University of Waterloo

Department of Applied Mathematics

AMATH 343

DISCRETE MODELS IN APPLIED MATHEMATICS

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Lecture Notes

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Lecture 1

Introduction: Some simple examples of discrete models

Discrete models are used to analyze or predict a property (or properties) of a system over discrete time units t_k, $k = 1, 2, \ldots$, as opposed to analyzing it over a continuous time variable $t \in \mathbb{R}$. A rather straightforward example is the modelling of the population of a particular species of perennial plant in a given ecosystem. It is natural to consider the number of plants in the system each year at a given time of the year, say in June. Starting at some year which will be designated by the number 0, we let $p(0)$ denote the number of plants in the system in June. (Of course, we have to count them!) We then count the number of plants in June for each year afterwards. This can be summarized mathematically by letting $p(n)$ be the number of plants in this system n years past this year, where $n \geq 0$. In this case, p is a function which maps the space of nonnegative integers $\mathbb{Z} = \{0, 1, 2, \cdots\}$ (or at least a finite subset) into the space $\mathbb{Z}$ itself, since the population must always be nonnegative.

Note: It doesn't necessarily follow that if $p(n) = 0$ for some $n > 0$, then $p(n+1) = 0$. It is possible that there are seeds in the ground that remained dormant during the year n and sprout the next year. We'll discuss this in more detail later.

Another note: Instead of writing $p(n)$ to denote the plant population at year $n \geq 0$, one often writes, for simplicity, "p_n." In other words, the populations p_n may be viewed as elements of a sequence $\mathbf{p} = \{p_0, p_1, \cdots\}$ which may or may not be infinite. (In practical situations, of course, the sequence is finite.)

Discrete modelling is also important in the financial world, for example, in the calculation of interest accrued on investments or loans on a yearly, monthly or even daily basis. We shall examine a couple of very simple interest schemes in a later section of this chapter. For the moment, however, we introduce the concept of discrete sampling/modelling in a seemingly convoluted way in order to give an idea of the of the complexities associated with the modelling of physical or biological systems.

Radioactive Decay

Suppose we have a rock sample that contains a radioactive element "X". In a book, we read that the radioactive "half-life" of X, denoted as $T_{1/2}$ is a certain time, say 10 days. This implies the following: **If our sample contains a units of X at some time t, then only one-half the original amount, $\frac{1}{2}a$ units, are present at time $t + T_{1/2}$.** Suppose that we reset our stop-watch to $t = 0$, at which time the amount of X in our sample is x_0. Then let x_k be the amount of X in our sample at the discrete times $t_k = kT_{1/2}$, $k = 0, 1, 2, \ldots$.

Note: Once again, we could have used $x(0)$ to denote the amount of X in our sample at time $t_0 = 0$ and $x(k)$ the amount at time t_k, $k \geq 0$. It is very convenient to use the sequence notation x_k.

The half-life property stated above gives us the simple relationship,

$$x_k = \tfrac{1}{2}x_{k-1}, \qquad k = 1, 2, 3, \ldots. \tag{1}$$

Eq. (1) is an example of a **difference equation** in the variables x_k, $k = 0, 1, 2, \ldots$.

> In these notes, we shall use the abbreviation, "**d.e.**," to denote **difference equation**. We use the periods since the term "**de**" could be confusing.

Given an initial amount x_0 of radioactive element X in the sample, we can easily find the amounts x_k from Eq. (1) by **iteration:**

$$\begin{aligned}
x_1 &= \tfrac{1}{2}x_0 \\
x_2 &= \tfrac{1}{2}x_1 = \tfrac{1}{2}\cdot\tfrac{1}{2}x_0 = \left(\tfrac{1}{2}\right)^2 x_0 \\
&\vdots \\
x_k &= \left(\tfrac{1}{2}\right)^k x_0.
\end{aligned} \tag{2}$$

In this case, it was easy to see the pattern to come up with the expression for x_k. Indeed, the expression

$$x_k = \left(\frac{1}{2}\right)^k x_0 \tag{3}$$

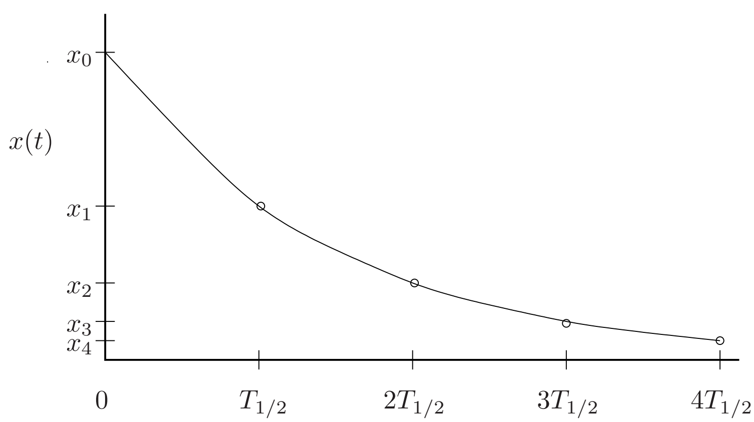
is the **solution** to the difference equation in (1) with initial condition x_0. We'll discuss the idea of solutions of d.e.'s later.

There is actually more information in the expression for the solution x_k in Eq. (3). It is often of great interest to determine the **long-term** or **asymptotic** behaviour of sequence, i.e., the behaviour of $\{x_k\}$ in the limit $k \to \infty$. In Eq. (3), the asymptotic behaviour is quite clear: $x_k \to 0$ as $k \to +\infty$. And this is, of course, what we would have expected in the case of a decay process. We'll be discussing asymptotic behaviour in much more detail later in this course.

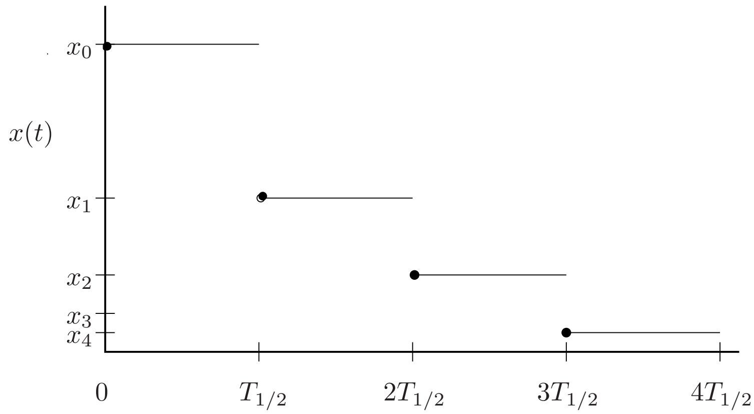
Now this description of radioactivity may be enough for some applications but it gives us no idea of what is actually going on in the rock. For example, one might be interested in how much of element X is present after a non-integral number of half lifes, e.g. $2.5T_{1/2}$. If we now assume that the amount of X in the rock can be represented as a function $x(t)$ of a **continuous** time variable $t \in \mathbb{R}$, then the quantities x_k, $k = 0, 1, 2, \ldots$ can be interpreted as the result of sampling or measuring the amount of X at the discrete times t_k:

$$x_k = x(t_k), \qquad k = 0, 1, 2, \ldots \,. \tag{4}$$

In this and other applications, the sampling can be viewed as a "stroboscopic" examination of a certain physical property $x(t)$ of a physical or biological system that evolves over time. A graphical interpretation of our rock sample problem could be as follows:



The curve $x(t)$ necessarily interpolates the sampled values x_k, $k = 0, 1, 2, \ldots$. Of course, one might ask, "How do you know that the curve behaves in this way? What if the X nuclei wait until just before you take a measurement at the times t_k, and then decay, producing the decrease in the amount of X?" If this were the case, then the actual behaviour of $x(t)$ over time would be as follows:



Note that this behaviour of $x(t)$ is still consistent with our "half-life" working definition of radioactive decay, i.e. we still find that

$$x(t + T_{1/2}) = \tfrac{1}{2}x(t), \qquad t \geq 0. \tag{5}$$

However, it is nonsensical from a physical viewpoint since there is nothing special about the times t_k that *we* have chosen to perform the sampling. (This is reminiscent of such questions as, "Is the moon there when you're not looking at it?") It is only when we use the true "radioactive decay law", i.e.,

[Rate of decay]   proportional to   [amount of radioactive substance present]

that we obtain the proper result. The above word problem translates to the mathematical statement

$$\frac{dx}{dt} = -kx, \qquad k > 0. \tag{6}$$

where $k > 0$ is the proportionally constant or **decay constant** specific to X. You will recognize this as a **differential equation** (DE) in the function $x(t)$. The solution to this DE satisfying the initial condition $x(0) = x_0$ is

$$x(t) = x_0 e^{-kt}. \tag{7}$$

You may verify that $x(t)$ in (7) satisfies (5) if $k = \ln 2/T_{1/2}$.

Let us repeat that Eq. (7) gives a continuous time description of the amount of radioactive X in our sample. It is a better model of the process since radioactive decay is going on in a continuous fashion in the sample. The function $x(t)$ is the continuous counterpart of the discrete sequence x_k in (2) that is the solution of the difference equation (d.e.) (1).

Population growth

In a first-year calculus course you probably saw mathematical models that describe the evolution of a population over a continuous time parameter $t \geq 0$. For example, the model of "Malthusian growth" is as follows:

[Rate of population growth]   is proportional to   [population at time t]

This turns out to be a reasonable model for some populations, e.g. single celled organisms and humans alike, when unlimited resources – food and space – are available. For example, the model describes well the population "explosion" of bacteria introduced into a Petri dish full of agar nutrient, at least until "crowding" begins to takes place.

So let $x(t)$ denote the population of a species of bacteria "X" (how unimaginative!) at time $t \geq 0$ and suppose that $x(0) = x_0$. However, we encounter another potential problem – $x(t)$ should be integer valued, shouldn't it? Organisms come in discrete units. In order to get around this "glitch", we can assume that $x(t)$ is large and that single increments in x are so small that they may be considered as infinitesimal changes. (We may also let $x(t)$ represent the **concentration** of organisms per unit volume of Petri dish solution.) The translation of the Malthusian model of growth into mathematics yields the following DE

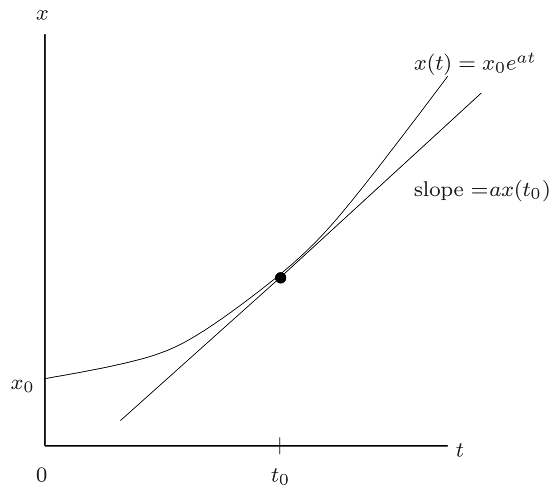
$$\frac{dx}{dt} = ax, \qquad a > 0. \tag{8}$$

The proportionality constant "a" is assumed to incorporate both birth and death processes. The solution to this DE satisfying the initial condition $x(0) = x_0$ is

$$x(t) = x_0 e^{at}. \tag{9}$$

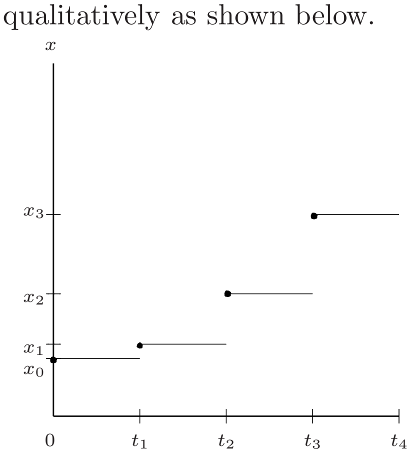
In other words, the Malthusian model predicts exponential growth in time. (If $x_0 = 0$, then $x(t) = 0$ for all $t \geq 0$, the rather uninteresting case of zero population.)

The DE in Eq. (8) represents a **continuous** dynamical model of population evolution. Once again, "continuous" refers here to the time variable - the fact that t assumes continuous real values, as opposed to discrete time values t_k, $k = 0, 1, 2, \ldots$. Without even knowing the solution of this DE (Eq. (9)) we can conclude that populations will increase: If $x_0 > 0$ then $x'(t)$ must be positive for $t > 0$, implying that $x(t)$ is an increasing function. As $x(t)$ increases, $x'(t)$ must also increase. (In fact, if we differentiate both sides of Eq. (8) with respect to t, we find that $x''(t) > 0$, implying that the graph of $x(t)$ is concave upward.) All of this proceeds in a continuous fashion in time. This reflects the basic underlying assumption of this population model: that birth/death processes are taking place continually in time.



Now suppose that, for some reason, this continuous model is unrealistic. For example, suppose that each bacterium possesses a "biological clock" and that the biological clocks of all cells are somehow synchronized, perhaps due to their synchronization with an external physical, periodic phenomenon, e.g. daylight/darkness over a 24 hour period. And, to add to our hypothetical model, suppose that all reproduction by cell division as well as death occurs only at discrete time steps that are separated by a constant time interval T specific to the bacteria under investigation. Assuming an initial population $x(0) = x_0$ at $t = 0$, we assume that birth and death occurring at times $t_k = kT$, $k = 1, 2, 3, \ldots$.

Assuming an initial population $x(0) = x_0$ at $t = 0$, and, for the moment, that the population grows, the curve $x(t)$ will behave qualitatively as shown below.



The mathematical definition of $x(t)$ will be

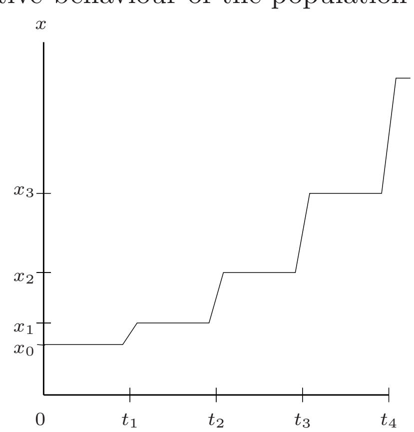
$$x(t) = x_k, \quad t_k \leq t < t_{k+1},\ k \geq 0. \tag{10}$$

In other words, $x(t)$ is a piecewise constant function with jump discontinuities at the times $t_k = kT$, $k = 1, 2, 3 \ldots$. The magnitudes of the jumps are

$$\Delta x_k = x_k - x_{k-1},\ k \geq 1. \tag{11}$$

Note that it is no longer necessary to keep track of $x(t)$ for all times $t \geq 0$. It suffices to keep the sequence of population values $\{x_k\}$ at the discrete times $t_k = kT$, $k = 0, 1, 2, \ldots$.

You may well argue that this is an over-idealized picture of our bacterium population, i.e., that it is unreasonable to assume that birth-death processes occur at instantaneous times t_k. It might be more reasonable to assume that an accelerated metabolism leading to division, hence population growth, takes place not instanteously but over time intervals $[t_k - h, t_k + h)$, where h is much smaller than T. A possible picture for the qualitative behaviour of the population $x(t)$ is given below.



However, given that the population $x(t)$ assumes the constant values x_k for *most* of the time, one may be content to develop mathematical models that relate these values to each other, not worrying about what happens during those small intervals of intense metabolic activity. (These latter processes may, however, be of particular interest to other researchers.)

Now you may well be thinking that this discussion of bacteria with synchronized biological clocks is a true stretch of the imagination. Yes, perhaps. But there are other biological systems whose birth/death process are better described by discrete models - for example, the propagation of annual plants. Each year, at a certain time, typically the fall, the number of plants in generation n, say

x_n, will produce seeds. Let us assume for simplicity that each plant produces M seeds. A fraction $0 \leq \alpha \leq 1$ of these seeds will survive the winter and a fraction β of these survivors will germinate to produce the plants of generation $n+1$. A fraction γ of these plants will survive until the fall and produce a new set of seeds, etc.. In this very simple picture, we can postulate the relationship

$$x_{n+1} = cx_n, \qquad \text{where} \;\; c = \alpha\beta\gamma M\,. \tag{12}$$

(Note that we've used the index n instead of k used earlier. It doesn't matter what letter we use for the index, as long as we maintain consistency.) As we shall see below, if $c > 1$, then the population x_n is guaranteed to increase with n. Of course, this is an oversimplified picture. It is possible that some seeds that survive over the winter will not germinate in the next season but rather after two winters of dormancy. Then x_{n+1} will depend not only on x_n but also on x_{n-1}, This model will be studied in detail in a later section.

General questions regarding discrete mathematical models

Let us now step back and consider a couple of general questions regarding discrete models:

(1) If we know the discrete (population) values $x_0, \dots x_n$, for some $n > 0$, can we determine x_{n+1} uniquely? In particular, how many of the previous values do we need to do so?

(2) What is the behaviour of sequences $\{x_n\}$ with respect to n, in particular as $n \to +\infty$? For example, do sequences increase, with $x_n \to +\infty$ as $n \to \infty$? Or do they decrease, with $x_n \to 0$ as $n \to \infty$? Perhaps some sequences, if not all, approach a limit $\bar{x}$. Or perhaps they oscillate, never approaching a limit.

With reference to (1) posed above, the simplest type of mathematical model is one in which each discrete value x_n is determined only by the previous value x_{n-1}, i.e.

$$x_n = f(x_{n-1}), \qquad n = 1, 2, 3, \dots. \tag{13}$$

Depending upon the particular physical or biological process being modelled, we typically require that the function f will obey some basic properties. For example, in the case of population models, we might require that:

1. f is not only continuous in x but it is also increasing in x at least for a suitable range of x-values, i.e., the greater the population x_{n-1}, the greater the population x_n of the next generation. (Recall the definition of an increasing function: $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.)

2. $f(0) = 0$: We can't expect a situation of zero population to give rise to a nonzero population in the future.

The simplest case of a function satisfying these two requirements is when f is linear in x, i.e.

$$f(x) = cx,$$

where c is a positive constant. In this case, our model becomes

$$x_n = cx_{n-1}, \tag{14}$$

the simple plant population model discussed above.

However, as we shall see later, such a model is too simplistic: It predicts unbounded populations ($x_n \to \infty$) which is unrealistic, given that resources (e.g., food) are limited. It might be more realistic to assume that $f(x)$ is **increasing** over some interval $(0, a)$ and then **decreasing** for $x > a$. (The value of a will depend upon the species and its environment.) In this way small populations may have the possibility to grow. However, excessively large populations, in which case members must compete for scarce resources, will be diminished. More on this later.

Eq. (14) is an example of a **difference equation**, a discrete model of population evolution. The term **discrete dynamical system** will be used regularly in a later section of this course to refer to such models. We now show that Question (2), regarding the asymptotic behaviour of sequences generated by this simple discrete model, can be answered completely.

Lecture 2

Introduction: Some simple examples of discrete models (cont'd)

Analysis of the asymptotic behaviour of sequences $x_n = cx_{n-1}$

We continue our discussion with an analysis of the asymptotic behaviour of solutions to the difference equation,

$$x_n = cx_{n-1}\,, \quad n \geq 1\,, \tag{15}$$

introduced at the end of the last lecture as a very simple discrete model of population growth.

Note that a knowledge of the initial value x_0 determines all elements of the sequence $\{x_k\}$:

$$\begin{aligned} n = 1: &\quad x_1 = cx_0 \\ n = 2: &\quad x_2 = cx_1 = c^2x_0 \\ n = 3: &\quad x_3 = cx_2 = c^3x_0. \end{aligned}$$

In general, we have

$$x_n = c^n x_0\,, \quad n \geq 0\,, \tag{16}$$

as the solution to the difference equation (14) with initial condition x_0. (It's always a good idea to check such results by substitution: LHS of (1.12) $= x_n = c^n x_0$. RHS of (1.12) $= cx_{n-1} = c \cdot c^{n-1}x_0 = c^n x_0$. Therefore Eq. (21) is satisfied.)

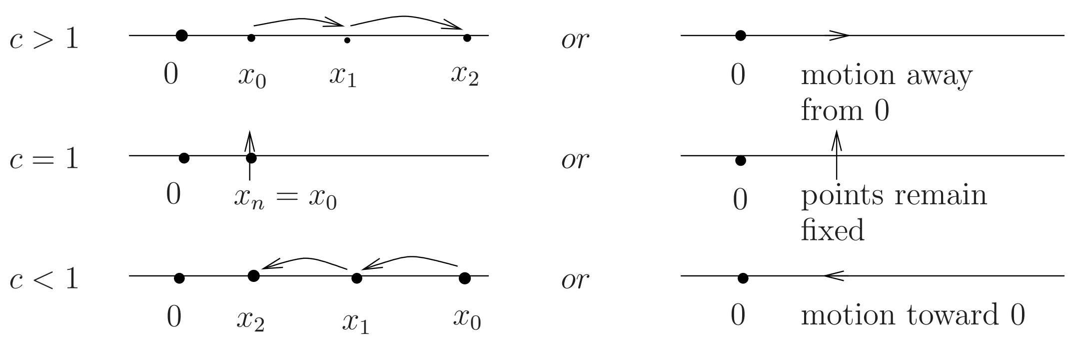
First note that if $x_0 = 0$, then $x_n = 0$ for $n = 1, 2, 3, \ldots$ regardless of the value of c. Although it may seem rather uninteresting, it is an important reference solution. For the moment, we continue to assume that the $\{x_n\}$ represent population values, so they must be non-negative. This is guaranteed if $x_0 \geq 0$ and $c > 0$. The behaviour of the x_n can then be classified as follows:

Case 1: $c > 1$, then $c < c^2 < c^3 \ldots$. Therefore $x_0 < x_1 < x_2 \ldots$, i.e. $x_n < x_{n+1}$ for $n = 0, 1, 2, \ldots,$ implying that $\{x_n\}$ is a strictly increasing sequence. Since $c^n \to \infty$ as $n \to \infty$, we have $x_n \to \infty$ as $n \to \infty$. In summary, the population grows monotonically without bound.

Case 2: $c = 1$, for which $x_n = x_0$. The population remains constant.

Case 3: $0 < c < 1$, for which $c^{n+1} < c^n$, implying that $x_{n+1} < x_n$, $n = 1, 2, , \ldots$. Thus $\{x_n\}$ is a strictly decreasing sequence and $x_n \to 0$ as $n \to \infty$. In summary, the population decreases monotonically with limit 0.

The qualitative behaviour of these three cases can be summarized pictorially as follows:



These sets of diagrams are known as **phase portraits** of the dynamical system in Eq. (14). They pictorially summarize the behaviour of sequence values $\{x_k\}$ for various situations. Note that the dynamical system in (14) involves the iteration of the following function,

$$f(x) = cx \,, \tag{17}$$

i.e.,

$$x_n = f(x_{n-1}), \qquad n = 1, 2, 3, \ldots \,. \tag{18}$$

Alternatively, we can write $x_1 = f(x_0)$, $x_2 = f(f(x_0)) = f^{\circ 2}(x_0)$, etc., so that, in general,

$$x_n = f^{\circ n}(x_0), \qquad n = 0, 1, 2, \ldots, \tag{19}$$

where $f^{\circ n}$ denotes the n-fold composition of f with itself.

As we shall see later, it will be convenient to generalize this dynamical system to include all real values of c as well as real values of the x_n. Once again, the solution of the dynamical system

$$x_n = cx_{n-1}, \tag{20}$$

where $c \in \mathbb{R}$, is given by

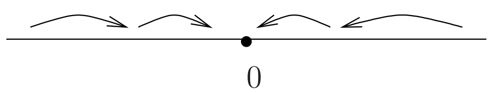
$$x_n = c^n x_0. \tag{21}$$

If $x_0 = 0$, then $x_n = 0$ for all n. The point $x = 0$ is a **fixed point** of the function $f(x) = cx$, i.e. $f(0) = 0$. Regardless of the value of c, if we start at the fixed point $x_0 = 0$, we remain there.

The behaviour of nontrivial sequences, i.e., $x_0 \neq 0$, is determined by the value of c. In the special case $c = 0$, then for any $x_0 \in \mathbb{R}$, $x_k = 0$, $k = 1, 2, \ldots$, i.e., all future iterates are mapped to 0. This seems to be a quite extreme and, admittedly, uninteresting case. We now consider nonzero values of $c \in \mathbb{R}$. In what follows, we assume that $x_0 \neq 0$, implying that $x_k \neq 0$, $k = 1, 2, \ldots$.

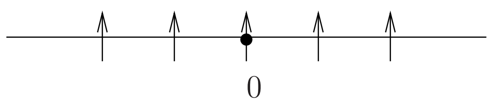
Case A: $\boldsymbol{c > 0}$ Since $x_n = cx_{n-1}$, it follows that x_n and x_{n-1} have the same sign. If $x_0 > 0$, then the behaviour of the iterates is summarized in the three phase portraits shown earlier. If $x_0 < 0$, the phase portraits will be mirror images. The resulting portraits are as follows:

$0 < c < 1$



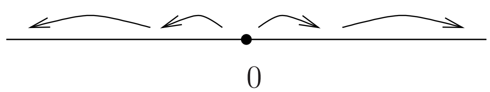
$x = 0$ is an *attractive* fixed point

$c = 1$



each point x is a fixed point

$c > 1$



$x = 0$ is a *repulsive* fixed point

Clearly, the fixed point $x = 0$ of the function $f(x) = cx$ is an important reference solution. For $0 < c < 1$, all (nonzero) iterates x_n approach $x = 0$ as $n \to \infty$. As such, it is referred to as an **attractive fixed point**. For $c > 1$, all iterates x_n – regardless of how close to $x = 0$ they may be for some $n > 0$ – move away from $x = 0$ as $n \to \infty$. As such, it is referred to as a **repulsive fixed point**.

Note: In the case $c = 1$, the fixed point $x = 0$ is neither attractive or repulsive. In many books, such a fixed point is called **neutral** or **indifferent**.

Case B: $\boldsymbol{c < 0}$ Since $x_n = cx_{n-1}$, it follows that x_n and x_{n-1} alternate in sign: If $x_{n-1} > 0$, then $x_n < 0$ and vice versa. Thus, the successive values $x_k, x_{k+1}, \ldots$ hop back and forth from one side of the origin to the other. Whether or not they approach the origin or travel away from it is dependent upon the magnitude of c. If we take absolute values of both sides of (21), then

$$|x_n| = |c|^n |x_0|.$$

Then letting $y_n = |x_n|$ $n = 1, 2, \ldots$ and $d = |c| > 0$, we have

$$y_n = dy_{n-1}, \qquad y_n > 0, \quad d > 0,$$

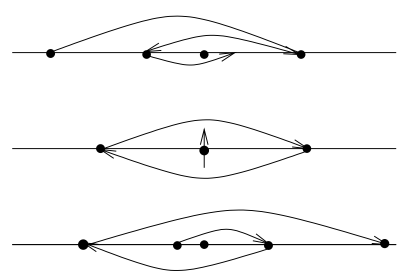
the dynamical system that modelled our bacterial population evolution earlier. The behaviour of the $y_n = |x_n|$ is easily found:

Case 1: $d = |c| > 1$, implying that $c < -1$. The sequence $\{y_k\}$ is monotonically increasing and $y_k = |x_k| \to \infty$ as $k \to \infty$.

Case 2: $d = |c| = 1$, implying that $c = -1$. Then $y_k = |x_k| = |x_0|$. Here $x_k = (-1)^k x_0$ so that the x_k values oscillate between x_0 and $-x_0$.

Case 3: $d = |c| < 1$, implying that $-1 < c < 0$. The sequence $\{y_k\}$ is monotonically decreasing and $y_k = |x_k| \to 0$ as $k \to \infty$.

We can combine the above information with the fact that the sequence $\{x_k\}$ oscillates in sign to obtain the following phase portraits for $c < 0$:



$-1 < c < 0$ — $x = 0$ is an *attractive* fixed point

$c = -1$ — $x = 0$ is a *neutral* fixed point

$c < -1$ — $x = 0$ is a *repulsive* fixed point

In summary, as pointed out earlier, the fixed point $x = 0$, i.e. $f(0) = 0$, is an important reference point for this dynamical system. For $|c| < 1$, $x = 0$ is an **attractive fixed point** - all sequences $\{x_n\}$ approach it in the limit $n \to \infty$, regardless of the starting point x_0. For $|c| > 1$, $x = 0$ is a **repulsive fixed point** - all sequences move away from it, regardless of how close x_0 is to it (assuming $x_0 \neq 0$). In the "boundary situation", $|c| = 1$, i.e. $c = \pm 1$, $x = 0$ is said to be a **neutral** or **indifferent fixed point**. Points move neither away from it nor closer to it.

We shall return to this "dynamical systems" picture of iteration in a later chapter.

Difference equations and some applications

As seen above, discrete models, also called **discrete dynamical systems**, will involve sequences $\{x_k\}_{k=0}^{\infty}$ of real numbers whose elements are related to each other. Most generally, this relation can be expressed as follows: For some $n \geq 1$,

$$x_k = f(k, x_{k-1}, x_{k-2}, \ldots, x_{k-n}), \quad k \geq n. \tag{22}$$

In other words, x_k may be dependent upon the index k as well as the n sequence values that precede it. Eq. (22) represents the general form of a **difference equation of order** n.

Note: At this point you may realize that you have seen equations of form in (22) elsewhere but that they were called **recursion relations**.

Note that in order to "start" a unique sequence $\{x_k\}_{k=0}^{\infty}$ defined by Eq. (22), we need to provide values of $x_0, x_1, \ldots, x_{n-1}$, i.e. n initial conditions. This is quite reminiscent of the situation for differential equations. Indeed, as will be shown below, the classification of difference equations follows closely that of differential equations – the order of a difference equation, linear vs. nonlinear, homogeneous vs. inhomogeneous, constant coefficients, etc..

One may well ask, "Where's the **difference** in the 'difference equation' (22)?" An answer is that we may simply have chosen to "hide" the difference in the function f. We'll see later how differences arise from discrete sampling of functions. We can rewrite Eq. (22) in the form

$$x_k = x_{k-1} + g(k, x_{k-1}, x_{k-2}, \ldots, x_{k-n}) \tag{23}$$

where

$$g(k, x_{k-1}, x_{k-2}, \ldots, x_{k-n}) = f(k, x_{k-1}, x_{k-2}, \ldots, x_{k-n}) - x_{k-1}\,, \tag{24}$$

so that Eq. (22) becomes

$$x_k - x_{k-1} = g(k, x_{k-1}, x_{k-2}, \ldots, x_{k-n}). \tag{25}$$

The LHS is the first difference of the sequence $\{x_k\}$.

Example: The simple dynamical system studied earlier,

$$x_k = cx_{k-1}\,, \tag{26}$$

is, from Eq. (22), a first order order difference equation, i.e., $n = 1$. It may be rewritten as

$$x_k - x_{k-1} = (c-1)x_{k-1}. \tag{27}$$

The LHS is the first difference of the sequence $\{x_k\}$.

Example: The following difference equation,

$$x_k - x_{k-1} - 2x_{k-2} + 3x_{k-3} = 0, \qquad k \geq 3, \tag{28}$$

may be rewritten as follows,

$$x_k = x_{k-1} + 2x_{k-2} - 3x_{k-3}, \qquad k \geq 3\,. \tag{29}$$

From Eq. (22) this is a linear, third order difference equation with constant coefficients, i.e., $n = 3$. In order to uniquely define x_k, we need the three previous elements of the sequence.

Example: The difference equation,

$$x_n = nx_{n-1}, \qquad n = 1, 2, 3\ldots, \tag{30}$$

is linear and first order but with nonconstant coefficients. Its solution is easily found: For an $x_0 \in \mathbb{R}$, $x_1 = x_0$, $x_2 = 2x_1 = 2 \cdot x_0$, $x_3 = 3x_2 = 3 \cdot 2 \cdot x_0$, so that

$$x_n = (n!)\, x_0\,, \quad n \geq 0\,. \tag{31}$$

OK, technically, we should provide a formal proof of this result even though it seems quite obvious. Indeed, the proof is as simple as the "finding-a-pattern" method used to obtain the formula. As you might expect, we'll use induction. The above formula (31) is true for the case $n = 0$. We assume it to be true for all $0 \leq n \leq N$ for some $N > 1$ and show it to be true for $n = N + 1$. Since the $\{x_n\}$ are assumed to be solutions of (30), we have

$$\begin{aligned} x_{N+1} &= (N+1)\, x_N \\ &= (N+1)(N!)\, x_0 \quad (\text{true for } n = N) \\ &= (N+1)!\, x_0\,. \end{aligned} \tag{32}$$

Since the result is true for $0 \leq n \leq N$ and $n = N + 1$, it is true for all natural numbers $n \geq 0$.

Example: The difference equation

$$x_n = x_{n-1} + x_{n-2}, \qquad n \geq 2, \tag{33}$$

is linear and second order. If we set $x_0 = 0$ and $x_1 = 1$, then $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$, $x_6 = 8\ldots$, and we have the famous "Fibonacci sequence".

Is there a closed-form expression for the Fibonacci numbers? Yes, there is:

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n\,, \quad n \geq 0\,. \tag{34}$$

This is known as "Binet's formula" although it was known to the mathematicians Abraham de Moivre and Daniel Bernoulli. (Source: Wikipedia "Fibonacci number".) It might seem rather bizarre to see the $\sqrt{5}$ terms in the above expression. How can positive integers be produced? It turns out that there is perfect cancellation of all terms containing $\sqrt{5}$. This can be shown by considering the binomial expansions for $(a+b)^n$ and $(a-b)^n$. We'll derive Eq. (34) a little later in the course when the formal solution of linear difference equations will be discussed.

Note that for the Fibonacci difference equation, we must supply two "initial conditions", x_0 and x_1, in order to produce a unique, well-defined sequence $\{x_k\}$. Once again, this is similar to the situation for differential equations as we shall show below.

Example: Consider the second order, linear homogeneous DE

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0. \tag{35}$$

The general solution of this DE is

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}, \tag{36}$$

where c_1 and c_2 are arbitrary constants. Recall that the exponential components e^{-t} and e^{-2t} are obtained by assuming solutions of the form $x(t) = e^{rt}$, where r is a constant to be determined. Substitution of this solution into Eq. (35) yields the quadratic "characteristic equation" in r:

$$r^2 + 3r + 2 = 0, \tag{37}$$

with roots $r = -2, -1$.

In order to extract a particular solution from (36) we need to impose two initial conditions, e.g. $x(0) = x_0$ and $x'(0) = v_0$. From these conditions, we may solve for c_1 and c_2 in terms of x_0 and v_0.

Difference equations/recursion relations associated with integrals

You may recall from first year Calculus that complicated integrals – definite and indefinite alike – can sometimes be computed with the help of recursion relations. The general approach is to consider the integral as a member of a general family of integrals, I_n, $n = 0, 1, 2\ldots$. Usually by means of a suitable integration-by-parts procedure, I_n can be related to I_{n-1} and possibly I_{n-2}, maybe even I_{n-3}. If one is able to compute I_0 (and I_1, etc. if necessary) directly, I_2, then I_3, etc. can be computed by recursion.

A classic example involves the integrals

$$I_n = \int_0^\infty t^n e^{-t} dt. \tag{38}$$

Integration by parts ($u = t^n, dv = e^{-t}dt \Rightarrow du = nt^{n-1}, v = -e^{-t}$) yields

$$I_n = \lim_{b\to\infty} -t^n e^{-t}\Big|_0^b + n\int_0^\infty t^{n-1}e^{-t}dt$$

or

$$I_n = nI_{n-1}, \tag{39}$$

which is precisely the difference equation (30), with solution $I_n = n!I_0$. Since $I_0 = 1$, we have $I_n = n!$, $n = 0, 1, 2, \ldots$, i.e.,

$$I_n = \int_0^\infty t^n e^{-t} dt = n!\,, \quad n = 0, 1, 2, \cdots, \tag{40}$$

a well-known result from Calculus.

A side note - for information only: A generalization of the integrals in (38) is given by the so-called Gamma function $\Gamma(p)$, defined as:

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt, \qquad p > 0. \tag{41}$$

(The above integral is undefined for $p \le 0$.) Note that, from (40), we have

$$\Gamma(n) = (n-1)! \qquad n = 1, 2, 3, \ldots$$

In other words, we know that values of the Gamma function for positive values. And if we replace n with $n + 1$, we have

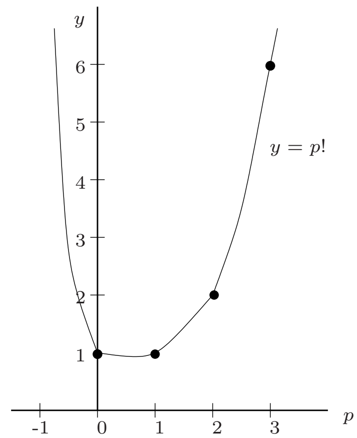
$$\Gamma(n+1) = I_n = n! \qquad n = 0, 1, 2, \ldots, \tag{42}$$

which is consistent with our earlier discussion of the integrals I_n. Now, however, we wish to consider **noninteger** values of the argument of the Gamma function in Eq. (41), i.e., replacing n integer with p real in Eq. (42), i.e.,

$$\Gamma(p+1) = p!\,, \tag{43}$$

whatever "$p!$" means for possibly noninteger values of p.

Indeed, the Gamma function in (41) may be seen as a way to produce a continuous curve that interpolates the discrete values of the usual integer-valued factorial function $n!$ for $n = 0, 1, 2, \ldots$. From a plot of $n!$ vs. n, along with the knowledge that the integral $\Gamma(0) = (-1)!$ diverges, one would expect the interpolation curve to behave qualitatively as shown on the right.



Now if we replace p by $p+1$ in (41),

$$\Gamma(p+1) = \int_0^\infty t^p e^t dt = p!, \tag{44}$$

which is, as expected, in agreement with Eq. (43). If we now integrate the RHS of (44) by parts, as we did for $p = n$ in (38), we obtain the relation,

$$\Gamma(p+1) = p\Gamma(p). \tag{45}$$

Therefore, if we are able to compute the value of $\Gamma(p)$ for some $p > -1$ by whatever means, e.g. numerical, algebraic, we can then compute the values $\Gamma(p+1)$, $\Gamma(p+2), \ldots$ by repeated application of (45). For example, consider $p = -\frac{1}{2}$, i.e.

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt.$$

If we let $t = r^2$ so that $dt = 2r\,dr$, then

$$\begin{aligned} \Gamma\left(\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right)! &= 2\int_0^\infty e^{-r^2} dr \\ &= \sqrt{\pi} \\ &\cong 1.7724\ldots\ . \end{aligned}$$

This allows us to compute $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \cong 0.8867\ldots$,

$$\left(\tfrac{1}{2}\right)! = \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi} \cong 0.8867\ldots,$$

which is consistent with the interpolation graph sketched above. Then,

$$\left(\tfrac{3}{2}\right)! = \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{4}\sqrt{\pi} \cong 1.3293\ldots$$

and

$$\left(\tfrac{5}{2}\right)! = \tfrac{5}{2}\left(\tfrac{3}{2}\right)! = \tfrac{15}{8}\sqrt{\pi} \cong 3.32335\ldots\,.$$

Note that we can also use Eq. (45) to compute $\Gamma(p)$ values in the other direction, i.e. for decreasing values of p, by rearrangement:

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1). \tag{46}$$

If we replace p by $p+1$ and use the factorial notation, then

$$p! = \frac{1}{p+1}(p+1)!\,.$$

This indicates that $p! \to \infty$ as $p \to -1^+$.

Difference equations/recursion relations associated with differential equations

If you have taken an advanced course in differential equations (DEs) (e.g., AMATH 351), you will have most probably explored the idea of finding solutions to more complicated DEs in the form of series expansions. (If you have not taken such a course, or seen such an idea, don't worry about it. The purpose of this section is only to provide an additional example of the use of recursion relations.)

Very briefly, as discussed in AMATH 351, linear second order DEs of the form,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0\,, \tag{47}$$

where the $a_i(x)$ are polynomials in x, are encountered in many applications in science and engineering (e.g., Laplace's equation $\vec{\nabla}^2 u = 0$ in various coordinate systems, solved using separation of variables, etc.). Such DEs are rarely solvable in "closed form," i.e., in terms of standard functions, e.g., $\sin x$, $\cos x$, e^x. The idea of series solutions is to assume a that the solution(s) of the DE in Eq. (47) can be expressed as a power series of the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n\,, \tag{48}$$

where x_0 is the point of expansion. (In many applications, $x_0 = 0$.) By doing this, we are assuming that the solution $y(x)$ to our DE has a Taylor series expansion. Sometimes this works; other times, it won't and we have to alter the form of the series expansion.

Substitution of the expansion (48) into Eq. (47) will yield a relation involving the expansion coefficients a_n which can often be written in the form,

$$a_n = f(n, a_{n-1}, a_{n-2}, \cdots, a_{n-K})\,, \tag{49}$$

where the constant K will be determined by the degrees of the polynomials $a_i(x)$ in (47). This is, of course, a **recursion relation** in the a_n which, from our earlier discussion, is a Kth order difference equation.

If our method works, then we can use this recursion relation to determine the coefficients a_n of the series expansion(s) in (48) up to a multiplicative constant. These coefficients then define a unique function $f(x)$ (up to a multiplicative constant).

Example: Consider the DE,

$$y'' + y = 0\,. \tag{50}$$

Of course, we know that two linearly independent solutions to this DE are $y_1(x) = \cos x$ and $y_2(x) = \sin x$ but let's pretend that we don't know this.) Substitution of the series in (48) with $x_0 = 0$ into the DE, collecting like terms in x^n, $n = 0, 1, 2, \cdots$ and imposing the condition that the resulting equation is satisfied for all $x \in R$ yields the following relation for the a_n,

$$n(n-1)a_n + a_{n-2} = 0\,, \quad n \geq 2\,, \tag{51}$$

which can be rearranged to yield the following recursion relation,

$$a_n = -\frac{1}{n(n-1)}\, a_{n-2}\,, \quad n = 2, 3, 4, \cdots\,. \tag{52}$$

This recursion relation shows that the coefficient a_0 determines a_2 which, in turn, determines a_4, etc.. Likewise, the coefficient a_1 determines a_3 which, in turn, determines a_5, etc.. This implies that a_0 and a_1 are arbitrary but that once they are fixed, they determine all other coefficients $a_{n,even}$ and $a_{n,odd}$ of their respective sequences. After a little work, we obtain the following series solutions of the DE in (50),

$$y(x) = a_0\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^2 - \cdots\right] + a_1\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right]\,. \tag{53}$$

One should be able to see the pattern. Indeed, from the recursion relation, one can determine the exact form of the a_n to be as follows,

$$a_{2n} = (-1)^n\frac{1}{(2n)!}\,, \quad a_{2n+1} = (-1)^n\frac{1}{(2n+1)!}\,, \quad n \geq 0\,. \tag{54}$$

The reader should recognize each of the expansions in Eq. (53) as the first three terms of the Taylor series of $\cos x$ and $\sin x$, i.e.,

$$y(x) = a_0 \cos x + a_1 \sin x\,. \tag{55}$$

Of course, this is in agreement with our previous knowledge of this DE.

There is a large number of more complicated DEs of the form in (47) which occur quite often in applications. In these cases, the series solutions are used to define well-known functions, e.g. Bessel functions, hypergeometric functions.