solutions will therefore be determined by the root m_1.

Recall from the previous section that the long-term behaviour of the solution to a difference equation is determined by the magnitudes of the roots to its characteristic equation - in particular, the root with the greatest magnitude. From Eq. (53), the root with the greatest magnitude is m_1. There are two situations to consider:

1. $|m_1| < 1$, which, from Eq. (53), implies that $|m_2| < 1$: Then all solutions $P_k \to 0$ as $k \to \infty$. The result is extinction of the plant population.

2. $m_1 > 1$, which obviously implies that $|m_1| > 1$: Then solutions P_k, in general, will not decay to zero but grow with k. This situation is obviously in the interest of the plant since it will not decay into extinction.

From Condition No. 2 above and Eq. (50), it follows that a sufficient (but not necessary) condition to ensure that the plant species does not decay is that

$$\boxed{\alpha\gamma\sigma > 1 .} \tag{54}$$

Note that $\alpha\gamma\sigma$ is the number of seeds produced by a given plant that survive a winter and germinate the following spring. In other words, the model is telling us that we need at least one seed per plant to germinate the following year in order that the population does not decrease, which makes sense. Note that the inequality in (54) essentially describes a "one-generation contribution," i.e., nondecay of the population based only on the production and germination of seeds from the previous generation. There is no "two-generation contribution" here, i.e., contribution to the population from seeds from two generations ago.

Note: Recall that the condition in (54) was obtained by the rather "gross" approximation in (49). This is why we wrote that the condition in (54) is a **sufficient** condition and **not** a necessary one. We'll see below that this condition on the parameters can be somewhat relaxed.

Let us now perform a more detailed or "fine-tuned" analysis of the roots m_1 and m_2, along the lines performed in the book by Edelstein-Keshet to see how the condition in Eq. (54) could be relaxed by including "two-generation contributions." First, we rewrite the equation for the roots in (43) as

$$m_{1,2} = \tfrac{1}{2}\alpha\gamma\sigma \left[1 \pm \sqrt{1+\delta}\right], \tag{55}$$

where

$$\delta = \frac{4\beta}{\alpha\gamma}\left(\frac{1}{\alpha} - 1\right). \tag{56}$$

Note that $\delta > 0$ since $0 < \alpha < 1$. It follows that the roots m_1 and m_2 are real and distinct.

In the above model, it is necessary that the dominant root satisfy the inequality

$$m_1 = \tfrac{1}{2}\alpha\gamma\sigma \left[1 + \sqrt{1+\delta}\right] > 1 \tag{57}$$

for the population not to decrease.

Since $\delta > 0$ (see above), it follows that

$$1 + \sqrt{1+\delta} \geq 2\,. \tag{58}$$

(We shall return shortly to the limiting case $\delta = 0$). Once again, it follows that Eq. (54) is a sufficient (but not necessary) condition to ensure the inequality in (57).

This is certainly true in the case that β, or perhaps more appropriately, the ratio β/α, is very small, i.e. few, if any, seeds germinate after two winters – once again, the "one-generation effect." When β/α is small, δ is close to zero – see Eq. (56) above – and the above inequality appears necessary.

If β/α is not so small, so that δ is not close to zero – a "two-generation effect" – then such a stringent requirement as (54) is not necessary. In fact, we can derive a less stringent requirement on the parameters by going back to Eq. (50) and demanding that $m_1 > 1$, i.e.,

$$m_1 = \tfrac{1}{2}\alpha\gamma\sigma + \tfrac{1}{2}\sqrt{(\alpha\gamma\sigma)^2 + 4\beta\sigma^2\gamma(1-\alpha)}\; > \;1\,. \tag{59}$$

From this inequality, we can already see that a little less "pressure" is placed on the term $\alpha\gamma\sigma$ so that it doesn't have to be greater than one. Now subtract the term $\frac{1}{2}\alpha\gamma\sigma$ from both sides of the second inequality,

$$\tfrac{1}{2}\sqrt{(\alpha\gamma\sigma)^2 + 4\beta\sigma^2\gamma(1-\alpha)} > 1 - \tfrac{1}{2}\alpha\gamma\sigma\,. \tag{60}$$

Now square both sides to obtain the result,

$$\boxed{\alpha\gamma\sigma + \beta\gamma\sigma^2(1-\alpha) > 1.} \tag{61}$$

This is the new condition that guarantees that $m_1 > 1$ – a kind of "relaxed version" of the condition in (54). One can see how an increased β value lessens the requirement on $\alpha\gamma\sigma$ to be greater than one. In other words, the second generation seeds are contributing to survival.

In order to illustrate the effects of these parameters, we examine two cases numerically. In both cases, we begin with a plant population $P_0 = 100$. P_1 is computed from P_0 using seeds from only the previous generation, i.e.

$$P_1 = \alpha\sigma\gamma P_0\,. \tag{62}$$

Future generations $P_2, P_3, \ldots$ are then computed using the second-order d.e. in Eq. (40).

Case 1: $\gamma = 2.0$**,** $\sigma = 0.8$**,** $\alpha = 0.5$**,** $\beta = 0.25$

Note that inequality (54) is not satisfied:

$$\alpha\gamma\sigma = 0.8 < 1\,.$$

Recall that this was the "gross approximation" – a sufficient but not necessary condition. There might be the hope that a second-generation contribution might help the species. However, the inequality in (61) is also not satisfied:

$$\alpha\gamma\sigma + +\beta\gamma\sigma^2(1-\alpha) = \frac{24}{25} < 1.$$

In fact, the roots of the characteristic equation are $m_1 \cong 0.847$, $m_2 \cong -0.047$. Since $|m_1| < 1$, $|m_2| < 1$, the solution P_n will tend to zero as $n \to \infty$. This is confirmed in the plot on the next page.

Case 2: $\gamma = 2.0$**,** $\sigma = 0.8$**,** $\alpha = 0.6$**,** $\beta = 0.3$

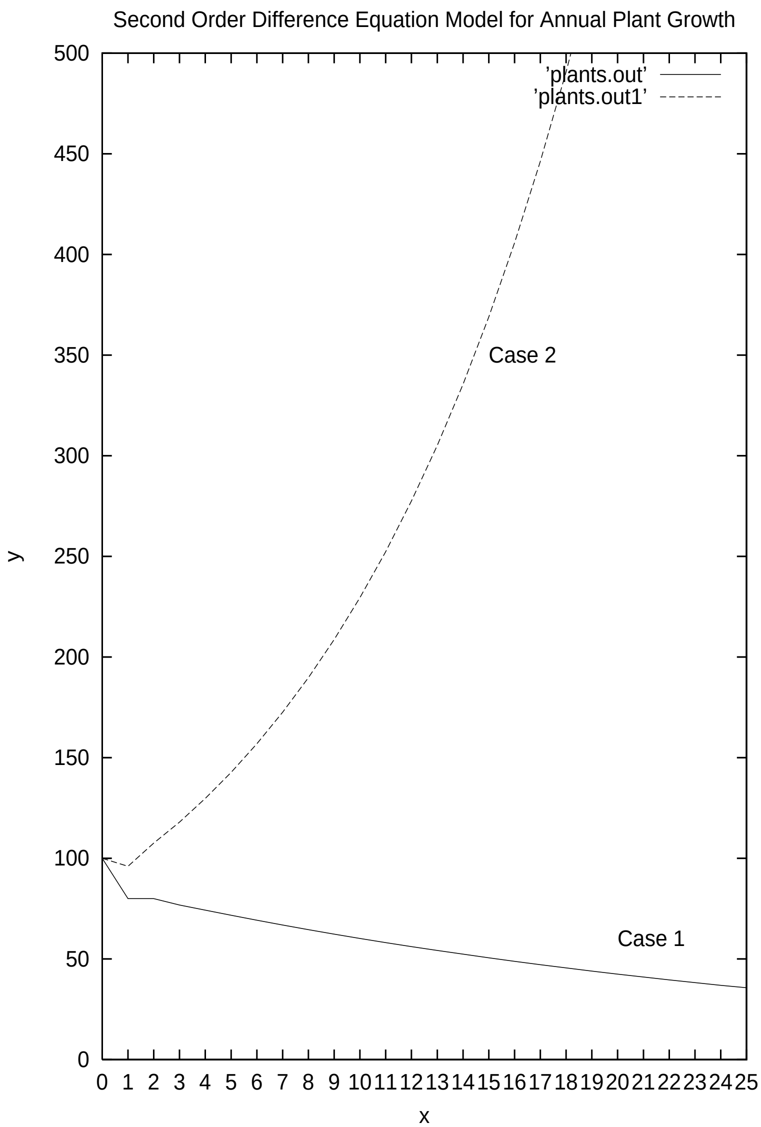
Note that γ and σ are as in Case 1. The fractions α and β have been increased slightly from Case 1. In this case, (54) remains unsatisfied:

$$\alpha\gamma\sigma = 0.96 < 1\,.$$

However, (61) is satisfied:

$$\alpha\gamma\sigma + \beta\gamma\sigma^2(1-\alpha) = \frac{696}{625} > 1.$$

The roots of the characteristic equation are $m_1 \cong 1.08$, $m_2 \cong -0.14$. The solution P_n is expected eventually to increase with n, as is seen in the plot on the next page. Contributions from the second generation save the day.



Solutions P_n to the annual plant propagation model for the two cases described in the text.

Linear difference equations with constant coefficients (cont'd)

Finding solutions to the inhomogeneous equation – The Method of Undetermined Coefficients

The previous sections were concerned with finding general solutions Y_k of first and second order homogeneous d.e.'s with constant coefficients. We now focus on finding particular solutions to the second order **inhomogeneous** d.e. with constant coefficients

$$y_{k+2} + py_{k+1} + qy_k = r_k, \quad k \geq 0. \tag{63}$$

for some standard forms of the inhomogeneous terms r_k. In these cases, a "method of undetermined coefficients" quite analogous to the method used for differential equations can be used. We assume a solution to (63) having a particular form that "matches" the r_k terms. This solution will have a number of unknown parameters, the "undetermined coefficients", that are then to be determined by substitution into (63).

With regard to a motivation to study inhomogeneous d.e.'s of the form (63), let us consider – perhaps with a little stretch of the imagination – the propagation model for annual plants studied in the previous section. Let us assume that a constant number of plants, say $B > 0$, are added to the environment each year. This term would have to be added to the right hand side of Eq. (39) to produce the modified d.e.,

$$P_{n+2} = \alpha\gamma\sigma P_{n+1} + \beta\gamma\sigma^2(1-\alpha)P_n + B\,, \tag{64}$$

which, after rewriting, becomes the following inhomogeneous d.e.,

$$P_{n+2} - \alpha\gamma\sigma P_{n+1} - \beta\gamma\sigma^2(1-\alpha)P_n = B\,. \tag{65}$$

Case 1: $r_k = a$ (Constant inhomogeneous term)

This represents the simplest possible inhomogeneous term, where the r_k are independent of k. The d.e. in (63) is

$$y_{k+2} + py_{k+1} + qy_k = a\,, \quad k \geq 0\,. \tag{66}$$

If we assume a **constant solution**, i.e. $y_k^{(p)} = A$, where A is the undetermined coefficient, substitution into (63) yields

$$A + pA + qA = A(1+p+q) = a. \tag{67}$$

We may solve for A:

$$A = \frac{a}{1+p+q}, \tag{68}$$

provided that $1 + p + q \neq 0$.

Example:

$$y_{k+2} - 7y_{k+1} + 12y_k = 5.$$

If we assume a particular solution $y_k^{(p)} = A$, then, substituting into the d.e., we obtain

$$A - 7A + 12A = 5 \quad \Longrightarrow \quad 6A = 5 \quad \Longrightarrow \quad A = \frac{5}{6}.$$

It is better to do it this way than to try to remember (68), i.e.,

$$A = \frac{a}{1+p+q} = \frac{5}{1-7+12} = \frac{5}{6}.$$

Since the general solution to the homogeneous d.e. is (Exercise)

$$Y_k = C_1 4^k + C_2 \cdot 3^k,$$

the general solution to this d.e. is

$$y_k = \tfrac{5}{6} + C_1 \cdot 4^k + C_2 \cdot 3^k.$$

In the case that $1 + p + q = 0$, the above method breaks down. This is because $m = 1$ is a root of the characteristic equation for the associated homogeneous d.e.. This, in turn, implies that $y_k = C_1 \cdot 1^k = C_1$ – the constant solutions – satisfies the homogeneous d.e. yielding 0 in the RHS. As such, it is impossible to produce a non-zero term on the RHS. In this case, we try a solution of the form $y_k^{(p)} = Ak$.

Example:

$$y_{k+2} - 3y_{k+1} + 2y_k = 10.$$

The characteristic equation

$$m^2 - 3m + 2 = 0$$

has roots $m_1 = 2$ and $m_2 = 1$. The general solution to the homogeneous d.e. is

$$Y_k = C_1 \cdot 2^k + C_2,$$

which, of course, includes the constant solution. Therefore, we cannot assume the particular solution to be a constant solution. If, however, we assume a particular solution $y_k^{(p)} = Ak$, substitution yields

$$A(k+2) - 3A(k+1) + 2Ak = 10, \quad k \geq 0$$

Collecting like terms in k:

$$(A - 3A + 2A)k + A(2-3) = 10.$$

The term in k vanishes (as it should – Exercise) and we find $A = -10$. Thus, the general solution is

$$y_k = -10k + C_1 \cdot 2^k + C_2.$$

Example:

$$y_{k+2} - 2y_{k+1} + y_k = 10.$$

The characteristic equation has $m_1 = m_2 = 1$ as a double root, so that its general solution is

$$Y_k = C_1 + C_2 k.$$

A constant particular solution of the form $y_k^{(p)} = A$ will not work since it coincides with the constant homogeneous solution C_1. However, a particular solution of the form $y_k^{(p)} = Ak$ will also not work since it coincides with the term $C_2 k$ in Y_k. We must try $y_k^{(p)} = Ak^2$. Substitution yields

$$A(k+2)^2 - 2A(k+1)^2 + Ak^2 = 10.$$

or

$$(A - 2A + A)k^2 + (4A - 4A)k + (4-2)A = 10\,,$$

which reduces to the equation,

$$2A = 10\,.$$

Therefore $A = 5$. Thus the general solution is

$$y_k = C_1 + C_2 k + 5k^2.$$

Note: If we had assumed a solution of the form $y_k^{(p)} = A_0 + A_1 k + A_2 k^2$, the terms A_0 and $A_1 k$ would have been "annihilated" on the left side of the d.e., making it impossible to determine A_0 and A_1 uniquely. (This is a consequence of the fact that these terms are solutions to the homogeneous d.e..) However, it never hurts to include extra terms. The mathematics will eventually tell you whether or not such terms were necessary.

Lecture 8

Linear difference equations with constant coefficients (cont'd)

Finding solutions to the inhomogeneous equation - The Method of Undetermined Coefficients (cont'd)

We continue our discussion from the previous lecture on finding particular solutions to second order **inhomogeneous** d.e. with constant coefficients,

$$y_{k+2} + py_{k+1} + qy_k = r_k, \quad k \geq 0\,, \tag{69}$$

for some standard forms of the inhomogeneous terms r_k. Recall that p and q are assumed to be (real) constants.

The following discussion of the so-called "Method of Undetermined Coefficients" is rather "short and sweet" – a kind of "recipe book" with recipes that get rather complicated as we move down the cases. The reader is advised **not** to try to memorize any of the formulas. The most important aspect of the procedures outlined below is to note exactly where the sequence index k (i.e., y_k) appears in the inhomogeneous term, i.e., appearing in the exponent (Case 2) or the "thing" being exponentiated (Case 3), or perhaps a combination of both (Case 4).

Case 2: $r_k = a^k$ **(or** Ca^k**)**

In this case, Eq. (69) has the form

$$y_{k+2} + py_{k+1} + qy_k = a^k, \quad k \geq 0\,, \tag{70}$$

where a is a constant. The inhomogeneous term, a^k, has the sequence index, k, in the exponent. If we assume a particular solution of the form $y_k^{(p)} = Aa^k$, then substitution into (70) yields

$$Aa^{k+2} + pAa^{k+1} + qAa^k = a^k. \tag{71}$$

Dividing both sides of the equation by a^k gives

$$A[a^2 + pa + q] = 1. \tag{72}$$

If a is not a root of the characteristic equation, then associated with (70), i.e., $a^2 + pa + q \neq 0$, we may solve for A:

$$A = \frac{1}{a^2 + pa + q}. \tag{73}$$

On the other hand, if a is a simple root of the characteristic equation, then we must try $y_k^{(p)} = Aka^k$. If a is a double root, i.e. $m_1 = m_2 = a$, then we must try $y_k^{(p)} = Ak^2a^k$.

Example:

$$y_{k+2} - 4y_{k+1} + 4y_k = 2^k.$$

The characteristic equation has a double root $m_1 = m_2 = 2$ so that the general solution to the homogeneous d.e. is

$$Y_k = C_1 \cdot 2^k + C_2 k 2^k.$$

We must assume a particular solution having the form $y_k^{(p)} = Ak^2 2^k$. Substitution yields

$$A(k+2)^2 2^{k+2} - 4A(k+1)^2 2^{k+1} + 4Ak^2 \cdot 2^k = 2^k.$$

Dividing out the term 2^k from both sides:

$$(4A - 8A + 4A)k^2 + (16A - 16A)k + 16A - 8A = 1$$

implying that $A = \frac{1}{8}$. Thus $y_k^{(p)} = \frac{1}{8}k^2 \cdot 2^k$ and the general solution is

$$y_k = C_1 \cdot 2^k + C_2 k 2^k + \tfrac{1}{8}k^2 \cdot 2^k.$$

Case 3: $r_k = k^n$ **(or** Ck^n**)**

In this case, Eq. (69) has the form

$$y_{k+2} + py_{k+1} + qy_k = k^n, \quad k \geq 0\,, \tag{74}$$

where the exponent n is a constant. The inhomogeneous term, k^n, has the sequence index, k, in as the number to be taken to the power n.

It is not sufficient to assume a particular solution of the form $y_k^{(p)} = Ak^n$. Because of the cross terms that are produced in the terms $(k+2)^n$ and $(k+1)^n$, it is necessary to include lower powers of k, i.e.

$$y_k^{(p)} = A_0 + A_1 k + A_2 k^2 + \cdots + +A_n k^n. \tag{75}$$

This form must again be modified if $m = 1$ is a root of the characteristic equation since the constant term A_0 will be a solution to the homogeneous d.e.. As a result, we must multiply the above expression by k, or equivalently use

$$y_k^{(p)} = A_1 k + A_2 k^2 + A_n k^n + A_{n+1} k^{n+1}. \tag{76}$$

(The author prefers the above form since the index j of the coefficient A_j matches the power of k it multiplies, i.e., $A_j k^j$, $1 \leq j \leq n$.) If $m = 1$ is a double root of the characteristic equation, then we must use

$$y_k^{(p)} = A_2 k^2 + A_3 k^3 + \cdots + A_n k^n + A_{n+1} k^{n+1} + A_{n+2} k^{n+2}. \tag{77}$$

Example:

$$y_{k+2} - 3y_{k+1} + 2y_k = k^2.$$

The roots of the characteristic equation are $m_1 = 2$, $m_2 = 1$, so that the general solution of the homogeneous d.e. is

$$Y_k = C_1 2^k + C_2.$$

Since $m = 1$ is a root of the characteristic equation, we assume a particular solution of the form in Eq. (76), i.e.,

$$y_k^{(p)} = A_1 k + A_2 k^2 + A_3 k^3.$$

Substitution into the d.e. yields, after some computation involving the collection of like powers of k,

$$-3A_3k^2 + (-2A_2 + 3A_3)k + (-A_1 + A_2 + 5A_3) = k^2.$$

We see immediately that $A_3 = -\frac{1}{3}$. We can use this result to obtain the following linear system of equations in the unknowns A_2 and A_3:

$$\begin{aligned} -2A_2 + 3A_3 &= 0 \\ -A_1 + A_2 &= \tfrac{5}{3}. \end{aligned}$$

The solution of this system is $A_2 = -\frac{1}{2}$ and $A_1 = -\frac{13}{6}$. The general solution is therefore

$$y_k = -\tfrac{13}{6}k - \tfrac{1}{2}k^2 - \tfrac{1}{3}k^3 + C_1 \cdot 2^k + C_2.$$

Case 4: $r_k = k^n a^k$

If "a" is not a root of the characteristic equation, then one may assume a solution of the form

$$y_k^{(p)} = a^k(A_0 + A_1 k + \cdots + A_n k^n). \tag{78}$$

As before, if "a" is a simple root of the characteristic equation, then the above solution will not work since the term $A_0 a^k$ will be a solution of the homogeneous d.e. We must then try

$$y_k^{(p)} = a^k(A_1 k + A_2 k^2 + \cdots + A_n k^n + A_{n+1} k^{n+1}). \tag{79}$$

If "a" is a double root, then we must assume

$$y_k^{(p)} = a^k(A_2 k^2 + \cdots + A_{n+2} k^{n+2}). \tag{80}$$

Case 5: $r_k = \sin bk$ **or** $\cos bk$ **(or a linear combination of both)**

Assume a particular solution having the form

$$y_k^{(p)} = A \cos bk + B \sin bk. \tag{81}$$

Example:

$$y_{k+2} - 3y_{k+1} + 2y_k = 7\sin\frac{k\pi}{2}.$$

Here, $b = \dfrac{\pi}{2}$. Substitution of the particular solution in (81) into the d.e. yields

$$\begin{aligned} A\cos\left(\frac{k+2}{2}\pi\right) + B\sin\left(\frac{k+2}{2}\pi\right) - 3A\cos\left(\frac{k+1}{2}\pi\right) - 3B\sin\left(\frac{k+1}{2}\pi\right) \\ + \quad 2A\cos\left(\frac{k\pi}{2}\right) + 2B\sin\left(\frac{k\pi}{2}\right) = 7\sin\left(\frac{k\pi}{2}\right). \end{aligned}$$

Note that

$$\begin{aligned} \cos\left(\frac{k+2}{2}\pi\right) &= \cos\left(\frac{k\pi}{2}+\pi\right) = -\cos\left(\frac{k\pi}{2}\right) \\ \sin\left(\frac{k+2}{2}\pi\right) &= \sin\left(\frac{k\pi}{2}+\pi\right) = -\sin\left(\frac{k\pi}{2}\right) \\ \cos\left(\frac{k+1}{2}\pi\right) &= \cos\left(\frac{k\pi}{2}+\frac{\pi}{2}\right) = -\sin\left(\frac{k\pi}{2}\right) \\ \sin\left(\frac{k+1}{2}\pi\right) &= \sin\left(\frac{k\pi}{2}+\frac{\pi}{2}\right) = \cos\left(\frac{k\pi}{2}\right). \end{aligned}$$

After some algebra, the above equation yields

$$(A-3B)\cos\left(\frac{k\pi}{2}\right) + (3A-B)\sin\left(\frac{k\pi}{2}\right) = 7\sin\left(\frac{k\pi}{2}\right).$$

Since this relation must hold for all k, we have

$$\begin{aligned} A - 3B &= 0 \\ 3A - B &= 7 \end{aligned}$$

with solution $A = \frac{21}{8}$, $B = \frac{7}{8}$.

The general solution is therefore

$$y_k = C_1 + C_2\cdot 2^k + \tfrac{21}{8}\cos\left(\tfrac{k\pi}{2}\right) + \tfrac{7}{8}\sin\left(\tfrac{k\pi}{2}\right).$$

Case 6: $r_k = a^k\sin bk$ or $a^k\cos bk$ (or a linear combination of both)

Assume

$$y_k^{(p)} = a^k(A\cos bk + B\sin bk). \tag{82}$$

The above discussion covers the most standard cases of inhomogeneous terms that would be of interest in this course. When solving such problems, it is best to first determine the general solution Y_k of the associated homogeneous d.e.. One can then see if the solutions assumed in the method of undetermined coefficients have to be slightly modified – by multiplication by k or k^2 – so that they no longer coincide with solutions making up the Y_k.

Finally, note that if $y_k^{(p,1)}$ is a particular solution to the inhomogeneous d.e.

$$y_{k+2} + py_{k+1} + qy_k = r_k^{(1)} \tag{83}$$

and $y_k^{(p,2)}$ is a solution to the d.e.

$$y_{k+2} + py_{k+1} + qy_k = r_k^{(2)} \tag{84}$$

then

$$y_k^{(p)} = y_k^{(p,1)} + y_k^{(p,2)} \tag{85}$$

is a solution of the d.e.

$$y_{k+2} + py_{k+2} + qy_k = r_k^{(1)} + r_k^{(2)}. \tag{86}$$

This is easily shown by the linearity of the d.e.. This property allows us to solve d.e.'s whose inhomogeneous terms r_k may be composed of a number of terms from the different cases studied above. For example, in the case

$$y_{k+2} - 5y_{k+1} + 6y_k = 4^k + k^2$$

one can use

$$y_k^{(p,1)} = A \cdot 4^k, \quad y_k^{(p,2)} = A_0 + A_1 k + A_2 k^2$$

and solve for A in one procedure and A_0, A_1 and A_2 in another. The net result (Exercise):

$$y_k = C_1 \cdot 3^k + C_2 \cdot 2^k + \frac{1}{2}4^k + 3 + \frac{3}{2}k + \frac{1}{2}k^2.$$

Problems

1. Find particular solutions of the following difference equations by the method of undetermined coefficients:

   (a) $y_{k+2} - 5y_{k+1} + 6y_k = 2$

   (b) $8y_{k+2} - 6y_{k+1} + y_k = 2^k$

   (c) $y_{k+2} - 3y_{k+1} + 2y_k = 1$

   (d) $y_{k+2} - y_{k+1} - 2y_k = k^2$

   (e) $y_{k+2} + y_k = \sin \frac{k\pi}{2}$

   (f) $4y_{k+2} - 4y_{k+1} + y_k = 2$

   (g) $y_{k+2} - 2y_{k+1} + y_k = 5 + 3k$

   (h) $y_{k+2} - 2y_{k+1} + y_k = 2^k(k-1)$

2. Find the general solutions of the difference equations in Problem 2. (First find the general solutions of the corresponding homogeneous equations and then use Theorem 2.6.)

3. Find the solutions of the difference equations in Problem 1 which satisfy the initial conditions $y_0 = 1$ and $y_1 = -1$. (*Hint:* Find appropriate values for the arbitrary constants in the general solutions found in Problem 2.)

4. For the difference equation in Problem 1(a), find the solution which satisfies the initial conditions $y_1 = -1$ and $y_2 = -9$. Note that your answer is the solution obtained in Problem 3 for the initial conditions $y_0 = 1$ and $y_1 = -1$. Why is this so?

Answers:

1. (a) Let $y_k^{(p)} = A$ and show that $A = 1$.
   (c) $y_k^{(p)} = A$ fails; let $y_k^* = Ak$ and find $A = -1$.
   (e) $y_k^{(p)} = A \sin \frac{k\pi}{2} + B \cos \frac{k\pi}{2}$ fails; let $y_k^{(p)} = Ak \sin \frac{k\pi}{2} + Bk \cos \frac{k\pi}{2}$ and find $A = -\frac{1}{2}$, $B = 0$.
   (g) Try $y_k^{(p)} = Ak^2 + Bk^3$ and find $A = 1$, $B = \frac{1}{2}$.

2. To the particular solutions obtained in Problem 7 add the general solution y of the homogeneous equation, where Y_k equals
   (a) $C_1 2^k + C_2 3^k$,
   (c) $C_1 + C_2 2^k$,
   (e) $A \cos\left(\frac{k\pi}{2} + B\right)$,
   (g) $C_1 + C_2 k$,
   (h) $C_1 + C_2 k$.

3. (a) $y_k = 2^{k+1} - 2 \cdot 3^k + 1$.

(c) $y_k = 2 - 2^k - k$.

(e) $y_k = \dfrac{\sqrt{5}}{2} \cos\left(\frac{k\pi}{2} + B\right) - \frac{k}{2} \sin \frac{k\pi}{2}$, where B is the acute angle for which $\sin B = \dfrac{1}{\sqrt{5}}$ and $\cos B = \dfrac{2}{\sqrt{5}}$.

(g) $y_k = \frac{1}{2}(2 - 7k + 2k^2 + k^3)$.

Linear Systems of Difference Equations

Introduction

A system being modelled may have several components whose time evolution are of interest. Moreover, it is quite likely that components will interact with each other. For example, in the biological world, consider a simple model involving the interaction of a population of predators, say foxes, that feed on a population of prey, say rabbits. If there were no foxes around, the rabbit population would be unchecked. If we assume, for simplicity, that a virtually infinite supply of food were available to the rabbits, then the rabbit population would grow exponentially according to our naive model in Chapter 1, $p_{n+1} = cp_n$. Now suppose that a small number of foxes appear on the scene. In the midst of so much food, the fox population would probably grow rapidly over the next few years. Foxes, however, have to eat quite regularly, so it is conceivable that the rate at which the rabbit population is growing, however high, will eventually be insufficient to replace the number of rabbits being eaten. As a result, the rabbit population decreases. As this happens, there is increased competition among the foxes for this dwindling food supply. The fox population will also decrease. Assuming that neither species is extinguished, it is possible that the fox population becomes sufficiently small so that the rabbits once again have a chance to add more to their population than are being taken away by the small number of foxes. As a result, the rabbit population grows again, and we have returned to the beginning of the cycle in which a small number of foxes is present amidst a large number of rabbits. The cycle then repeats itself.

In what follows, we shall be looking at linear systems of difference equations. It turns out that a linear system cannot produce a robust population cycle as sketched above. Nevertheless, for purposes of illustration, let us assume that we can at least set up a linear model for this fox-rabbit interaction.

If we let x_n and y_n denote the populations of, respectively, rabbits/prey and foxes/predators, in a given part of a year (e.g. summer), we assume that the populations in the next year will be given as follows:

$$\underbrace{x_{n+1}}_{\text{prey}} = \underbrace{a_1 x_n}_{\substack{\text{net growth of}\\ \text{prey due to}\\ \text{birth/death,}\\ a_{11} > 1}} + \underbrace{a_{12} y_n}_{\substack{\text{the prey eaten}\\ \text{by the}\\ \text{predators}\\ a_{12} < 0}}$$

$$\underbrace{y_{n+1}}_{\text{predator}} = \underbrace{a_{21} x_n}_{\substack{\text{the positive effect}\\ \text{of prey population}\\ a_{21} > 0}} + \underbrace{a_{22} y_n}_{\substack{\text{net growth of}\\ \text{predators due to}\\ \text{birth/death,}\\ a_{22} > 0}}$$

This is an example of a two-dimensional linear homogeneous difference equation with constant coefficients:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad n \geq 0. \tag{87}$$

It is also convenient to write the above system in vector-matrix notation as follows,

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n \tag{88}$$

where $\mathbf{x}_n = (x_n, y_n)^T$. Either of these two equations, (87) or (88), may be viewed as defining a **first order** system in $\mathbb{R}^2$: We now consider a sequence of ordered pairs or vectors $\{\mathbf{x}_n\}$, where the vector $\mathbf{x}_{n+1}$ is determined from $\mathbf{x}_n$ and no previous vector in the sequence, e.g., $\mathbf{x}_{n-1}$.

In Chapter 1, we studied the scalar form of (88) in the single variable x_n, $x_{n+1} = ax_n$ and arrived at the general solution $x_n = a^n x_0$. In an analogous way, we can arrive the solution to (88). Setting $n = 0$ and 1 in (88):

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 \\ \mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{A}\mathbf{x}_0 = \mathbf{A}^2\mathbf{x}_0\,. \\ \mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 = \mathbf{A}\mathbf{A}^2\mathbf{x}_0 = \mathbf{A}^3\mathbf{x}_0\,. \\ &\vdots \end{aligned} \tag{89}$$

The pattern is clear: The general solution is

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0\,, \quad n \geq 0\,. \tag{90}$$

Here $\mathbf{x}_0 = (x_0, y_0)^T \in \mathbb{R}^2$ represents the initial condition necessary to uniquely specify a sequence.

In the scalar case, it was quite easy to determine the asymptotic behaviour of the iterates x_n, i.e. their behaviour as $n \to \infty$. The first feature to look for was whether $|a|$ was greater than one, equal to one or less than one. Then it remained to determine the sign of a. The situation, in general, is not as straightforward with the products $\mathbf{A}^n$ in (90). In addition to being complicated to compute, these matrices do not generally provide a clear picture of the behaviour of the iterates $\mathbf{x}_n$.

In the special case that $\mathbf{A}$ is diagonal, i.e.

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{91}$$

the analysis is quite simple. In this case,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{92}$$

so that

$$x_{n+1} = ax_n, \qquad y_{n+1} = by_n. \tag{93}$$

In other words, the difference equations for x_n and y_n are *decoupled*, or independent of each other. Their solutions are simply

$$x_n = a^n x_0, \qquad y_n = b^n y_0. \tag{94}$$

This result also follows from the fact that

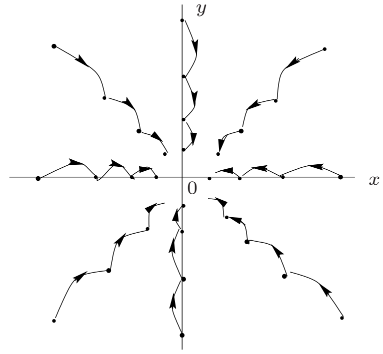
$$\mathbf{A}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}.$$

It is instructive to analyze the possible behaviours of solutions (x_n, y_n) in (94). The behaviour of (x_n, y_n) is clearly dependent upon the multipliers a and b. We examine only a couple of cases below and leave it as an exercise for the reader to study the others. It will also be helpful to look at "phase portraits", i.e. the general motion of solutions (x_n, y_n) in the plane or a given case.

Case 1: $|a| < 1$, $|b| < 1$

From (94), it is clear that $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$. In other words, $(x_n, y_n) \to (0, 0)$ as $n \to \infty$. Note that if $y_0 = 0$, then $y_n = 0$: If our starting point (x_0, y_0) lies on the x-axis, then all points $(x_n, 0)$ lie on the axis. Similarly, $x_0 = 0$ implies that $x_n = 0$. We say that the x-axis and the y-axis represent **invariant sets** under the action of the diagonal matrix $\mathbf{A}$. (Later, we shall see how these invariant sets generalize to other lines in the plane.) And let us not forget the special nature of the point $(0, 0)$, the point of intersection of these two invariant sets. The point $(0, 0)$ is a *fixed point* of the iteration procedure in (92). It is the only constant solution of this difference equation.

In the particular case that both a and b are positive, i.e., $0 < a, b < 1$, then points (x_n, y_n) move toward the origin without being "flipped" about the origin or either of the coordinate axes. A generic sketch of the behaviour of the sequences (x_n, y_n) – a "phase portrait" – is sketched below:

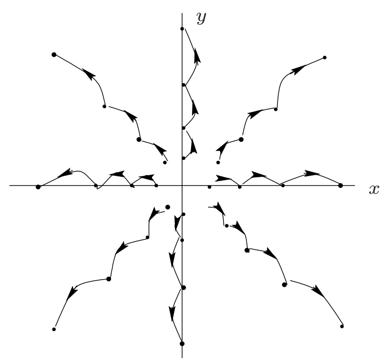


Clearly, all points $(x_n, y_n) \to (0,0)$ as $n \to \infty$. The fixed point $(0,0)$ is said to be *attractive.*

If a is positive and b is negative, i.e $0 < a < 1$, $-1 < b \leq 0$, then $x_n \to 0$ monotonically whereas the $y_n \to 0$ in a damped oscillatory manner. The points (x_n, y_n) are flipped from above the x-axis to below it and vice versa as they approach $(0,0)$. If a is negative and b is positive, the $y_n \to 0$ and the $x_n \to 0$ in a damped oscillatory manner. If both a and b are negative, then both the x_n and y_n approach zero in a damped oscillatory manner. Points in the first quadrant $(x_n, y_n > 0)$ are mapped to the third quadrant $(x_{n+1}, y_{n+1} < 0)$ and vice versa. Similarly, points in the second and fourth quadrants are exchanged under iteration of $\mathbf{A}$. The phase portraits for these cases of flipping are quite complicated and are not sketched here.

Case 2: $|a| > 1$**,** $|b| > 1$

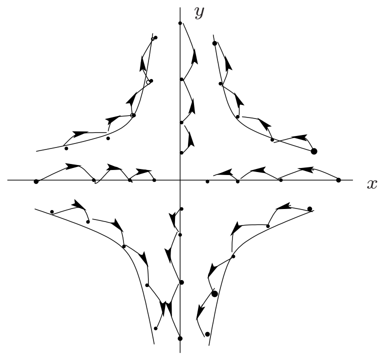
The picture here is equivalent to "running the camera backwards" in Case 1. Points are repelled away from the fixed point $(0,0)$ and $|x_n|$, $|y_n| \to +\infty$ as $n \to \infty$.



When a and/or b are negative, then consecutive points are "flipped" about appropriate coordinate axes (x-axis if $a < 0$, $b > 0$; y-axis if $a > 0$, $b < 0$) or the origin ($a, b < 0$). The fixed point $(0,0)$ is *repulsive.*

Case 3 $|a| < 1$, $|b| > 1$

The dynamics are more interesting in this case. Clearly, $|x_n| \to 0$ and $|y_n| \to +\infty$ for any starting point $(x_0, y_0) \neq (0, 0)$. The x-axis and y-axes are again invariant sets. Since $|x_n| \to 0$, the y-axis plays the role of an asymptote for the long-term trajectories of points (x_n, y_n). A phase portrait is shown below for the case in which both a and b are positive.



Note that only those sequences (x_n, y_n) with initial points (x_0, y_0) on the x-axis remain bounded as $n \to \infty$. In fact, for these sequences, $(x_n, y_n) = (x_n, 0) \to (0, 0)$ as $n \to \infty$. As a result, we may state that "almost all" sequences (in the sense of the "area" of initial conditions in the plane) are unbounded.